Dynamic equations of massless-like particles in five-dimensional space-time derived by variation of inertial mass

Robert M.Yamaleev

Department of Physics
Universidad Nacional Autonoma de Mexico
Facultad de Estudios Superiores
Cuautitlán Izcalli Av. 1-Mayo, Campo 1
C.P.54740, México.
Email:iamaleev@servidor.unam.mx

ABSTRACT.
The paper consists of two parts. In the first part, we derive relativistic equations of motion for a massless particle from Newtonian equations by variation the inertial mass for which the module of three-velocity is fixed by the speed of light. In the second part, same algorithm is used in order to derive relativistic equations of motion with variation of proper mass. For that purpose we start from Stuckelberg-Horwitz-Piron "off mass-shell" equations which are analogous to Newtonian equations in 4D space-time. By variation the inertial mass for which the pseudo-module of four-velocity is fixed by the speed of light, we derive 5D equations of motion for *penta-* massless and *penta* -massive particles.

1 Introduction

In 1891 F.Klein [1] generalized the Hamilton-Jacobi theory for $D > 3$ spaces pointing out that in the spaces of higher dimensions a trajectory of the massive particle can be considered as a light trajectory in the corresponding medium. In order to unify gravitation with electromagnetism in 1921 T.Kaluza [2] suggested to introduce a fifth component of momentum related with the charge. The ideas of F.Klein and T.Kaluza had been deeply developed by Yu.Rumer [3]. Later 5D space-time formulations appeared for different purposes in quantum field theories [4], in the classical relativistic dynamics of charged spinning particles [5], and in the density functional theory [6].

In 1941 Stueckelberg [7] proposed a form of relativistic and quantum mechanics for one particle providing a manifestly covariant description of dynamical processes, including pair production and annihilation, and in which the particle rest mass is not constant. In 1950 Feynman [8] showed how a Schrödinger type equation, coinciding with that of Stueckelberg, could be derived from a path integral technique for the motion of a point in space-time as a function of an invariant parameter along the path. In the sequel, there has been much study of this theory by Horwitz and collaborators in which the evolution of the system was parameterized by invariant world (or historical) time τ. Investigations of several aspects of the structure of this theory have been carried out. Horwitz and Piron [9] extended the ideas of Stueckelberg to the formulation of a theory applicable to many particles with electromagnetic interaction. Saad, Horwitz and Arshansky [10] have shown that the requirement of local gauge invariance leads to five compensation fields. These fields, which have been called pre-Maxwell fields, are defined on a five-dimensional manifold with coordinates (x, τ). The interpretations of these fields through an examination of the Lorentz force and the structure of the energy-momentum tensor was done in Ref.[11].

In recent years we observe an emergence of theories overcoming the assumption that a particle's rest mass is constant (see, for instance, in addition to references cited above, [12] and references therein). In view of the problem of self-energy divergence and the field's singularities in relativistic dynamics, A.Vankov [13] suggested to consider the proper-mass of the relativistic particle as a variable depending on an external field. This assumption has been complemented with constancy of pseudo-module of the four-velocity. However, variability of the proper mass is not compatible with the main feature of the Minkowski-force to be orthogonal to four-velocity. Therefore the usage of these assumptions claims essential changes of the structure of relativistic dynamics. In this paper we explore enlarged relativistic equations of motion derived on the basis of these assumptions.

The main objective of the present paper is to show that the relativistic proper-mass variation leads us to 5D formulation of relativistic dynamics. The paper can be divided into two parts. In the first part, we elaborate a pathway from Newtonian to relativistic equations of motion of a massless particle. This pathway is passed by assuming that the mass of the Newtonian particle is not a constant, but is a variable whereas the module of the three-vector of velocity is a constant of motion.

In the second part, we elaborate an algorithm to extend the relativistic dynamics. Within the framework of the relativistic dynamics of the massive particle instead of constancy of module of the three-vector of velocity we have constancy of the peudo-module of the four-vector of velocity. In order to obtain an evolution equation for the variable proper-mass we should extend the Minkowski four-force. Therefore we start with Stueckelberg-Piron-Horwitz equations of motion, which analogous to the Newtonian equations in 4D space-time. In this case the assumption of constancy of the pseudo-module of four-velocity is a constraint. In a manner similar to that carried out in the first part, the 4D Newton-Lorentz equations are transformed into 5D relativistic equations. These equations describe charged *penta*-massless particles in the enlarged electromagnetic fields. The class of penta-massless particles includes, as a special case, the relativistic particles. In addition to the penta-massless particles we introduce also the class of *penta*-massive particles.

The paper is organized as follows.

In Section 2, we outline some features of classical relativistic dynamics. In particularly, we observe that the notion of masslessness does not bind one to take as zero the mass parameter of the massless particle.

In Section 3, we build the passage from Newtonian equations into relativistic equations of motion under two assumptions: 1) the mass of the particle is variable, and 2) the module of the velocity is a constant of motion. These assumptions unequivocally lead one to the equations for a relativistic *massless* particle if the module of velocity is taken as a speed of light.

In Section 4, the mass-variation algorithm is employed within Stueckelberg-Piron-Horwitz "off mass-shell" equations where the proper-mass is considered as a variable whereas the pseudo-module of four-velocity is constrained by the speed of light. In this way we arrive at the notion of *penta*- particles.

In Section 5, we explore a motion of charged penta-particles in interaction with 5D electromagnetic fields.

The paper closes with a summary.

2 Classical equations of motion of relativistic particle

The Lorentz-covariant form of the relativistic equations are usually written with respect to an invariant evolution parameter — the proper time of the particle:

$$\frac{d}{d\tau}p_\mu = K_\mu, \quad m_0\, g_{\nu\mu}u^\nu = p_\mu, \quad u^\nu = \frac{dx^\nu}{d\tau}. \tag{2.1}$$

where K_μ is, so-called, Minkowski-force. The main feature of the Minkowski-force is orthogonality with the four-velocity, valid if we constrain the motion to mass shell

$$K_\mu u^\mu = 0. \tag{2.2}$$

One of the concrete examples of the Minkowski-force is the Lorentz-force:

$$\frac{d}{d\tau}p_\mu = \frac{e}{c}\ F_{\mu\nu}u^\nu, \text{ with } \ F_{\mu\nu}u^\nu u^\mu = 0, \tag{2.3}$$

where $F_{\mu\nu}$ is a tensor of the electromagnetic (e.m.) field composed of the three-vectors of electric field strength $\vec{E}$ and the magnetic flux density $\vec{B}$.

It is relevant to draw an analogy between three-dimensional *gyroscopic (or ponderemotive)* force $\vec{G}$ and the four-dimensional Minkowski force. The force $\vec{G}$ does no work along the trajectory

$$\int(\vec{G}\cdot d\vec{l}) = 0. \tag{2.4}$$

The Minkowski force does not do "work" along the world-line trajectory

$$\int(K_\mu dl^\mu) = 0. \tag{2.5}$$

The G- force does not exchange the kinetic energy of the particle. The Minkowski force does not contribute to the change of mass, which is a constant of motion

$$g^{\nu\mu}p_\nu p_\mu := p_0^2 - p^2 = M^2c^2. \tag{2.6}$$

The relationship (2.6) has fundamental importance for the relativistic dynamics. In order to interpret the constant of motion M one has to reduce Eqs.(2.1) into the Newton equations of motion. For that purpose rewrite (2.1) with respect to the coordinate time by using the formula

$$u^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau}. \tag{2.7}$$

By taking into account (2.6), we get

$$\frac{d\vec{p}}{dt} = \vec{K}, \quad \frac{d\vec{r}}{dt} = c\frac{\vec{p}}{\sqrt{(Mc)^2+p^2}}. \tag{2.8}$$

In the case when

$$\frac{(Mc)^2}{p^2} >> 1$$

Eqs.(2.8) give arise Newton equations if one takes $M^2 = m_0^2$. This simple exercise suggests that the constant of motion M has to be taken equal to the proper mass of the particle m_0. Obviously, the procedure of reduction is valid only if $M^2 \neq 0$. This restriction separates the class of *massive particles* from the class of *massless particles*; the latter are defined by the condition $M = 0$. Observe, however, the condition $M = 0$ does not bind one to set m_0 equal to zero. Indeed, we have seen that an interpretation of M as a mass of the particle comes from the principle of correspondence with the Newton equations. When one takes $M = 0$ this correspondence breaks down. All the more that in this case an interpretation of M is needless.

Consider a stationary motion obeying the relativistic equations in the potential field $V(r)$:

$$\frac{d\vec{p}}{d\tau} = -\vec{\nabla}V \; p_0 \frac{1}{m_0 c}, \quad \frac{dp_0}{d\tau} = -(\vec{\nabla}V \cdot \vec{p}) \frac{1}{m_0 c}. \tag{2.9}$$

This equations come from the Lorentz-force equations in absence of the vector part of the electromagnetic potential. As a consequence of property (2.2) we get the first constant of motion :

$$p_0^2 - p^2 = M^2 c^2, \tag{2.10}$$

where $M^2 = m_0^2$ for a massive particle and $M = 0$ for a massless particle. Another constant of motion, the energy, emerges due to stationarity of the potential field:

$$cp_0 + V(r) = \mathcal{E}. \tag{2.11}$$

Now, let us apply this system of equations in order to describe the motion of the massless particle. In this case $M = 0$ and, consequently, $p = \pm p_0$. The pair of Eqs.(2.9) are reduced into one equation for the three-vector of momentum:

$$(a)\; \frac{d\vec{p}}{d\tau} = -\vec{\nabla}V \; p \frac{1}{m_0 c}, \quad \text{with} \quad (b)\; \vec{p} = m_0 \frac{d\vec{r}}{d\tau}, p = m_0 c \frac{dt}{d\tau}. \tag{2.12}$$

One may re-formulate Eqs.(2.12) with respect to the coordinate time. One gets

$$\frac{d\vec{p}}{dt} = -\vec{\nabla}V, \quad \frac{d\vec{r}}{dt} = c\vec{n}, \quad \vec{n} = \frac{\vec{p}}{p}. \tag{2.13}$$

This result remains valid needless of the limit $m_0 \to 0$ in order to formulate equations of motion of the massless particle. The mass parameter does not participate in Eqs.(2.13).

Our purpose was to show what kind of possibilities arise from relativistic equations of motion. From these equations, we have seen, there follows the existence two classes of particles defined by the choice of the constant of motion in (2.10). The first class, with $M^2 \neq 0$, is called as a class of *massive particles*, because the principle of correspondence with the Newton equations permits one to interpret M as the inertial mass at rest. The second class, with $M^2 = 0$, is called as a class of *massless particles*. Does there exist a massless particle in the Nature? This is a problem of the interpretation of the experimental data. Usually in physics the massless particles are identified with the photon, which moves with the speed of light. In the present paper we don't touch the problem of interpretation of the classical equations of the massless particles.

In the direction of the vector of momentum Eqs.(2.9) admit a *polar representation* [14]:

$$\frac{dp}{d\phi} = p_0, \quad \frac{dp_0}{d\phi} = p, \tag{2.14}$$

where ϕ obeys the equation

$$\frac{d\phi}{d\tau} = \frac{e}{m_0 c}(\vec{E} \cdot \vec{n}). \tag{2.15}$$

Formal solutions of (2.14) are given by

$$p_0 = A(cosh(\phi) + B\ sinh(\phi)), \quad p = A(sinh(\phi) + B\ cosh(\phi)), \tag{2.16}$$

with

$$A^2(1 - B^2) = M^2 c^2,$$

where $B^2 < 1$ for massive particles and $B^2 = 1$ for massless particles.

3 Passage from Newtonian to relativistic equations of motion

The gap between Newtonian and relativistic mechanics usually is overcome only in one direction: from the relativistic to the Newton equations by taking the speed of light to infinity. In this limit, the relativistic equations tend to the system including besides the Newton equations an equation for the kinetic energy. In order to overcome the passage in

the opposite direction the textbooks recount the history of the construction of relativistic dynamics. Meanwhile, there exists a simple way of the passing between these two parts of the classical dynamics. In this section we build a shortest pathway from Newton equations to the relativistic equations for the massless particle. Before proceeding, notice that the module of the velocity of the massless particle is a constant of motion. In fact, $v^2 = c^2$ is a consequence of Eqs.(2.13). However, the potential field inside of which the particle is moving produces work. Then the following inquiry arises:
What kind of dynamic variable accumulates kinetic-like energy?
Let us settle this question starting from Newton equations of motion.

Consider Newton equations of motion given by

$$(a)\ \frac{d\vec{p}}{dt} = -\vec{\nabla}V, \quad (b)\ m\vec{v} = m\frac{d\vec{r}}{dt} = \vec{p}. \tag{3.1}$$

From these equations the equation for the kinetic energy is derived

$$\frac{d}{dt}\frac{p^2}{2m} = -(\vec{v}\cdot\vec{\nabla}V), \tag{3.2}$$

which due to the stationarity of the potential is nothing else than an energy-conservation law:

$$\frac{d}{dt}(\frac{p^2}{2m} + V) = 0, \quad \mathcal{E} = \frac{p^2}{2m} + V. \tag{3.3}$$

In these equations we supposed that the mass of the particle m is a constant. Then the work of the potential field contributes to the kinetic energy, or more precisely, to the module of velocity. Now, let us to fix the module of the velocity simultaneously freeing the mass m. Thus, we obtain the situation when the mass depends of the time $m = m(t)$ whereas the module of the vector of velocity is a constant of motion:

$$v^2 = c^2. \tag{3.4}$$

Consequently, the velocity is orthogonal to the acceleration

$$(\frac{d\vec{v}}{dt}\cdot\vec{v}) = 0. \tag{3.5}$$

This condition helps us to derive an equation for the mass from Eqs.(3.1). We get

$$c^2\frac{dm}{dt} = -(\vec{v}\cdot\vec{\nabla}V). \tag{3.6}$$

Compare this equation with (3.2). Since we deal with a stationary potential, from (3.6) we get the following energy-conservation law:

$$\frac{d}{dt}(mc^2 + V) = 0, \quad \mathcal{E} = mc^2 + V. \tag{3.7}$$

Thus, the kinetic part of the energy of the particle with variable mass and the top line module of the velocity is proportional to the mass of the particle.

Now suppose that the variation of the mass begins at the moment $t = t_0$. Let us denote the initial value of the mass by $m_0 = m_{t=t_0}$. Introduce a new evolution parameter τ by

$$\frac{dt}{d\tau} = \frac{m}{m_0}. \tag{3.8}$$

Define the velocity with respect to the new evolution parameter

$$\vec{u} = \frac{d\vec{r}}{d\tau}.$$

The velocities $\vec{v}$ and $\vec{u}$ are related as follows

$$\vec{u} = \frac{dt}{d\tau}\frac{d\vec{r}}{dt} = \frac{m}{m_0}\vec{v}. \tag{3.9}$$

Correspondingly, formulae of the momentum are given by

$$\vec{p} = m_0\vec{u} = m\vec{v}. \tag{3.10}$$

Introduce a new component of momentum corresponding to variation of the mass by

$$p_0 = mc. \tag{3.11}$$

By using the definitions (3.10) and (3.11) we can equivalently re-write Eq.(3.6) as

$$\frac{d}{dt}\frac{p_0^2}{2} = -(\vec{\nabla} \cdot \vec{p}). \tag{3.6a}$$

In order to formulate equations (3.1) and (3.6a) with respect to τ we should multiply these equations by derivative $\frac{dt}{d\tau}$ and use (3.8). Then, by using definitions (3.10) and (3.11), Eqs.(3.1), (3.6a) equivalently are transformed into the following set of equations

$$\frac{d\vec{p}}{d\tau} = -\frac{1}{m_0 c}\vec{\nabla}V\ p_0, \quad \frac{dp_0}{d\tau} = -\frac{1}{m_0 c}(\vec{\nabla}V \cdot \vec{p}), \tag{3.12a}$$

with

$$\vec{p} = m_0 \frac{d\vec{r}}{d\tau}, \quad p_0 = m_0 c \frac{dt}{d\tau}. \tag{3.12b}$$

These equations are nothing else than the relativistic equations of motion (2.9) with the first constant of motion

$$p_0^2 - p^2 = C. \tag{3.12c}$$

Notice, the same formula for C follows from Eqs.(3.1), (3.6a) written with respect to t.

Remember that Eqs.(3.12) have been derived under the assumption (3.4). Multiply (3.4) by m and take into account definitions (3.10) and (3.11). Then we get $p_0^2 - p^2 = 0$ which corresponds to $C = 0$. Thus, in (3.12c) we have to take $C = 0$.

Now, let us try to arrive at the relativistic equations for a massive particle. In this case we have to suppose variability of the mass without restricting the module of velocity. For that purpose we start from the Newton-Lorentz equations with external electromagnetic fields

$$\frac{d\vec{p}}{dt} = e(\vec{E} + [\vec{v} \times \vec{B}]), \quad \vec{p} = m\vec{v}. \tag{3.13}$$

Suppose $m = m(t)$ and $m_0 = m(t)_{t=0}$. Define a new evolution parameter τ by (3.8) and re-define velocity and momentum according to (3.9), (3.10). With respect to τ Eq.(3.13) takes the following form

$$\frac{d\vec{p}}{d\tau} = e(\vec{E}\frac{m}{m_0} + [\vec{u} \times \vec{B}]), \quad \vec{p} = m_0\vec{u}. \tag{3.14}$$

An equation for the mass variable now cannot be derived from (3.13). This equation has to be postulated independently. If one makes a postulate of form of Eq.(3.6), then one arrives at the relativistic Lorentz-force equations (2.3). Eq.(3.6) in the external electromagnetic field is written by replacing $\vec{\nabla}V$ with $-e\vec{E}$. One gets

$$c^2 \frac{dm}{dt} = e(\vec{v} \cdot \vec{E}), \tag{3.15}$$

which with respect to the proper time as an evolution parameter takes the following form

$$c^2 \frac{dm}{d\tau} = e(\vec{u} \cdot \vec{E}). \tag{3.16}$$

By taking into account formulae (3.10) and (3.11), Eqs.(3.14), (3.16) one may equivalently rewrite the equations of motion as follows

$$\frac{d\vec{p}}{d\tau} = \frac{e}{m_0 c}(\vec{E}\, p_0 + [\vec{p} \times \vec{B}]), \quad \frac{dp_0}{d\tau} = \frac{e}{m_0 c}(\vec{p} \cdot \vec{E}),$$

which are nothing else than the Lorentz-force equations (2.3) written in the components.

Let us close this section with the following remarks.
1. The Newtonian particle with the variable mass and with the velocity constrained by $v^2 = c^2$, where c is not obliged to be speed of light, will imitate behavior of the massless particle.
2. The celebrated Einstein formula $\mathcal{E} = mc^2$ is derived within the framework of Newtonian mechanics where constancy of the module of the velocity compensates by variation of the mass.
3. The constant of motion $C \neq 0$ is defined by $C = m_0^2 c^2 - p(0)^2$, where $p(0)$ is momentum at the initial moment of the motion.

4 Extension of Stuckelberg-Horwitz-Piron "off-shell" equations by variation of mass parameter

In the previous section we have seen that variation of the mass enables us to transform Newton equations of motion into relativistic equations of motion. In this algorithm the law of variation of the mass is derived by fixing the module of the velocity. In the relativistic mechanics of the massive particle the pseudo-module of the four-vector of velocity is equal to the speed of light. This gives an idea to use the algorithm of variation of the proper mass within the framework of the relativistic dynamics. However, as we have mentioned above, the variation of the proper mass contradicts with the assumption that the Minkowski-force has to be orthogonal to the four-velocity. Let us explore this situation more in detail. Let K_μ be a Minkowski-force. Then

$$K_\nu u^\nu = 0, \tag{4.1}$$

whereas we suppose that the proper-mass is a variable. Denote the variable proper-mass by m. Then from Eqs.(2.1) we obtain an equality

$$\frac{d}{d\tau}(m^2 g_{\mu\nu} u^\mu u^\mu) = 0.$$

That is to say, the expression within the brackets is a constant of motion

$$m^2 u_\mu u^\mu = C.$$

Consequently, the pseudo-module of four-velocity is not a constant of motion, because

$$u_\mu u^\mu = \frac{C}{m^2}.$$

From these formulae we come to the following conclusion:
If four-force is the Minkowski-force then both the proper-mass and the pseudo-module of four-velocity have to be constants, or the both have to be variables.

Now, let us abandon the condition (4.1). We must keep in mind, however, removing this main property of the Minkowski-force we indeed abandon the usual framework of relativistic dynamics. Denote the four-force by F_μ, for which the inequality holds true

$$F_\nu u^\nu \neq 0. \tag{4.2}$$

This four-force will produce work along the world-line, that is

$$\int_b^a (F_\nu dl^\nu) \neq 0. \tag{4.3}$$

Consequently, the four-kinetic energy

$$\mathcal{E}_{kin} = \frac{1}{2m} g^{\mu\nu} p_\mu p_\mu, \tag{4.4}$$

will change. Thus, the inequality (4.2) lead us out of the conventional framework of the relativistic dynamics. Since all four of the components of energy- momentum are kinematically independent, the theory is the Newtonian-like dynamics in 4D Minkowski space. This is very dynamics introduced by Stuckelberg and developed by Horwitz and Piron and collaborators (see, references). In order to describe a system in interaction, it is necessary to consider states which are not restricted to the "mass shell". Therefore it is postulated that the states of a particle are described by eight independent variables $x^\mu,\ p_\mu$. The usual point of view is that the motion is described by a relation between all these variables which defines a trajectory on the phase space $x^\mu,\ p_\mu$. In the "off mass shell" theory a parameter of the evolution τ is not a proper time of the particle. Horwitz and Piron [9] interpreted this parameter as *historical time.*

The canonical 4D-equations of motion are written as

$$\frac{d}{d\tau} p_\mu = F_\mu, \quad m\, g_{\nu\mu} u^\nu = p_\mu, \quad u^\nu = \frac{dx^\nu}{d\tau}. \tag{4.5}$$

These equations fall into the usual framework of the relativistic dynamics as soon as the force F_μ is taken as a Minkowski-force. In Hamiltonian form these equations are written as follows

$$\frac{dp_\mu}{d\tau} = -\frac{\partial H}{\partial x^\mu}, \quad \frac{dx^\mu}{d\tau} = +\frac{\partial H}{\partial p_\mu}, \tag{4.6}$$

where H is the Hamiltonian of the system. Consider a canonical form of this Hamiltonian

$$H = \frac{1}{2m} g^{\mu\nu} p_\mu p_\mu + V(x^\mu).$$

If the external Lorentz-scalar potential field $V(x^\mu)$ is not trivial then the kinetic energy $\mathcal{E}_{kin}$ is a variable. This means that the theory is "off mass shell". One may try to return to the usual framework by using the constraint

$$g_{\mu\nu} u^\mu u^\mu = c^2. \tag{4.7}$$

Obviously, this constraint in the external Lorentz-scalar potential field $V(x^\mu)$ has to be considered together with the possibility for the proper-mass to be variable. In this way we arrive at the situation just explored in the previous section for 3D case. On making use of the constraint (4.7) from Eqs.(4.6) we derive an equation for the variation of mass parameter m:

$$\frac{dp_4}{d\tau} = \frac{1}{c} u^\mu F_\mu, \tag{4.8}$$

where

$$p_4 := mc, \quad F_\mu := -\frac{\partial V}{\partial x^\mu}.$$

Further, denote the initial value of the mass at the initial moment $\tau = 0$ by $\mu = m(\tau = 0)$ and introduce a new evolution parameter s via the relation

$$\frac{d\tau}{ds} = \frac{m}{\mu}. \tag{4.9}$$

Define the velocity and the momentum with respect to s:

$$w^\mu = \frac{dx^\mu}{ds}, \quad p_\mu = \mu\, g_{\mu\nu} w^\nu. \tag{4.10}$$

The following relations hold true

$$w^\mu = \frac{d\tau}{ds}\frac{dx^\mu}{d\tau} = \frac{m}{\mu}\, u^\mu, \quad p_\mu = \mu\, g_{\mu\nu} w^\nu = m\, g_{\mu\nu} u^\nu. \tag{4.11}$$

Employing s as an evolution parameter we cast Eqs. (4.5) and (4.8) into the following form (with 4.11)

$$\frac{d}{ds}p_\mu = \frac{m}{\mu}\ F_\mu, \quad \frac{dp_4}{ds} = \frac{1}{c\mu}\ g^{\nu\mu}p^\nu F_\mu. \tag{4.12}$$

These equations imply the following integral of motion:

$$p^\nu p_\nu - p_4^2 = C. \tag{4.13}$$

The passage from Eqs.(4.6) to Eqs.(4.8) is similar to the passage from non-relativistic equations to the relativistic one. Notice, Eqs.(4.8) and (4.12) are equivalent to equations (4.6) with constraint (4.7) if only if $C = 0$. In this way we come to the following relationship

$$p^\nu p_\nu = m^2 c^2, \tag{4.14}$$

which looks like as mass-shell equation, but now the mass m is a variable. By substituting relationships (4.11) into (4.14) we get null 5D metric interval

$$(cd\tau)^2 - (cdt)^2 + (dr)^2 = 0.$$

In addition to the case with $C = 0$, Eqs.(4.8),(4.12) admit also a constant of motion different from zero, i.e. $C \neq 0$. This constant has to be equal to the term $p^\nu p_\nu - m^2c^2$ at the initial moment $\tau = 0$. That is,

$$C := p^\nu(0)p_\nu(0) - \mu^2c^2.$$

This case corresponds to the non-trivial 5D interval

$$(cd\tau)^2 - (cdt)^2 + (dr)^2 = ds^2.$$

Further, consider the variable $p_4 = mc$ as a *penta*-component of the momentum: $p_4 = mc$. In these terms Eqs.(4.10) are written as follows

$$\frac{d}{ds}p_\mu = \frac{1}{\mu c}\ p_4 K_\mu, \quad \frac{d}{ds}p_4 = \frac{1}{\mu c}\ g^{\mu\nu}p_\mu K_\nu. \tag{4.15}$$

Like for the case of relativistic particles, we introduce two class of particles characterized by the value of the constant of motion.
The first class (the class of *penta*-massless particles) is defined by the condition (4.14). The second class (the class of *penta*-massive particles) is defined by the relationship

$$p^\nu p_\nu - p_4^2 = \mu^2c^2. \tag{4.16}$$

The relativistic massive particles form a subset of the class of *penta*-massless particles.

5 5D Lorentz-force equations for charged penta-particles

In this section we explore a motion of charged penta-particles in the presence of 5D electromagnetic fields.

Before proceeding, let us recall a motion of the Newtonian particle inside stationary magnetic field

$$\frac{dp_k}{dt} = e\ v^l\ B_{kl}, \tag{5.1}$$

where B_{kl} antisymmetric tensor of the magnetic flux density. The force $G = e\ v^l\ B_{kl}$ is orthogonal to the three-velocity. We remark that the magnetic field does not contribute to the change of the kinetic energy. Now, consider Newton-Lorentz force with external electromagnetic fields,

$$\frac{d\vec{p}}{dt} = e(\vec{E} + [v \times \vec{B}]), \quad \vec{p} = m\vec{v}. \tag{5.2a}$$

In this force the electric field contributes to the change of the kinetic energy according to the law

$$\frac{1}{2m}\frac{dp^2}{dt} = e(\vec{E}\cdot\vec{v}). \tag{5.2b}$$

The relativistic Lorentz-force is analogous to the magnetic field, whereas the Stuckelberg-Horwitz-Piron 4D "off-shell" Lorentz-force [11] analogous to 3D Newton-Lorentz force (5.2a).

In the framework of a covariant relativistic quantum theory, in which the evolution of the system is parameterized by invariant world (historical) time, Saad, Horwitz, Arshansky [10] have shown that the requirement of local gauge invariance leads to five compensation fields. These fields have been called pre-Maxwell fields.

Consider the "off-shell" Schrödinger equation

$$i\frac{\partial}{\partial\tau}\Psi = \frac{1}{2m}p_\mu p^\mu \Psi. \tag{5.3}$$

This equation may be made locally gauge-invariant under the transformations

$$\Psi \to exp(iq\Lambda(x,\tau))\Psi,$$

through introduction of compensation fields. Land and Horwitz [11] have formulated the Lorentz-force associated with the pre-Maxwell fields $f_{\mu 4}, f_{\nu\mu}$:

$$\frac{d}{d\tau}p_\mu = e\ f_{\mu 4} + e\ u^\nu f_{\nu\mu}, \quad p_\mu = mg_{\mu\nu}u^\nu. \tag{5.4}$$

This force generates changes in the 4D-kinetic energy according to the law

$$\frac{1}{2m}\frac{d}{d\tau}(p_\mu p^\mu) = e\ f_{\mu 4} u^\mu. \tag{5.5}$$

In the absence of the term $e\ f_{\mu 4}$ Eqs.(5.3) are reduced into relativistic Lorentz-force equations (2.3). In this case the Lorentz-force does not produce a 4D-work, consequently, the 4D-kinetic energy is a constant of motion and can be interpreted via proper mass of the particle.

The field tensor satisfies the field equations

$$\partial_\beta f^{\alpha\beta}(x,\tau) = ej^\alpha(x,\tau).$$

Integrating the $\alpha = \mu$ components of this equation on $(-\infty,\infty)$ over τ, with the condition that $f_{\mu 4}$ vanishes at $\tau \to \pm\infty$, one recovers the Maxwell equation [11]

$$\partial_\nu F^{\mu\nu}(x) = eJ^\mu(x).$$

The integration over τ has been called "concatenation" and provides the link between the notion of an *event* along a world line and the notion of a *particle*, whose support in space-time is the world line. The representation of $f^{\alpha\beta}(x,\tau)$ as the antisymmetric derivative of a five-field is equivalent to the homogeneous equation

$$\partial^\alpha \epsilon_{\alpha\beta\gamma\delta\sigma} f^{\alpha\beta}(x,\tau) = 0,$$

analogous to the homogeneous Maxwell equations.

In a manner similar to the way the 3D Newton-Lorentz equations were transformed into relativistic Lorentz-force equations, the 4D "off mass shell" Lorentz equations can be enlarged into *penta*-Lorentz-force equations by variation of the mass parameter. With respect to s as an evolution parameter the *penta*-Lorentz-force equations are written as follows

$$\frac{d}{ds}p_\alpha = e\ w^\beta f_{\beta\alpha},\ \ p_\alpha = \mu\ g_{\alpha\beta} w^\beta,\ \ \alpha,\beta = 0,1,2,3,4, \tag{5.6}$$

where

$$f_{\beta\alpha} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}, \tag{5.7}$$

and

$$p_\mu = \mu\ g_{\mu\nu} w^\nu,\ (\mu,\nu = 0,1,2,3),\ p_4 = \mu c\ \frac{d\tau}{ds}. \tag{5.8}$$

The last relationship in (5.8) admits re-formulation of Eqs.(5.6) with respect to τ. This leads us to the following set of equations

$$\frac{d}{d\tau}p_\mu = e\ F_{\mu 4} + e\ u^\nu F_{\nu\mu}, \tag{5.9a}$$

$$p_\mu = \mu\ g_{\mu\nu}\frac{u^\nu}{\sqrt{1 - g_{\mu\nu}\frac{u^\nu u^\mu}{c^2}}}. \tag{5.9b}$$

Compare these equations with Eqs.(2.8). In particlularly, in the "static" potential field for which there is no s dependence the penta-equations take on the form

$$\frac{d}{d\tau}p_\mu = -\frac{\partial\Phi}{\partial x^\mu},\quad \frac{d}{d\tau}p_4 = -u^\mu\frac{\partial\Phi}{\partial x^\mu}. \tag{5.10}$$

These equations imply two constants of motion:

$$(a)\ p^2 - p_4^2 = C, \quad (b)\mathcal{E} = cp_4 + \Phi.$$

The case $C = 0$ corresponds to the 5D massless particle, whereas the case $C = \mu^2 c^2$ corresponds to 5D the massive particle, we have denominated, respectively, by *penta*-massless and *penta*-massive particles.

Notice that the equations of motion for the *penta*-massless particle written with respect to τ do not contain a mass parameter μ:

$$\frac{d}{d\tau}p_\mu = F_\mu,\quad \frac{d}{d\tau}x^\mu = \frac{p^\mu}{p},\quad p = \sqrt{p^\nu p_\nu}.$$

6 Summary

We have started by outlining some peculiar features of the classical relativistic dynamics. In particular, we have explored the distinction between massive and massless particles. The actual proposed theory is applied to charged particles with mass m_0 and the charge e; such particles experience an electric/magnetic field and, thus, a Lorentz-force coded into a Minkowski force term(with the consequent orthogonality between velocity and acceleration). One of the important conclusions of this section is that the relativistic condition of masslessness does not oblige one to set the inertial mass of the particle to zero. Furthermore, we have shown that the relativistic equations of the massless particle, as well as the celebrated Einstein formula, can be derived within Newtonian mechanics by variation of the inertial mass. In order to derive an evolution law for the mass we have fixed the module of the velocity by the speed of light. In this way we derive the relativistic equations of motion for the massless particle.

The Stueckelberg-Piron-Horwitz "off- mass-shell" equations play the role, in Minkowski space, of Newtonian type equations. Relativistic equations in higher dimension have been derived from these equations with proper-mass variation by using the relativistic relationship for the pseudo-module of four-velocity as a constraint. In this way we arrived at the equations of the *penta*-massless particles. When an external force is the Minkowski force the penta-massless particles give arise the relativistic massive particle. Like in the relativistic dynamics, beside the class of penta-massless particles we have introduced also the class of *penta*-massive particles.

ACKNOWLEDGMENTS

I thank Jaime Keller and Adan Rodriguez for encouragement and discussions, and Valery Dvoeglazov for pointing out the paper of A.Vankov [13]. I am grateful for the referees for helpful critique of the manuscript.

References

[1] F.Klein, Gesamelte Math.Abh.-Berlin:J.Springer (1921).
F.Klein, *Elementarmathematik Vom Hoheren Standpunkte Aus Geometrie.* Vol.2 Berlin: J.Springer (1925).

[2] Th. Kaluza, Sitzungsberg. d. Berlin Acad. Klasse, 966, (1921).
L.O'Raifertaigh and N.Strumann, Rev. Mod. Phys. **72**, 1, (2000).

[3] Yu.B.Rumer, J. Exp. Theor. Phys. (Russian), **19**, 86, (1949); **19**, 207, (1949); **19**, 869, (1949).

[4] V.G.Kadyshevsky, J. of Particles and Nuclei (Rassian), **11**, 5, (1980).
J.R.Fanchi, *Parametrized Relativistic Quantum Theory* Kluwer, Dordrecht, (1993).

[5] J.Keller and R.M.Yamaleev, J. Adv.in Appl.Clifford Algebras, **7**, (2), 141, (1997).
J.Keller , R.M.Yamaleev, A.Rodriguez, J. Adv. in Appl. Clifford Algebras, **6**, (2), 275, (1996).

[6] J.Keller, J. Rev.Soc.Quim.Mex., **44**, No.1, 22, (2000).

[7] E.C.G.Stueckelberg, Helv. Phys. Acta, **14**, 316, (1941).

[8] R.P.Feynman, Phys.Rev., **80**, 440, (1950).

[9] L.P.Horwitz and C.Piron, Helv. Phys. Acta, **46**, 316, (1973).
L.P.Horwitz, C.Piron and F.Reuse, Helv. Phys. Acta, **48**, 546, (1975).

[10] D.Saad, L.P.Horwitz, and R.I.Arshansky, J.Found.Phys., **19**, 1126, (1989).
R.I.Arshansky and L.P.Horwitz, J.Math.Phys., **30**, 66, (1989).

[11] M.C.Land and L.P.Horwitz, J. Found.Phys.Lett., **Vol.4**, no.1, 61, (1993).

[12] Calin Galeriu, Ann. Found.L.de Broglie, **Vol.23**, no.1, 61, (2003).

[13] A.Vankov, *On Problem of Mass Origin and Self-Energy Divergence in Relativistic Mechanics and Gravitational Physics*, arXiv:gr-qc/0311063 v1 (2003).

[14] R.M.Yamaleev, Ann. of Phys. (N.Y.), **285**, no.2, 1, (2000).
R.M.Yamaleev, Ann. of Phys.(N.Y.), **292**, no.2, 157 (2001).

(Manuscrit reçu le 10 Septembre 2004)