Are Abelian and Nonabelian Gauge Field Quantum Theories Elementary?

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1 Introduction

Are gauge theories the ultimate means for the description of the dynamics of matter or can these theories be explained by the action of still more elementary entities?

Seventy years ago a first step in this direction was made by de Broglie. In his theory of fusion de Broglie showed that classical electrodynamics can be deduced by considering photons as composite particles, the constituents of which are neutrinos, i.e., in this way electrodynamics is not an elementary but an effective theory, [1],[2].

In the meantime nonabelian gauge theories were invented which in general are only used on the quantum level, [3],[4]. Thus with respect to their elementarity the same question can be posed as in the case of de Broglie's derivation of electrodynamics.

However, such a problem cannot be treated within de Broglie's original fusion theory. In order to reveal that current quantum field theories of matter are effective theories, a quantum field theoretic version of de Broglie's theory of fusion is needed.

Such a model based on a nonlinear spinor field was developed in the past decades, [5],[6],[7], and, in particular, nonabelian gauge field quantum theories were derived in [8]-[14]. In this article an updated and improved summary of these treatments will be given.

2 Algebraic representation of the spinor field

The algebraic representation is the basic formulation of the spinor field model and the starting point for its evaluation. We give only some basic formulas of this formalism and refer for details to [5],[6], [7]. The corresponding Lagrangian density should not be identified with similar (effective) Lagrangians of nuclear physics and reads, see [6], eq. (2.52)

$$\mathcal{L}(x) := \sum_{i=1}^{3} \lambda_i^{-1} \bar{\psi}_{A\alpha i}(x)(i\gamma^\mu \partial_\mu - m_i)_{\alpha\beta} \delta_{AB} \psi_{B\beta i}(x) \tag{1}$$
$$- \frac{1}{2} g \sum_{h=1}^{2} \delta_{AB}\delta_{CD} v^h_{\alpha\beta} v^h_{\gamma\delta} \sum_{i,j,k,l=1}^{3} \bar{\psi}_{A\alpha i}(x)\psi_{B\beta j}(x)\bar{\psi}_{C\gamma k}(x)\psi_{D\delta l}(x)$$

with $v^1 := \mathbf{1}$ and $v^2 := i\gamma^5$. The field operators are assumed to be Dirac spinors with index $\alpha = 1,2,3,4$ and additional isospin with index $A = 1,2$ as well as auxiliary fields with index $i = 1,2,3$ for nonperturbative regularization. The algebra of the field operators is defined by the anticommutators

$$[\psi^+_{A\alpha i}(\mathbf{r},t)\psi_{B\beta j}(\mathbf{r}',t)]_+ = \lambda_i \delta_{ij}\delta_{AB}\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}') \tag{2}$$

resulting from canonical quantization. All other anticommutators vanish.

If the adjoint spinors are replaced by formally charge conjugated spinors

$$\psi^c_{A\alpha i}(x) = C_{\alpha\beta}\bar{\psi}_{A\beta i}(x) \tag{3}$$

a uniform transformation property with respect to Lorentz transformations can be obtained which leads with the index Λ to the introduction of superspinors:

$$\psi_{A\Lambda\alpha i}(x) = \begin{pmatrix} \psi_{A\alpha i}(x); \Lambda = 1 \\ \psi^c_{A\alpha i}(x); \Lambda = 2 \end{pmatrix} \tag{4}$$

Then the set of indices is defined by $Z := (A, \Lambda, \alpha, i)$.

For obtaining definite results, a state space is needed in which the dynamical equations resulting from (1) can be formulated. This is achieved by the use of the algebraic Schroedinger representation. For a detailed discussion we refer to [5],[6],[7].

To ensure transparency of the formalism we use the symbolic notation

$$(\psi_{I_1}...\psi_{I_n}) := \psi_{Z_1}(\mathbf{r}_1,t)...\psi_{Z_n}(\mathbf{r}_n,t) \tag{5}$$

with $I_k := (Z_k, \mathbf{r}_k, t)$. Then in the algebraic Schroedinger representation a state $|a\rangle$ is characterized by the set of matrix elements

$$\tau_n(a) := \langle 0|\mathcal{A}(\psi_{I_1}...\psi_{I_n})|a\rangle, \qquad n = 1...\infty \tag{6}$$

where $\mathcal{A}$ means antisymmetrization in $I_1...I_n$.

By means of this definition the calculation of an eigenstate $|a\rangle$ is transferred to the calculation of the set of matrix elements (6) which characterize this state. For a compact formulation of this method, generating functionals are introduced and the set (6) is replaced by the functional state

$$|\mathcal{A}(j;a)\rangle := \sum_{n=1}^{\infty} \frac{i^n}{n!} \sum_{I_1...I_n} \tau_n(I_1...I_n|a) j_{I_1}...j_{I_n}|0\rangle_F \tag{7}$$

where $j_I := j_Z(\mathbf{r})$ are the generators of a CAR-algebra with corresponding duals $\partial_I := \partial_Z(\mathbf{r})$ which satisfy the anticommutation relations

$$[j_I, \partial_{I'}] = \delta_{ZZ'}\delta(\mathbf{r} - \mathbf{r}') \tag{8}$$

while all other anticommutators vanish. With $\partial_I|0\rangle_F = 0$ the basis vectors for the generating functional states can be defined. The latter are not allowed to be confused with creation and annihilation operators of particles in physical state spaces. According to (7) to each state $|a\rangle$ in the physical state space we associate a functional state $|\mathcal{A}(j;a)\rangle$ in the corresponding functional space. The map is biunique and the symmetries of the original theory are conserved. For details see [5],[6].

In order to find a dynamical equation for the functional states, we apply to the operator products (5) the Heisenberg formula

$$i\frac{\partial}{\partial t}\mathcal{A}(\psi_{I_1}...\psi_{I_n}) = [\mathcal{A}(\psi_{I_1}...\psi_{I_n}), H]_- \qquad n = 1...\infty \tag{9}$$

where H is the Hamiltonian of the system under consideration.

If $|0\rangle$ as well as $|a\rangle$ are assumed to be eigenstates of H, then between both states the matrix elements of (9) can be formed and subsequent evaluation of these expressions leads to the functional equation

$$\begin{aligned} E_0^a|A(j;a)\rangle = [&K_{I_1I_2}j_{I_1}\partial_{I_2} - W_{I_1I_2I_3I_4}j_{I_1}(\partial_{I_4}\partial_{I_3}\partial_{I_2}+ \\ &A_{I_4J_1}A_{I_3J_2}j_{J_1}j_{J_2}\partial_{I_2})]|A(j;a)\rangle \end{aligned} \tag{10}$$

with $E_0^a = E_a - E_0$. For details of the derivation, see [5],[6].

The symbols which are used in (10) are defined by the following relations

$$K_{I_1I_2}:=iD^0_{I_1I}(D^k\partial_k - m)_{II_2}, \qquad W_{I_1I_2I_3I_4} := -iD^0_{I_1I}V_{II_2I_3I_4} \quad (11)$$

$$D^\mu_{I_1I_2}:=i\gamma^\mu_{\alpha_1\alpha_2}\delta_{A_1A_2}\delta_{\Lambda_1\Lambda_2}\delta_{i_1i_2}\delta(\mathbf{r}_1-\mathbf{r}_2) \quad (12)$$

$$m_{I_1I_2}:=m_{i_1}\delta_{\alpha_1\alpha_2}\delta_{A_1A_2}\delta_{\Lambda_1\Lambda_2}\delta_{i_1i_2}\delta(\mathbf{r}_1-\mathbf{r}_2) \quad (13)$$

$$V_{I_1I_2I_3I_4}:=\sum_{h=1}^{2} g\lambda_{i_1}B_{i_2i_3i_4}v^h_{\alpha_1\alpha-2}\delta_{A_1A_2}\delta_{\Lambda_1\Lambda_2}(v^hC)_{\alpha_3\alpha_4}\delta_{A_3A_4}\delta_{\Lambda_31}\delta_{\Lambda_42} \quad (14)$$
$$\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_1-\mathbf{r}_3)\delta(\mathbf{r}_1-\mathbf{r}_4)$$
$$B_{i_2i_3i_4}:=1; \qquad i_2,i_3,i_4=1,2,3$$

and the anticommutator matrices

$$A_{I_1I_2} := \lambda_{i_1}(C\gamma^0)_{\alpha_1\alpha_2}\delta_{A_1A_2}\sigma^1_{\Lambda_1\Lambda_2}\delta_{i_1i_2}\delta(\mathbf{r}_1-\mathbf{r}_2) \quad (15)$$

So far this functional equation holds for any algebraic representation. A special representation can be selected by specifying the corresponding vacuum. This can be achieved by the introduction of normal ordered functionals which are defined by

$$|\mathcal{F}(j;a)\rangle := exp[\frac{1}{2}j_{I_1}F_{I_1I_2}j_{I_2}]|\mathcal{A}(j;a)\rangle \quad =: \sum_{n=1}^{\infty}\frac{i^n}{n!}\varphi_n(I_1...I_n|a)j_{i_1}...j_{I_n}|0\rangle_F \quad (16)$$

where the two-point function

$$F_{I_1I_2} := \langle 0|\mathcal{A}\{\psi_{Z_1}(\mathbf{r}_1,t)\psi_{Z_2}(\mathbf{r}_2,t)\}|0\rangle \quad (17)$$

contains an information about the groundstate and thus fixes the representation.

The normal ordered functional equation then reads, see [5], Theorem 3.13

$$E^a_0|\mathcal{F}(j;a)\rangle = \mathcal{H}_F(j,\partial)|\mathcal{F}(j;a)\rangle \quad (18)$$

with

$$\mathcal{H}_F(j,\partial):=J_{I_1}K_{I_1I_2}\partial_{I_2} + W_{I_1I_2I_3I_4}[j_{I_1}\partial_{I_4}\partial_{I_3}\partial_{I_2} - 3F_{I_4K}j_{I_1}j_K\partial_{I_3}\partial_{I_2}$$
$$+(3F_{I_4K_1}F_{I_3K_2}+\frac{1}{4}A_{I_4K_1}A_{I_3K_2})j_{I_1}j_{K_1}j_{K_2}\partial_{I_2} \quad (19)$$
$$-(F_{I_4K_1}F_{I_3K_2}+\frac{1}{4}A_{I_4K_1}A_{I_3K_2})F_{I_2K_3}J_{I_1}j_{K_1}j_{K_2}j_{K_3}]$$

Equation (19) is the algebraic Schroedinger representation of the spinorfield Lagrangian (1), written in functional space with a fixed algebraic state space. With respect to the physical interpretation of this formalism two comments have to be added:

i)The algebraic Schroedinger representation is the formulation of the Hamilton formalism for quantum fields independently of perturbation theory. Nevertheless for the justification of this procedure the findings of perturbation theory are indispensible.

From perturbation theory one learns that during the time interval where mutual interactions between the particles take place, these particles cannot be kept on their mass shell. This means that in order to get nontrivial interactions one is not allowed to enforce the particles on their mass shell all the time.

The formal treatment of perturbation theory starts with the Schroedinger picture, i.e., the Hamilton formalism which explicitly avoids the on shell fixing of particle masses. Hence if this treatment of the perturbation theory is extended to the case of composite particle interactions, the use of the Hamilton operator leads to the formalism introduced above.

Then one will ask whether in consequence of this procedure relativistic covariance is completely lost. In perturbation theory this is not the case, because the equivalence of the Hamilton formalism and the covariant formulation can be shown. A similar result can be obtained for composite particle theory in algebraic Schroedinger representation: The corresponding effective theories for composite particle reactions can be likewise covariantly formulated.

ii) In general, in the literature, spinor field models, like the NJL-model in nuclear physics, are considered as effective, low energy theories. If spinor field models are to play a more fundamental role, they need a special preparation to suppress divergencies. In the case under consideration this preparation is expressed by the introduction of auxiliary fields with constants λ_i in (1),(2) and (14),(15) which are designed to generate a nonperturbative intrinsic regularization of the theory.

This regularization is closely connected with the probability interpretation of the theory. As the λ_i are indefinite an indefinite state space of the auxiliary fields results. Hence these auxiliary fields are unobservable and a special definition of a corresponding physical state space is needed. This definition is identical with the intrinsic regularization prespription and leads in turn to probability conservation in physical state space.

Furthermore it can be shown that the latter property can be transferred to the corresponding effective theories themselves. For details we refer to the literature [7]

3 Bosonization in functional space

For showing the non-elemental character of gauge theories, we treat the simple case of collective modes where bound states are represented by fermion pairs.

In nuclear physics this treatment has a long history which in a general way can be summarized as follows: Given a fermionic Hamiltonian in Fock space, a transformed Hamiltonian has to be derived which describes the system dynamics in terms of an equivalent independent Fock space of boson operators associated to the fermion pairs, cf. [15],[16]

In the case under consideration the algebraic Schroedinger representation of the spinor field replaces the ordinary Schroedinger equation in Fock space and the corresponding equations are defined by a functional equation (18),(19).

A transformation of the latter functional equation to describe fermion pairs leads to a corresponding bosonic functional equation in algebraic Schroedinger representation. For this case exact mapping theorems were derived in [17],[18] and [13]. In [13] the functional mapping formalism was formulated in close analogy to ordinary bosonization in Fock space by Kerschner.

In the algebraic Schroedinger representation the central formula which defines the map into the boson representation was originally given by the expansion, see [5],[6].

$$\varphi_n(I_1...I_{2n}|a) = \frac{1}{(2n)!}\varrho(k_1...k_n)C_{k_1}^{\{I_1 I_2}...C_{k_n}^{I_{2n-1}I_{2n}\}} \tag{20}$$

where the wave functions $C_k^{II'}$ are single time functions which result from fully covariant wave functions $\varphi_k(x_1, x_2)$ by formation of the symmetric limit $t_1 \to t$, $t_2 \to t$, and by separation into parts which describe the wave functions of collective variables, see Section 4. So the wave functions $C_k^{II'}$ reflect in their structure their relativistic origin by containing the relativistic deformations, but no genuine energy eigenvalue equation can be derived for them. According to ii) the state space of the auxiliary fields is indefinite, see [5],[6].

From these facts it follows that in general C_k^* is not the dual of C_k. In contrast to the Fock space mapping methods in nuclear physics, it is thus necessary to introduce dual states $R^k_{II'}$, [8], where both types of states are assumed to be antisymmetric functions

$$C_k^{II'} = -C_k^{I'I}; \qquad R^k_{II'} = -R^k_{I'I} \tag{21}$$

and where the relations

$$\sum_{I_1 I_2} C_k^{I_1 I_2} R^{k'}_{I_1 I_2} = \delta_k^{k'} \tag{22}$$

$$\sum_k C_k^{I_1 I_2} R^k_{I_1' I_2'} = \frac{1}{2}(\delta^{I_1}_{I_1'}\delta^{I_2}_{I_2'} - \delta^{I_1}_{I_2'}\delta^{I_2}_{I_1'}) \tag{23}$$

have to be satisfied by definition, provided the wave functions $\{C_k\}$ are a complete set of antisymmetric functions in $I_1 - I_2$ -space.

In order to study the equivalence of both sides of (20), we study the inversion of (20) by multiplying (20) with the corresponding duals and summing over $I_1 ... I_{2n}$. This gives

$$R^{k_1}_{I_1 I_2} ... R^{k_n}_{I_{2n-1} I_{2n}} \varphi(I_1 ... I_{2n}|a) = \tilde{\varrho}(k_1 ... k_n|a) \tag{24}$$

with

$$\tilde{\varrho}(K_1 ... k_n|a) = S^{k_1 ... k_n}_{k_1' ... k_n'} \varrho(K_1' ... k_n'|a) \tag{25}$$

where

$$S^{k_1 ... k_n}_{k_1' ... k_n'} = \frac{1}{(2n)!} \sum_{I_1 ... I_{2n}} R^{k_1}_{I_1 I_2} ... R^{k_n}_{I_{2n-1} I_{2n}} C^{\{I_1 I_2}_{k_1'} ... C^{I_{2n-1} I_{2n}\}}_{k_n'} \tag{26}$$

The latter tensors have projector properties

$$S^{k_1 ... k_n}_{k_1' ... k_n'} S^{K_1' ... K_n'}_{l_1 ... l_n} = S^{k_1 ... k_n}_{l_1 ... l_n} \tag{27}$$

Although (20) is admitted, see Prop. 4.2 in [5], a bijective mapping for the functions appearing in (20) can only be achieved if (20) is replaced by

$$\varphi_n(I_1 ... I_{2n}|a) = \frac{1}{(2n)!} \tilde{\varrho}(k_1 ... k_n|a) C^{\{I_1 I_2}_{k_1} ... C^{I_{2n-1} I_{2n}\}} \tag{28}$$

Then according to (27) formula (24) is indeed the inverse of (28).

In the next step we define the generating functional state

$$|\mathcal{B}(b|a)\rangle = \sum_{n=1}^{\infty} \frac{1}{n!} \tilde{\varrho}(k_1...k_n|a) b_{k_1}...b_{k_n}|0\rangle_B \tag{29}$$

with functional creation operators b_k and the functional Fock vacuum $|0\rangle_B$, where with the duals ∂_k^b of b_k the commutation relations

$$[b_k, \partial_{k'}^b]_- = \delta_{kk'} \tag{30}$$

are assumed, while all other commutators vanish.

Furthermore for later use we define the projektor

$$\mathcal{P} = \sum_{n=1}^{\infty} \frac{1}{n!} b_{l_1}...b_{l_n}|0\rangle_B S^{l_1...l_n}_{l'_1...l'_n}\, {}_B\langle 0|\partial^b_{l'_n}...\partial^b_{l'_1} \tag{31}$$

which leaves the boson state (29) invariant

$$\mathcal{P}|\mathcal{B}(b|a)\rangle = |\mathcal{B}(b|a)\rangle \tag{32}$$

Therefore by the definition (16) of the fermion functional state, by the definition of the mapping relation (28) and the definition of the boson functional state (29) a bijective map between (16) and (29) is established.

It remains the task to transform the eigenvalue equation (18) for the fermion state (16) into a corresponding eigenvalue equation for the boson state (29).

The first step must be to replace the mapping definition in configuration space (28) by a corresponding mapping definition in functional space.

By generalizing the Usui- transformations of nuclear physics, [16], the following definition was introduced by Kerschner, [13]:

$$\mathcal{T}(b, \partial^f) := \sum_{n=1}^{\infty} \frac{1}{n!} b_{k_1}...b_{k_n}|0\rangle_B R^{k_1}_{I_1 I_2}...R^{k_n}_{I_{2n-1} I_{2n}}\, {}_F\langle 0|\partial^f_{I_1}...\partial^f_{I_{2n}} \tag{33}$$

which leads to the mapping definition

$$|\mathcal{B}(b|a) = \mathcal{T}(b, \partial^f)|\mathcal{F}(j|a)\rangle \tag{34}$$

and owing to the general definition of duals (22),(23) the inverse relation of (34) reads

$$|\mathcal{F}(j|a)\rangle = \mathcal{S}(j,\partial^b)|\mathcal{B}(b|a)\rangle \tag{35}$$

with

$$\mathcal{S}(j,\partial^b) := \sum_{n=1}^{\infty} \frac{1}{(2n)!} j_{I_1}...j_{I_{2n}}|0\rangle_F C_{k_1}^{I_1 I_2}...C_{k_n}^{I_{2n-1}I_{2n}}{}_B\langle 0|\partial^b_{k_n}...\partial^b_{k_1} \tag{36}$$

In contrast to nuclear physics the inverse operator of $\mathcal{T}$ is not its Hermitean conjugate, but (36).

For these operators the following relation can be derived, [13]:

$$\mathcal{T}(b,\partial^f)\mathcal{S}(j,\partial^b) = \mathcal{P} \tag{37}$$

and

$$\mathcal{S}(j,\partial^b)\mathcal{T}(b,\partial^f) = \mathbf{1}(j,\partial^f) \tag{38}$$

Using these relations equation (18) can be mapped into equation

$$E_0^a|\mathcal{B}(b|a)\rangle = \mathcal{H}_B(b,\partial^b)|\mathcal{B}(b|a)\rangle \tag{39}$$

for the boson functional states (29) with

$$\mathcal{H}_B(b,\partial^b) = \mathcal{T}(b,\partial^f)\mathcal{H}_F(j,\partial^f)\mathcal{S}(j,\partial^b) \tag{40}$$

To evaluate equation (40) the commutator between $\mathcal{T}$ and $\mathcal{H}_F$ has to be derived. This can be achieved by the combination of two special commutation relations, [13]:

$$\mathcal{T}(b,\partial^f)j_I = 2R^k_{IK} b_k \mathcal{T}(b,\partial^f)\partial^f_K \tag{41}$$

and

$$\mathcal{T}(b,\partial^f)\partial^f_I\partial^f_K = C_k^{KI}\partial^b_k\mathcal{T}(b,\partial^f) \tag{42}$$

and repeated application. Similar commutator relations were evaluated and applied in nuclear physics, [15],[16]

For the case under consideration with the functional energy operator (19) one obtains

$$\begin{aligned} E_0^a|\mathcal{B}(b|a)\rangle := \{ & K^{kk'} b_k\partial^b_{k'} + W_1^{kll'} b_k \partial^b_l \partial^b_{l'} + W_2^{kk'}(b_k + \Gamma^{ll'}_{kk''} b_l b_{l'}\partial_{k''})\partial^b_{k'} + \\ & W_3^{kk'l}(b_k + \Gamma^{l'l''}_{kk''} b_{l'} b_{l''}\partial^b_{k''}) b_{k'}\partial^b_l + \\ & W_4^{k_1k_2}(b_{k_1} + \Gamma^{l_1l_1'}_{k_1k_1'} b_{l_1} b_{l_1'}\partial^b_{k_1'})(b_{k_2} + \Gamma^{l_2l_2'}_{k_2k_2'} b_{l_2} b_{l_2'}\partial^b_{k_2'})\}|\mathcal{B}(b|a)\rangle \end{aligned} \tag{43}$$

where the various tensor coefficients are defined by the following relations

$$
\begin{aligned}
K^{kk'} &:= 2K_{I_1I_2}R^k_{I_1K}C^{I_2K}_{k'} \qquad (44)\\
W_1^{kll'} &:= 2U_{I_1I_2I_3I_4}R^k_{I_1K}C^{I_4K}_{l}C^{I_2I_3}_{l'}\\
W_2^{kk'} &:= -6U_{I_1I_2I_3I_4}F_{I_4I}R^k_{I_1I}C^{I_2I_3}_{k'}\\
W_3^{kk'l} &:= 4U_{I_1I_2I_3I_4}(3F_{I_4I}F_{I_3I'}+\frac{1}{4}A_{I_4I}A_{I_3I'})R^k_{I_1I}R^{k'}_{I'K}C^{I_2K}_{l}\\
W_4^{k_1k_2} &:= -4U_{I_1I_2I_3I_4}(F_{I_4I}F_{I_3I'}+\frac{1}{4}A_{I_4I}A_{I_3I'})F_{I_2I''}R^{k_1}_{I_2I}R^{k_2}_{I'I''}\\
\Gamma^{ll'}_{kk'} &:= -2C^{I_1I_2}_{k}R^{l}_{I_2I_3}C^{I_3I_4}_{k'}R^{l'}_{I_4I_1}
\end{aligned}
$$

The effective boson functional equation (43) is identical with those equations in [17],[18], which were earlier derived without using the formalism introduced above.

4 Boson states and their duals

The basis of the fermion-boson map is the definition of a set of boson states and their duals (21) which have to be explicitly known. A suitable starting point for the derivation of this set is given by the set of solutions of corresponding generalized de Broglie-Bargmann-Wigner equations. These equations are relativistically invariant quantum mechanical many-body equations with nontrivial interaction, selfregularization and probability interpretation and result from the general field theoretic formalism.

In accordance with de Broglie's fusion theory abelian and nonabelian gauge bosons are assumed to have a partonic substructure with two fermionic constituents. For this case exact vector boson states can be derived as solutions of the generalized de Broglie-Bargmann-Wigner equations.

The quantum numbers of these states can be calculated and lead to a $SU(2) \oplus U(1)$ classification with fermion number $f = 0$. The complete set of solutions of these equations contains apart from bound state solutions also scattering state solutions. For heavy parton masses the scattering states decouple from the bound state dynamics and need not to be discussed here. Furthermore in [11],[12] the dynamics of a combined treatment of all possible boson bound states was studied, and it was shown that one is allowed to treat the dynamics of the set of abelian and nonabelian vector bosons separately from all other solutions. Hence we can confine ourselves to the treatment of these vector bosons only.

With respect to the explicit form of the exact vector boson states we refer to the literature [19],[20],[21]. In its general form these functions are given by the following expression

$$\varphi_2 := T^a_{\kappa_1\kappa_2} exp[-ik(x_1+x_2)/2] A^\mu_a(k)\chi_\mu(x_1-x_2)^{i_1i_2}_{\alpha_1\alpha_2} \tag{45}$$

with $k :=$ four-momentum of the boson, and T^j superspin-isospin matrices. This function transforms as a relativistic spin tensor and with

$$F^{\mu\nu}_a(k) := \frac{i}{2} A^\mu_a k^\nu_{as\{\mu\nu\}} \tag{46}$$

it can be decomposed into two parts

$$\varphi_2 = F^{\mu\nu}_a(k) T^a_{\kappa_1\kappa_2}\varphi_{\mu\nu}(x_1,x_2|k)^{i_1i_2}_{\alpha_1\alpha_2} + A^\mu_a(k) T^a_{\kappa_1\kappa_2}\varphi_\mu(x_1,x_2|k)^{i_1i_2}_{\alpha_1\alpha_2} \tag{47}$$

We interprete the coefficients $F^{\mu\nu}_a(k)$ and $A^\mu_a(k)$ as the field strength tensor and the vector potential of the vector boson states. This field strength tensor is completely fixed for single composite vector boson states (45) which are exact solutions of the corresponding generalized de Broglie-Bargmann-Wigner equations. But, if we switch on the full dynamics which is described by the functional energy operator (19) and which leads to interactions between the vector boson states, these states will be deformed under the influence of their interactions. In order to express this deformation we assume that the coefficient functions A and F become the field variables of the effective theory. Hence for the map we consider the functions $\varphi_{\mu\nu}$ and φ_μ as independent quantities.

In accordance with Section 3, for the transition from the fermion dynamics to the effective boson dynamics only the single time wave functions are admitted. If the symmetrical limit of t_1 and t_2 to a single time t is performed, this limit preserves the antisymmetry of the wave functions and leads to the set of independent single time boson functions

$$C^{I_1I_2}_q := \{T^a_{\kappa_1\kappa_2}\varphi_{\mu\nu}(\mathbf{r}_1,\mathbf{r}_2|\mathbf{k})^{i_1i_2}_{\alpha_1\alpha_2}, T^a_{\kappa_1\kappa_2}\varphi_\mu(\mathbf{r}_1\mathbf{r}_2|\mathbf{k})^{i_1i_2}_{\alpha_1\alpha_2}\} \tag{48}$$

where the set of matrices which represent the $SU(2)\oplus U(1)$ states are explicitly given in [19],[20],[21]. Corresponding to these definitions we specify the general set of boson generators and their duals by

$$\{b_k\} := \{b_{\mu,a}(\mathbf{k}), b_{\mu\nu,a}(\mathbf{k})\}, \qquad \{\partial^b_k\} := \{\partial_{\mu,a}(\mathbf{k}), \partial_{\mu\nu,a}(\mathbf{k})\} \tag{49}$$

Furthermore for the evaluation of the fermion-boson mapping the duals to the set of boson states (48) have to be constructed. The general formula for this construction is given in [22]. For brevity we refer for details to the literature.

5 Evaluation of the map

To grasp the physical content and meaning of equation (43), it is necessary to separate the terms representing the field dynamics from those terms which result from field quantization. In the original functional equation (10) the latter terms are given by the last term on the right hand side of (10) containing only the anticommutators $A_{II'}$.

If by the normal transformation (16) a special vacuum is selected, the resulting equation (18) with energy operator (19) contains in the last two terms of (19) superpositions of anticommutators and propagators. Obviously these terms have to be interpreted as generalized quantization terms, where the anticommutators (15) which govern and define the abstract field operator algebra are modified by the influence of the vacuum and in addition by the influence of the composite particle structure i.e., by the influence the special representation under consideration. This means: for an effective theory one cannot expect that for the corresponding effective fields exact canonical commutation relation hold.

The influence of the vacuum and of the composite particle structure on the quantum properties of the effective fields as expressed in the terms W_3 and W_4 needs a special investigation which, for instance, was performed in [5], Section 6.5. For brevity we refer to this investigation and consider in this paper only the map of the terms describing the dynamics of the system. In this case the effective functional equation (43) is reduced to the equation

$$E_0^a|\mathcal{B}(b|a)\rangle^d := \{K^{kk'} b_k \partial^b_{k'} + W_1^{kll'} b_k \partial^b_l \partial^b_{l'} \\ W_2^{kk'}(b_k + \Gamma^{ll'}_{kk''} b_l b_{l'} \partial^b_{k''})\partial^b_{k'}\}|\mathcal{B}(b|a)\rangle^d \tag{50}$$

where $|\mathcal{F}(b|a)\rangle^d$ is the functional state exclusively referred to the dynamical terms. If the corresponding equation is evaluated then one gets an information how the original fermion system dynamics is mapped into the effective boson system dynamics.

If by means of the set of boson functions (48) and their duals the coefficient functions (44) of the effective theory are calculated and substituted into (43), one obtains the explicit form of this effective boson theory. It is, however, not possible to perform this mapping in a gauge invariant way:

The physical state space of the original fermion theory is positive definite, [7], and allows a probability interpretation and conservation, [7]. It can be shown that this property of the original fermion theory is transferred to the effective boson theory by exact mappings, [7]. On the other hand in various gauges the phenomenological boson state spaces are indefinite. In consequence a mapping from the fermion state space into these indefinite boson state spaces would lead to contradictions. Hence only a mapping into gauges with positive boson state spaces is allowed which ends up in gauge fixing. Indeed the formalism itself enforces the use of the temporal gauge which is positive definite. In the following all calculations are done in this gauge.

Under these premises one obtains for (50) the following expression, see [10],[14]

$$\begin{aligned}\mathcal{H}_b^{eff}:=&i\frac{4}{5}\int d^3r b_{l,a}^A(\mathbf{r})\partial_{l0,a}^F(\mathbf{r}) + i\int d^3r b_{i0,a}^F(\mathbf{r})\epsilon_{ijk}\epsilon_{klm}\partial_j(\mathbf{r})\partial_{lm,a}^F(\mathbf{r})_{|l>m}\\ &-i\int d^3r b_{ij,a}^F(\mathbf{r})\epsilon_{ijk}\epsilon_{klm}\partial_l(\mathbf{r})\partial_{m0,a}^F(\mathbf{r})_{|i>j} \qquad (51)\\ &+iG\int d^3r\varepsilon^{abc}\epsilon_{ijk}\epsilon_{klm}b_{ij,a}^F(\mathbf{r})\partial_{l0,b}^F(\mathbf{r})\partial_{m,c}^A(\mathbf{r})_{|i>j}\\ &+iG\int d^3r\varepsilon^{abc}\epsilon_{ijk}\epsilon_{klm}b_{l0,a}^F(\mathbf{r})\partial_{ij,b}^F(\mathbf{r})\partial_{m,c}^A(\mathbf{r})_{|i>j}\end{aligned}$$

where ε^{abc} are the structure constants of the group SU(2) and and $b(\mathbf{r})$ and $\partial(\mathbf{r})$ are the Fourier transforms of the set (49). In performing this evaluation the last term of (50) was estimated to give a vanishing contribution compared with the leading terms in (50). Furthermore the coupling constant of the original fermion theory was fixed in order to compensate a mass renormalzation term which leads to a vanishing effective mass of the vector bosons.

In this form (51) can only be compared with the corresponding phenomenological counterpart in functional formulation which is not a familiar expression. Thus we apply some rearrangements which clarify the physical content of (51).

Due to the time-translational invariance of the algebraic Schroedinger representation the effective energy operator (51) and the generating functional are referred to an arbitratry time t. For the further evaluation of

(51) it is important that all quantities are explicitly referred to this time, i.e., we replace $b_k(\mathbf{r})$ and $\partial_k^b(\mathbf{r})$ by $b_k(\mathbf{r},t)$ and $\partial_k^b(\mathbf{r},t)$ in (51). In this case in analogy to quantum mechanics the time dependent algebraic Schroedinger representation can be derived from (50). It reads in functional formulation, [5],[6]:

$$\mathcal{H}_b^{eff}|\mathcal{B}(b|a)\rangle = i\int d^3r[b_k(\mathbf{r},t)\frac{\partial}{\partial t}\partial_k^b(\mathbf{r},t)|\mathcal{B}(b|a)\rangle \tag{52}$$

Furthermore we decompose the field strength generators into non-abelian electric and magnetic fields by the following definitions, [10],[14]:

$$\begin{aligned} b_{i,a}^E(\mathbf{r},t) &:= b_{i0,a}^F(\mathbf{r},t) \qquad & \partial_{i,a}^E(\mathbf{r},t) &:= \partial_{i0,a}^F(\mathbf{r},t) \\ b_{i,a}^B(\mathbf{r},t) &:= \epsilon_{ijk}b_{jk,a}^F(\mathbf{r},t) \qquad & \partial_{i,a}^B(\mathbf{r},t) &:= \epsilon_{ijk}\partial_{jk,a}^F(\mathbf{r},t) \end{aligned} \tag{53}$$

and substitute these definitions into (58). Then it is easily seen that one obtains a solution of equation (52) if the following equations are satisfied by the common state $|\mathcal{B}\rangle$

$$\begin{aligned} &\frac{\partial}{\partial t}\partial_{i,a}^A(\mathbf{r},t)|\mathcal{B}\rangle{:=}{-}\partial_{i,a}^E(\mathbf{r},t)|\mathcal{B}\rangle \\ &\frac{\partial}{\partial t}\partial_{i,a}^E(\mathbf{r},t)|\mathcal{B}\rangle{:=}[\epsilon_{ijk}\partial_j\partial_{k,a}^B(\mathbf{r},t) + G\varepsilon^{abc}\epsilon_{ijk}\partial_{j,b}^A(\mathbf{r},t)\partial_{k,c}^B(\mathbf{r},t)]|\mathcal{B}\rangle \\ &\frac{\partial}{\partial t}\partial_{i,a}^B(\mathbf{r},t)|\mathcal{B}\rangle{:=}[-\epsilon_{ijk}\partial_j\partial_{k,a}^E(\mathbf{r},t) - G\varepsilon^{abc}\epsilon_{ijk}\partial_{j,b}^E(\mathbf{r},t)\partial_{k,c}^A(\mathbf{r},t)]|\mathcal{B}\rangle \end{aligned} \tag{54}$$

In order to illustrate the meaning of these functional equations, we consider the classical limit which corresponds to the neglection of higher order quantum correlations in statistical mechanics. This limit is expressed by the ansatz, [5],[6],[10],[14]

$$|\mathcal{B}\rangle := exp[Z(b)]|0\rangle \tag{55}$$

with

$$Z(b) := \sum_X\sum_{k,a}\int d^3rX_a^k(\mathbf{r},t)b_{k,a}^X(\mathbf{r},t) \tag{56}$$

where X is given by $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{B}$.

If this ansatz is substituted into (54), one gets a solution of these functional equations if the following classical equations for the amplitudes are satisfied:

$$\begin{aligned}\dot{\mathbf{A}}_a &= -\mathbf{E}_a \\ \dot{\mathbf{E}}_a &= \nabla \times \mathbf{B}_a + G\varepsilon^{abc}\mathbf{A}_b \times \mathbf{B}_c \\ \dot{\mathbf{B}}_a &= -\nabla \times \mathbf{E}_a - G\varepsilon^{abc}\mathbf{E}_b \times \mathbf{A}_c\end{aligned} \tag{57}$$

where for brevity we suppressed the arguments of these fields.

In phenomenological theory the dynamical equations are supplied by constraints. These constraints can be directly derived from the system (57). If this is done our result can be summarized by the following system of equations

$$\begin{aligned}\dot{\mathbf{A}}_a &= -\mathbf{E}_a \\ \dot{\mathbf{E}}_a &= \nabla \times \mathbf{B}_a + G\varepsilon^{abc}\mathbf{A}_b \times \mathbf{B}_c\end{aligned} \tag{58}$$

while the constraints are given by

$$\begin{aligned}\mathbf{B}_a &= \nabla \times \mathbf{A}_a + \frac{1}{2}G\varepsilon^{abc}\mathbf{A}_b \times \mathbf{A}_c \\ 0 &= \nabla \cdot \mathbf{E}_a + G\varepsilon^{abc}\mathbf{A}_b \cdot \mathbf{E}_c\end{aligned} \tag{59}$$

Equations (58) and (59) represent the exact formulation of a nonabelian $SU(2)$ gauge field theory expressed in terms of its canonical variables in temporal gauge. An analogous derivation can be performed for the abelian case which for short will not be explicitly given here. For a discussion of the quantization terms we refer to [5], section 6.5.

In deriving this result, apart from the quantization terms, the last term in (50) can be estimated and neglected with respect to the other leading terms, if the parton masses are sufficiently large. So one can argue that by weak mapping one gets a version of this nonabelian theory which in the high enery range loses its gauge invariance owing to this small correction term.

Is therefore the elemental character of such gauge theories saved? Surely not, because even the most ardent adherent of the elemental character of these gauge theories must admit that the high energy range will never be so well explored as to secure the valitidy of such gauge theories up to infinite large energies.

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