

On the Generalized Dirac Equation for Fermions with Two Mass States

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ABSTRACT. The generalized Dirac equation of the second order, describing particles with spin 1/2 and two mass states, is analyzed. The projection operators extracting states with definite energy and spin projections are obtained. The first order generalized Dirac equation in the 20-dimensional matrix form and the relativistically invariant bilinear form are derived. We obtain the canonical energy-momentum tensor and density of the electromagnetic current expressed through the 20-component wave function. Minimal and non-minimal electromagnetic interactions of fermions are considered, and the quantum-mechanical Hamiltonian is found. It is shown that there are only causal propagations of waves in the approach considered.

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1 Introduction

One of the modern problems is to explain the number of quark and lepton generations, and their mass spectrum. The standard model (SM) of electroweak interactions contains many free parameters: m_e , m_μ , m_τ , m_u , m_d , m_c , m_s , m_t , m_b , four mixing angles and three coupling constants. One of the ways to reduce the number of free parameters is to explore the Grand Unification Theories (GUTs) with different gauge groups. At the GUTs scale, we have only one coupling constant, and masses of leptons and quarks are generated below the GUTs scale by the spontaneous gauge symmetry breaking with the help of the Higgs mechanism. As a result of the symmetry breaking, in the framework of the GUT, the generations of fermions appear. Nevertheless, the Higgs Lagrangians also contain free parameters. Recent observations of neutrino

oscillations show that SM should be extended, and possibly a fourth generation exists. The deeper insight into the dynamics of the mass generation mechanism is needed to understand physics beyond the SM.

We pay attention here on the Barut work [1], (see also [2]) who suggested a mass formula for e -, μ -leptons based on the generalized Dirac equation of the second order (see [3], [4]) describing particles with two mass states (see [5], [6] for the case of bosonic fields with two mass states). This equation may be considered as an effective one for partly “dressed” fermions, i.e. some radiation corrections are taken into account. This scheme represents the non-perturbative approach to quantum electrodynamics.

The goal of this paper is to formulate the mentioned second order equation in the form of the first order generalized Dirac equation (FOGDE), and investigate it.

The paper is organized as follows. In Sec. 2, we define the mass spectrum of the generalized Dirac equation of the second order and find the projection operator extracting the solutions in the momentum space for free particles. The FOGDE is derived in Sec. 3. The relativistically invariant bilinear form, the Lagrangian formulation, and spin projection operators are given. In Sec. 4, we obtain the canonical energy-momentum tensor and the electromagnetic current density from the Lagrangian. Sec. 5 is devoted to the introduction of minimal and non-minimal electromagnetic interactions of fermions in the approach discussed. The quantum-mechanical Hamiltonian is obtained. We make a conclusion in Sec. 6. The system of units $\hbar = c = 1$ is chosen and notations as in [7] are used.

2 Spin-1/2 Field Equation of the Second Order

Let us consider the second order (in derivatives) field equation describing spin-1/2 particles introduced in [3] (see also generalizations in [8]):

$$(\alpha_1 \gamma_\nu \partial_\nu + \alpha_2 \partial_\mu^2 + \kappa) \psi(x) = 0, \quad (1)$$

where $\partial_\nu = \partial/\partial x_\nu = (\partial/\partial x_m, \partial/\partial(it))$, $\psi(x)$ is a bispinor. Repeated indices imply a summation. The Dirac matrices γ_μ obey the commutation relations $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}$. As physical values depend only on two parameters, it is convenient to introduce new variables $\kappa/\alpha_1 = m$, $\alpha_2/\alpha_1 = -a/m$, where m is a parameter with the dimension of the

mass, and a is a massless parameter. With these arrangements, Eq. (1) is rewritten as

$$\left(\gamma_\nu \partial_\nu - \frac{a}{m} \partial_\mu^2 + m\right) \psi(x) = 0. \tag{2}$$

In momentum space Eq. (2) for positive energy is given by

$$\left(i\hat{p} + \frac{ap^2}{m} + m\right) \psi(p) = 0, \tag{3}$$

where $\hat{p} = \gamma_\mu p_\mu$. One has to make the replacement $p \rightarrow -p$ when considering antiparticles. Applying the operator $(-i\hat{p} + ap^2/m + m)$ to both sides of Eq. (3), we obtain the equation

$$[a^2 p^4 + m^2 (2a + 1) p^2 + m^4] \psi(p) = 0, \tag{4}$$

where $p^2 = \mathbf{p}^2 - p_0^2$. In order to have a non-zero wave function in Eq. (4), the diagonal elements need to be zero:

$$p^2 = -m^2 \left(\frac{2a + 1 \pm \sqrt{4a + 1}}{2a^2}\right). \tag{5}$$

Eq. (5) defines the squared masses of fermions:

$$m_1^2 = m^2 \left(\frac{2a + 1 - \sqrt{4a + 1}}{2a^2}\right), \quad m_2^2 = m^2 \left(\frac{2a + 1 + \sqrt{4a + 1}}{2a^2}\right). \tag{6}$$

From Eqs. (6), one finds the restriction on the parameter a : $a \geq -1/4$. At $a = -1/4$ both masses are equal, $m_1 = m_2$. We obtain from Eqs. (6) masses of spin-1/2 particles

$$m_1 = -m \left(\frac{1 - \sqrt{4a + 1}}{2a}\right), \quad m_2 = -m \left(\frac{1 + \sqrt{4a + 1}}{2a}\right). \tag{7}$$

In addition, we have a solution with opposite signs of masses. The signs of masses in Eqs. (7) are chosen to be positive for the negative parameter a . The negative mass solutions of Eqs. (6) can be incorporated in a description of antiparticles. When the parameter a approaches to zero, and the parameter m is fixed, masses of the fermionic states, m_1, m_2 , become unlimited. Therefore, the limit $a = 0$ should be made in Eqs (2), (3), and one recovers the Dirac equation which describes fermions with the mass m .

Eqs. (7) lead to the mass formula $m_1 + m_2 = -m/a = \alpha_1/\alpha_2$ obtained in [3]. The authors [3] interpreted the state of fermion with the mass m_1 as an electron, and the state of fermion with the mass m_2 as a muon.

Now we construct the projection operators extracting solutions of Eq. (3). It is easy to verify that projection operators (see the general method of projection operators in [9])

$$\Lambda_{\pm} = \frac{ap^2 + m^2 \mp im\hat{p}}{2(ap^2 + m^2)}, \quad (8)$$

which obey the necessary equation $\Lambda_{\pm}^2 = \Lambda_{\pm}$, are solutions of equations

$$\left(\pm i\hat{p} + \frac{ap^2}{m} + m \right) \Lambda_{\pm} = 0.$$

The projection operator Λ_+ corresponds to positive energy, and Λ_- - to negative energy of particles.

3 First Order Generalized Dirac Equation

In order to formulate the FOGDE, we introduce the 20-dimensional function

$$\Psi(x) = \{\psi_A(x)\} = \begin{pmatrix} \psi(x) \\ \psi_{\mu}(x) \end{pmatrix} \quad (\psi_{\mu}(x) = -\frac{1}{m}\partial_{\mu}\psi(x)), \quad (9)$$

where $A = 0, \mu$. The function $\Psi(x)$ in Eq. (9) is the direct sum of a bispinor $\psi(x)$ and a vector-bispinor $\psi_{\mu}(x)$. Therefore, the wave function $\Psi(x)$ realizes the reducible representation $[(1/2, 0) \oplus (0, 1/2)] \oplus \{(1/2, 1/2) \otimes [(1/2, 0) \oplus (0, 1/2)]\}$ of the Lorentz group.

Let us introduce the elements of the entire algebra $\varepsilon^{A,B}$ [10] with the properties

$$(\varepsilon^{M,N})_{AB} = \delta_{MA}\delta_{NB}, \quad \varepsilon^{M,A}\varepsilon^{B,N} = \delta_{AB}\varepsilon^{M,N}, \quad (10)$$

$A, B, M, N = 0, 1, 2, 3, 4$. The $\varepsilon^{M,N}$ represent 25 basis elements corresponding to the different values of the superscripts M, N , and they are 5x5 matrices whose elements are labelled by the subscripts. The elements of the matrix $\varepsilon^{M,N}$ consist of zeros and only one element is unity, where row M and column N cross.

With the help of Eqs. (9), (10), Eq. (2) can be rewritten in the form of the first order equation

$$\partial_\nu (\varepsilon^{\nu,0} + a\varepsilon^{0,\nu} + \varepsilon^{0,0}\gamma_\nu)_{AB} \Psi_B(x) + m [\varepsilon^{\mu,\mu} + \varepsilon^{0,0}]_{AB} \Psi_B(x) = 0, \tag{11}$$

where we imply that γ -matrices act on the bispinor subspace. All repeated indices such as in Eq. (11) imply a summation even in the superscripts of the elements of the algebra. After introducing 20-dimensional matrices

$$\alpha_\nu = (\varepsilon^{\nu,0} + a\varepsilon^{0,\nu}) \otimes I_4 + \varepsilon^{0,0} \otimes \gamma_\nu, \tag{12}$$

where unit four-dimensional matrix I_4 acts on bispinor subspace and unit five-dimensional matrix $1 \equiv I_5 = \varepsilon^{\mu,\mu} + \varepsilon^{0,0}$ acts on scalar-vector subspace, Eq. (11) takes the form of the FOGDE:

$$(\alpha_\nu \partial_\nu + m) \Psi(x) = 0. \tag{13}$$

Eq. (13) is convenient for investigations of spin-1/2 fermions with two mass states. According to the general theory [11], the masses of particles described by the first order relativistic wave equation (13) are m/λ_i , where λ_i are the eigenvalues of the matrix α_4 . In our case the matrix $\alpha_4 = (\varepsilon^{4,0} + a\varepsilon^{0,4}) \otimes I_4 + \varepsilon^{0,0} \otimes \gamma_4$ obeys the minimal matrix equation

$$\alpha_4^4 - (1 + 2a)\alpha_4^2 + a^2 = 0. \tag{14}$$

One can obtain from Eq. (14) four eigenvalues of the matrix α_4 :

$$\pm\lambda_1, \quad \pm\lambda_2, \quad \lambda_1 = \frac{-1 - \sqrt{4a + 1}}{2}, \quad \lambda_2 = \frac{-1 + \sqrt{4a + 1}}{2}. \tag{15}$$

It is easy to verify that masses of fermion states of 20-dimensional matrix Eq. (13): $m_1 = m/\lambda_1$, $m_2 = m/\lambda_2$ coincide with those in Eq. (7). Two additional masses possess opposite signs.

To build the spin operators, one needs the generators of the Lorentz group in the 20-dimensional representation. Such generators are given by [12], [13]

$$J_{\mu\nu} = J_{\mu\nu}^{(1)} \otimes I_4 + I_5 \otimes J_{\mu\nu}^{(1/2)}, \tag{16}$$

$$J_{\mu\nu}^{(1)} = \varepsilon^{\mu,\nu} - \varepsilon^{\nu,\mu}, \tag{17}$$

$$J_{\mu\nu}^{(1/2)} = \frac{1}{4} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu). \tag{18}$$

Operators $J_{\mu\nu}^{(1)}$, $J_{\mu\nu}^{(1/2)}$ are the generators of the Lorentz group in four-dimensional vector and bispinor spaces, respectively. Generators (16)-(18) obey the commutation relations

$$[J_{\mu\nu}, J_{\alpha\beta}] = \delta_{\nu\alpha}J_{\mu\beta} + \delta_{\mu\beta}J_{\nu\alpha} - \delta_{\nu\beta}J_{\mu\alpha} - \delta_{\mu\alpha}J_{\nu\beta}. \quad (19)$$

The antisymmetric parameters of the Lorentz group ω_{mn} ($m, n = 1, 2, 3$) are real, and ω_{m4} are imaginary in our metric. The commutation relation

$$[\alpha_\lambda, J_{\mu\nu}] = \delta_{\lambda\mu}\alpha_\nu - \delta_{\lambda\nu}\alpha_\mu \quad (20)$$

holds and guarantees the relativistic form-invariance of Eq. (13).

Now we consider operators of the spin projections on the direction of the momentum \mathbf{p} :

$$\sigma_p = -\frac{i}{2|\mathbf{p}|}\epsilon_{abc}\mathbf{P}_a J_{bc}, \quad (21)$$

where $|\mathbf{p}| = \sqrt{p_1^2 + p_2^2 + p_3^2}$. Using Eq. (20), it is not difficult to prove that the operator of Eq. (13) in the momentum space ($i\alpha_\mu p_\mu + m$) commutes with the spin operator (21): $[i\alpha_\mu p_\mu + m, \sigma_p] = 0$. Therefore, the operators ($i\alpha_\mu p_\mu + m$) and σ_p have the common eigenfunction in the momentum space. It is not difficult to verify that the minimal matrix equation

$$\left(\sigma_p^2 - \frac{1}{4}\right)\left(\sigma_p^2 - \frac{9}{4}\right) = 0$$

holds. With the aid of the method [9], we find the projection operators

$$P_{\pm 1/2} = \mp \frac{1}{2}\left(\sigma_p \pm \frac{1}{2}\right)\left(\sigma_p^2 - \frac{9}{4}\right) \quad (22)$$

which extract spin projections $\pm 1/2$, so that

$$\sigma_p P_{\pm 1/2} = \pm \frac{1}{2}P_{\pm 1/2}.$$

The operators $P_{\pm 1/2}$ acting on the arbitrary 20-dimensional function $\Psi_0(p)$ produce the eigenfunction $\Psi_{\pm 1/2}(p) = P_{\pm 1/2}\Psi_0(p)$ of the operator σ_p : $\sigma_p\Psi_{\pm 1/2} = \pm(1/2)\Psi_{\pm 1/2}$.

To obtain the Lagrangian, one has to find the Hermitianizing matrix η in our 20-dimensional representation space which obeys the relations [11]

$$\eta\alpha_m = -\alpha_m\eta, \quad \eta\alpha_4 = \alpha_4\eta \quad (m = 1, 2, 3). \quad (23)$$

Such a matrix exists, and is given by

$$\eta = (\varepsilon^{m,m} - \varepsilon^{4,4} - \varepsilon^{0,0}) \otimes \gamma_4. \tag{24}$$

Using Eqs. (23), we obtain from Eq. (13) the equation

$$\bar{\Psi}(x) \left(\alpha_\mu \overleftarrow{\partial}_\mu - m \right) = 0, \tag{25}$$

where $\bar{\Psi}(x) = \Psi^+(x)\eta$, and $\Psi^+(x)$ is the Hermitian-conjugate wave function. The derivative $\overleftarrow{\partial}_\mu$ in Eq. (25) acts on the left-standing function. The relativistically invariant bilinear form is given by $\bar{\Psi}(x)\Psi(x) = \Psi^+(x)\eta\Psi(x)$. Thus, the Lagrangian reads:

$$\mathcal{L} = -\frac{1}{2} \left[\bar{\Psi}(x) (\alpha_\mu \partial_\mu + m) \Psi(x) - \bar{\Psi}(x) \left(\alpha_\mu \overleftarrow{\partial}_\mu - m \right) \Psi(x) \right]. \tag{26}$$

Variation of the action $S = \int \mathcal{L} d^4x$ with respect to the independent fields $\Psi(x)$, $\bar{\Psi}(x)$, corresponding to Lagrangian (26), gives Euler-Lagrange equations (13), (25).

4 The Energy-Momentum Tensor and Electromagnetic Current

The energy-momentum tensor can be found with the help of the standard procedure, using the general formula [14]

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi(x))} \partial_\nu \Psi(x) + \partial_\nu \bar{\Psi}(x) \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\Psi}(x))} - \delta_{\mu\nu} \mathcal{L}. \tag{27}$$

With the help of Eq. (27), one may obtain from the Lagrangian (26) the canonical energy-momentum tensor as follows

$$T_{\mu\nu} = \frac{1}{2} (\partial_\nu \bar{\Psi}(x)) \alpha_\mu \Psi(x) - \frac{1}{2} \bar{\Psi}(x) \alpha_\mu \partial_\nu \Psi(x). \tag{28}$$

We took here into consideration that $\mathcal{L} = 0$ for functions $\Psi(x)$, $\bar{\Psi}(x)$ obeying Eqs. (13), (25). Canonical energy-momentum tensor (28) is conserved but is not symmetric. It can be symmetrized [4] by adding the divergence term to the Lagrangian (26) which does not change Euler-Lagrange equations (13), (25). With the help of Eqs. (9), (12), one may obtain from Eq. (28) the expression

$$T_{\mu\nu} = \frac{1}{2} \bar{\psi}(x) \gamma_\mu \partial_\nu \psi(x) - \frac{1}{2} (\partial_\nu \bar{\psi}(x)) \gamma_\mu \psi(x) + \frac{1}{2m} (\partial_\mu \bar{\psi}(x)) \partial_\nu \psi(x)$$

$$-\frac{1}{2m} (\partial_\mu \partial_\nu \bar{\psi}(x)) \psi(x) + \frac{a}{2m} (\partial_\nu \bar{\psi}(x)) \partial_\mu \psi(x) - \frac{a}{2m} \bar{\psi}(x) \partial_\nu \partial_\mu \psi(x).$$

The first two terms in this equation correspond to the ordinary Dirac equation. The energy density and the momentum density are given by $\mathcal{E} = T_{44}$, $P_m = iT_{m4}$.

The conserved electric current density may be obtained by the relationship [14]

$$j_\mu(x) = i \left(\bar{\Psi}(x) \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\Psi}(x))} - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi(x))} \Psi(x) \right). \quad (29)$$

Substituting Eq. (26) into Eq. (29), we get the electric current density

$$j_\mu(x) = i \bar{\Psi}(x) \alpha_\mu \Psi(x). \quad (30)$$

From Eqs. (13), (25), one may prove the conservation of the four-vector current density: $\partial_\mu j_\mu(x) = 0$. The canonical energy-momentum tensor (28) and electric current density (30) are expressed through 20-component wave function $\Psi(x)$. Using Eqs. (9), (12), we obtain from Eq. (30) the electric current density only in terms of the bispinor $\psi(x)$:

$$j_\mu(x) = -i \bar{\psi}(x) \gamma_\mu \psi(x) + i \frac{a}{m} \bar{\psi}(x) \partial_\mu \psi(x) - \frac{i}{m} (\partial_\mu \bar{\psi}(x)) \psi(x).$$

It follows from this expression that the current (30) is related to Barut's original anomalous magnetism. So, the $j_\mu(x)$ includes the usual Dirac current as well as convective terms.

5 Electromagnetic Interactions of Fermions

It follows from Eq. (14) that the inverse matrix α_4^{-1} exists, as there are no zero eigenvalues of the matrix α_4 . One may find from Eq. (14) the inverse matrix

$$\alpha_4^{-1} = \frac{2a+1}{a^2} \alpha_4 - \frac{\alpha_4^3}{a^2}. \quad (31)$$

This indicates that all components of the wave function $\Psi(x)$, Eq. (9), are canonical, and contain time derivatives. Thus, there are no subsidiary conditions in the first order wave equation (13), and it can be rewritten in the Hamiltonian form. The minimal interaction with electromagnetic field in the first order equation (13) can be obtained by the substitution $\partial_\mu \rightarrow D_\mu = \partial_\mu - ieA_\mu$ (A_μ is the four-vector potential

of the electromagnetic field). We also introduce non-minimal interaction by adding terms linear in the strength of the electromagnetic field $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. So, we postulate the matrix equation

$$\left[\alpha_\mu D_\mu + \frac{i}{2} (\kappa_0 P_0 + \kappa_1 P_1) \alpha_{\mu\nu} \mathcal{F}_{\mu\nu} + m \right] \Psi(x) = 0, \quad (32)$$

where the projection operators P_0, P_1 , are given by

$$P_0 = \varepsilon^{0,0} \otimes I_4, \quad P_1 = \varepsilon^{\mu,\mu} \otimes I_4, \quad (33)$$

and

$$\begin{aligned} \alpha_{\mu\nu} &= \alpha_\mu \alpha_\nu - \alpha_\nu \alpha_\mu \\ &= \varepsilon^{0,0} \otimes (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) + \varepsilon^{\mu,0} \otimes \gamma_\nu - \varepsilon^{\nu,0} \otimes \gamma_\mu \\ &\quad + \varepsilon^{0,\nu} \otimes \gamma_\mu - \varepsilon^{0,\mu} \otimes \gamma_\nu + a (\varepsilon^{\mu,\nu} - \varepsilon^{\nu,\mu}) \otimes I_4. \end{aligned} \quad (34)$$

The projection operators P_0, P_1 extract scalar, vector subspaces and obey the relations: $P_0^2 = P_0, P_1^2 = P_1, P_0 + P_1 = 1$. It is easy to verify that Eq. (32) is form-invariant under the Lorentz transformations. We have introduced two parameters κ_0, κ_1 which characterize anomalous electromagnetic interactions of fermions. It should be noted that the relativistic invariance allows us to introduce only two parameters because there are two subspaces corresponding to the projection operators P_0 and P_1 .

With the help of Eqs. (9), (10), (12), we obtain the tensor form of Eq. (32):

$$(\gamma_\nu D_\nu + i\kappa_0 \gamma_\mu \gamma_\nu \mathcal{F}_{\mu\nu} + m) \psi(x) + (aD_\mu + i\kappa_0 \gamma_\nu \mathcal{F}_{\nu\mu}) \psi_\mu(x) = 0, \quad (35)$$

$$(D_\mu + i\kappa_1 \gamma_\nu \mathcal{F}_{\mu\nu}) \psi(x) + (m\delta_{\mu\nu} + i\kappa_1 a \mathcal{F}_{\mu\nu}) \psi_\nu(x) = 0. \quad (36)$$

It should be stressed that Eq. (36) is an analog of the gradient equation (see Eq. (9)) in the case of interacting fields. System of Eqs. (35), (36) defines bispinor $\psi(x)$ as well as vector-bispinor $\psi_\nu(x)$. Eq. (36) may be considered as a matrix equation for vector-bispinor $\psi_\nu(x)$. After finding the inverse matrix $(m\delta_{\mu\nu} + i\kappa_1 a \mathcal{F}_{\mu\nu})^{-1}$, one can express the $\psi_\nu(x)$ from Eq. (36) and replace into Eq. (35) to get an equation for bispinor $\psi(x)$ describing the interaction of fermions with electromagnetic fields. This equation includes the interaction with the anomalous magnetic moment of particles. Eqs. (35), (36) can also be applied for phenomenological descriptions of composite fermions.

Let us find the quantum-mechanical Hamiltonian from Eq. (32). Eq. (32) can be rearranged as

$$i\alpha_4\partial_t\Psi(x) = \left[\alpha_a D_a + m + eA_0\alpha_4 + \frac{i}{2}(\kappa_0 P_0 + \kappa_1 P_1)\alpha_{\mu\nu}\mathcal{F}_{\mu\nu} \right] \Psi(x). \quad (37)$$

One can obtain from Eq. (37) the Hamiltonian form of the equation

$$i\partial_t\Psi(x) = \mathcal{H}\Psi(x), \quad (38)$$

$$\mathcal{H} = \alpha_4^{-1} \left[\alpha_a D_a + m + eA_0\alpha_4 + \frac{i}{2}(\kappa_0 P_0 + \kappa_1 P_1)\alpha_{\mu\nu}\mathcal{F}_{\mu\nu} \right],$$

where the matrix α_4^{-1} is given by Eq. (31).

To investigate the consistency problem, in accordance with the method [15], we replace the derivatives in Eq. (32) by the four-vector n_μ to get the characteristic surfaces. One can obtain the normals to the characteristic surfaces, n_μ , by solving the equation

$$\det(\alpha_\mu n_\mu) = 0. \quad (39)$$

Using the frame of reference where $n_\mu = (0, 0, 0, n_4)$, Eq. (39) becomes $\det(\alpha_4 n_4) = 0$. It follows from Eq. (14) that there are no zero eigenvalues of the matrix α_4 . Therefore, equation $\det(\alpha_4 n_4) = 0$ possesses only the trivial solution $n_4 = 0$, that indicates: there are only causal propagations of waves as in the Dirac equation, and no superluminal speed of waves.

6 Conclusion

The projection operators extracting states with definite energy and spin projections were constructed for the generalized Dirac equation of the second order, describing particles with spin 1/2 and two mass states. We have obtained the FOGDE in the 20-dimensional matrix form, the relativistically invariant bilinear form, and the Lagrangian. The FOGDE is convenient for different applications. The canonical energy-momentum tensor and density of the electromagnetic current have expressed through the 20-component wave function. We have introduced two parameters characterizing non-minimal electromagnetic interactions of fermions. As a particular case, they include the interaction of the anomalous magnetic moment of particles. The quantum-mechanical Hamiltonian has

obtained and the Hamiltonian form of the equation was given. It was shown that there are only causal propagations of waves in the approach considered.

In order to get the third generation of fermions, one needs to introduce a third-order (in derivatives) generalized Dirac equation. Using the technique described, such equation can be reduced to first-order equation such as Eq. (13). There is not a natural limit to these procedures, corresponding to a finite number of generations in the approach considered.

The scheme investigated can be applied for a consideration of the two flavour generations of leptons and quarks. Possibly, Eqs. (35), (36) may be considered as effective equations for electromagnetic interactions of hadrons.

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