Current status of Yang's theory of gravity

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ABSTRACT. The historical route and the current status of a curvature-squared model of gravity, in the affine form proposed by Yang, is briefly review. Due to its inherent scale invariance, it enjoys some advantage for quantization, similarly as internal Yang-Mills fields. However, the exact vacuum solutions with double duality properties exhibit a 'vacuum degeneracy'. By modifying the duality via a *scale breaking* term, we demonstrate that only the Einstein equations with induced cosmological constant emerge for the *classical* background, even when coupled to matter sources.

1 Introduction

In 2004 we commemorate not only the 50th anniversary of the Yang-Mills equation [46] but also the 30th anniversary of Yang's theory of gravity [45]. The historical route to the $SU(2)$ gauge theory is beautifully laid out by Mills [30], with several ramifications [28] mainly due to the paper of Schrödinger [35], in which the compact 'Clifform' formula [25] for the Riemannian curvature anticipated the concept of *gauge* curvature, see also the letter of the late Bob Mills to one of the authors reprinted in Appendix A.

Here we will concentrate on the gravitational aspect: In 1974 Yang [45] considered the possible replacement of Einsteins general relativity (GR) by an affine gauge theory with a Yang-Mills type action. In fact, curvature-squared Lagrangians had been considered before, first in 1919 by Weyl [43] with the emphasis on scale invariance, and then later by Stephenson [38], Higgs [12], as well as Kilmister and Newman [15].

When written in differential forms, the Stephenson-Kilmister-Yang (SKY) Lagrangian is given by

$$V_{\rm SKY} = -\frac{1}{2} R^{\alpha\beta} \wedge {}^*R_{\alpha\beta}\,, \tag{1}$$

cf. Appendix B and C for the notation [11] which follows closely Cartan's exterior calculus. The short-time initial value problem of Yang's equations

$$D\,{}^*R_{\alpha\beta} = 0 \tag{2}$$

is well-posed [7]. Moreover, it does not depend on any length scale, i.e. it is scale invariant as envisioned by Weyl [44] and therefore a good starting point for quantization. The complexified version of the teleparallelism equivalent of Einstein's GR has also received a Yang-Mills type reformulation [24, 26], but will not be considered here.

Due to its scale invariance, Yang's theory can be regarded as a fundamental theory of (quantum) gravity in the high-energy limit [10], without invoking extra dimensions or supersymmetry, cf. Ref. [14]. In this paper we investigated its classical limit, corresponding to the most probable, extremal 'trajectories' in the Feynman path integral. Here, these are classical configurations with self- or anti-self dual curvature. In order to lift this 'vacuum degeneracy', we consider a modified duality ansatz which explicitly breaks scale invariance. For torsionless configurations, we demonstrate that only Einstein's GR, consistenly coupled to the symmetrized energy-momentum current of matter fields, surface as low-energy (long range) effective theory.

2 Selfdual SKY gravity

It is rather instructive to supplement the SKY Lagrangian by the topological Euler term (30) as a boundary term, i.e.,

$$\begin{aligned} V^{(\star)}_{\rm SKY} &= -\frac{1}{2} R^{\alpha\beta} \wedge {}^*R_{\alpha\beta} - \frac{(-1)^s}{2} R^{\alpha\beta} \wedge R^{(\star)}_{\alpha\beta} \\ &= -\frac{1}{4} \left(R_{\alpha\beta} + {}^*R^{(\star)}_{\alpha\beta} \right) \wedge {}^* \left(R^{\alpha\beta} + {}^*R^{\alpha\beta(\star)} \right)\,, \end{aligned} \tag{3}$$

where we distinguish between the Hodge dual * and the Lie dual $^{(\star)}$ in a space(-time) of signature s. It is obvious from the equivalent binomial form of the Lagrangian that anti-selfdual solutions [21]

$$R_{\alpha\beta} = -\,{}^*R^{(\star)}_{\alpha\beta} \tag{4}$$

i.e., Einstein spaces, *annihilate* the corresponding partially topological action, whereas the selfdual spaces

$$R_{\alpha\beta} = {}^*R^{(\star)}_{\alpha\beta} \tag{5}$$

i.e. Thompson spaces [39], are extrema. Both satify Yang's equation (2) due to the Bianchi identity (27) for the Lie dual of the curvature. This 'vacuum degeneracy' of the 'Yang-Mielke' theory is discussed in more detail in Ref [41].

Concentrating on topological terms such as those of Pontrjagin (21) and Euler (30), related self–dual modifications are more recently advocated as *topological 4D selfdual gravity* by Nakamichi et al. [31]. There, self- or anti-selfdual solutions are 'living' on Einstein spaces, as well. The addition of the Pontrjagin term with respect to the Riemannian curvature $R^{\{\}}_{\alpha\beta}$ and the axial torsion one-form $A := {}^*(\vartheta_\alpha \wedge T^\alpha)$ is motivated by the the *axial anomaly* $\langle dj_5 \rangle = 2im\langle \overline{\psi}\, {}^*\gamma_5 \psi\rangle - \big(R^{\{\}}_{\alpha\beta} \wedge R^{\{\}\alpha\beta} + \frac{1}{2} dA \wedge dA\big)/48\pi^2$ in the coupling to Dirac fields ψ, cf. [16].

3 Gravitational gauge field equations

Let us slightly generalize our gemetrical framework: The total action of interacting matter and gravitational gauge fields is assumed to be a functional of suitable matter fields Ψ and of the metric $g_{\alpha\beta}$, the coframe ϑ^α, as well as the linear connection $\Gamma_\alpha{}^\beta$ as geometrical variables. Besides the Euler-Lagrange equation $\delta L/\delta\Psi = 0$ for matter, their independent variations yield the following *nonlinear field equations* [11]:

$$DH_\alpha - E_\alpha = \Sigma_\alpha, \qquad \text{(1st)} \tag{6}$$

$$DH^\alpha{}_\beta + \vartheta^\alpha \wedge H_\beta = \Delta^\alpha{}_\beta. \qquad \text{(2nd)} \tag{7}$$

In general, the *gauge field momenta* are defined by the two–forms:

$$H_\alpha := -\frac{\partial V}{\partial T^\alpha}, \qquad \text{and} \qquad H^\alpha{}_\beta := -\frac{\partial V}{\partial R_\alpha{}^\beta}. \tag{8}$$

The translational momenta H_α have dimension [length]. In addition to the material currents of energy–momentum $\Sigma_\alpha := \partial L/\partial\vartheta^\alpha$ and dynamical hypermomentum $\Delta^\alpha{}_\beta := \partial L/\partial\Gamma_\alpha{}^\beta$, generalizing the spin current three-form $\tau_{\alpha\beta} = \Delta_{[\alpha\beta]}$ there occur the three–form of *energy–momentum*

$$E_\alpha := \partial V/\partial\vartheta^\alpha = e_\alpha \rfloor V + (e_\alpha \rfloor T^\beta) \wedge H_\beta + (e_\alpha \rfloor R_\beta{}^\gamma) \wedge H^\beta{}_\gamma \tag{9}$$

of the gravitational gauge fields themselves. Note that E_α is the generalization of the usual Einstein current three-form $G_\alpha := R^{\{\}\beta\gamma} \wedge \eta_{\alpha\beta\gamma}/2$ which is dual to the usual Einstein tensor $G_{ij} := Ric^{\{\}}_{ij} - \frac{1}{2} g_{ij}$. In exterior form notation, the symmetric Ricci tensor is the holonomic version of the zero-form $Ric_{\alpha\beta} := (-1)^s \, {}^*(R_{(\alpha}{}^{\delta} \wedge \eta_{\delta|\beta)})$.

The 0th field equation arising from the variation of the metric $g_{\alpha\beta}$ is omitted here because it is known to be *redundant* 'on shell', i.e., once the matter equation is fulfilled.

3.1 Quadratic curvature Lagrangians

In the restricted Poincaré gauge framework, the most general *quadratic* curvature Lagrangian reads

$$V_{\rm QR} = -\frac{1}{2} R^{\alpha\beta} \wedge H_{\alpha\beta} \,, \qquad H_{\alpha\beta} := -{}^* \left(\sum_{N=1}^{6} b_{(N)} \, {}^{(N)}R_{\alpha\beta} \right) , \tag{10}$$

for which the propagating modes and its particle content has been determined by Sezgin and van Nieuwenhuizen [36], cf. Ref. [17].

In metric-affine extensions [11], there are not 6 but 11 irreducible pieces: Five more quadratic terms have been proposed by Esser [5] in an interesting decomposition. However, they may be partially related to the irreducible components of the topological Pontrjagin and Euler invariants. For instance, the Ricci squared term (27) of Ref. [42] is known [4] to be part of the Euler invariant (30). From the corresponding 2nd Noether identity there arises the generalized Bach-Lanczos identity (A.3.7) of Ref. [11] which relates some of the a priori independent quadratic curvature pieces in the first of the two vacuum field equation as equivalent terms.

4 Classical GR from modified double duality

In order unfold the classical correspondence of quadratic curvature Lagrangians to GR, let us now consider a variational principle with the constraint of vanishing torsion, consistently implemented by Lagrange multipliers:

$$\widetilde{V}_{\rm QR} = V_{\rm QR} + \lambda_\alpha \wedge T^\alpha \,. \tag{11}$$

Then obviously $T^\alpha = 0$ emerge and the 2nd field equation (7) amounts to an algebraic equation for the Lagrange multiplier two-form λ_α. After a resolution, it converts the 1st field equation (6) into (5.8.25) of Ref. [11], i.e.

$$2D\left(e^{\beta}\rfloor DH_{\alpha\beta}-\frac{1}{4}\vartheta_{\alpha}e^{\gamma}\rfloor e^{\delta}\rfloor DH_{\gamma\delta}\right)-E_{\alpha}=\Sigma_{\alpha}-D\mu_{\alpha}\,, \tag{12}$$

where $\mu_\alpha = \frac{1}{4}\,{}^*j_5\wedge\vartheta_\alpha$ is the *spin-energy potential.* In the case of Dirac spinors, it is dual to the axial current $j_5 := \overline{\psi}\gamma_5\,{}^*\gamma\psi$, cf. Ref. [27].

Let us consider now the modified *double duality ansatz*

$$H_{\alpha\beta}(**)=\theta^{\star}_{\rm L}\,R^{(\star)}_{\alpha\beta}+\frac{\theta^{*}_{\rm T}}{2\ell^{2}}\,\eta_{\alpha\beta} \tag{13}$$

for the rotational field momenta [22, 24, 47], where $\theta^*_{\rm T}$, and $\theta^\star_{\rm L}$ are dimensionless constants related to the individual coupling constants in the θ–type boundary terms (29) and (30). (The instanton solutions of Yang's theory of gravity, classified [21] already 1981, are a special case of the ansatz (13) for the choice $\theta^*_{\rm T}=0$ and $\theta^\star_{\rm L}=\mp 1$.)

Since $DR^{(\star)}_{\alpha\beta}\equiv 0$ and $D\eta_{\alpha\beta}=0$ in a Riemannian spacetime, the higher derivative Cotton type three-form in (12) drops out. Moreover, the Lie dual $R^{(\star)}_{\alpha\beta}$ of the curvature does not contribute in (9), due to the Bach-Lanczos identity (A.3.7) of Ref. [11] for Riemannian spacetimes.

Then we are left with (5.8.29) of Ref. [11], i.e.

$$-E_{\alpha}=\frac{\theta^{*}_{\rm T}}{2}R^{\{\}\beta\gamma}\wedge\eta_{\alpha\beta\gamma}-\theta^{*}_{\rm T}\Lambda_{\theta}\,\eta_{\alpha}=\ell^{2}\,\sigma_{\alpha}\,. \tag{14}$$

For $\theta^*_{\rm T}=1$ we obtain the classical Einstein equations

$$G_{\alpha}-\Lambda_{\theta}\,\eta_{\alpha}=\ell^{2}\,\sigma_{\alpha} \tag{15}$$

for the Riemannian background with the *symmetric* Belinfante-Rosenfeld three-form $\sigma_\alpha := \Sigma_\alpha - D^{\{\}}\mu_\alpha$ as source.

5 Discussion

Our main result is that the modification of the double duality relation (13) eliminates the 'vacuum ambiguity' for the exact solutions of SKY gravity, such that only Einstein spaces remain as classical background. Due to the explicit appearance of a length scale $\langle\varphi\rangle \propto 1/\ell$ in the ansatz (13), it is suggestive to associate this with a (sponteneous) *symmetry* *breaking* of the scale or Weyl invariance of the original Lagrangian (10), for instance in a model [10] dynamically coupled to a dilaton field φ, cf. Ref. [3]. Then an *induced* cosmological constant

$$\Lambda_\theta = -\frac{3\theta^*_{\rm T}}{2\ell^2(\theta^\star_{\rm L} + b_6)} \tag{16}$$

of microscopic origin [22] is unavoidable with an interesting (Anti-) de Sitter background, similarly as in the intriguing AdS/CFT correspondence.

The proposed duality could be extremely important for the path integral approach to quantum gravity where the quantum-mechanical transition amplitude $\int \mathcal{D}\Gamma \exp\left[-\int V^{(\star)}_{\rm SKY} d^4x/\hbar\right]$ is evaluated in an imaginary 'spacetime' with Euclidean signature, cf. Ref. [23]. For anti-selfdual SKY gravity, *instanton* type configurations [6, 13] near the classical ones, i.e. Einstein spaces, are more probable then the 'spurious' Thomson spaces, as one would expect naively. For the modified duality with a breaking of scale invariance, the transition amplitude peaks at classical Einstein spaces only. Alternatively, in a four-dimensional Yang-Mills theory gauging the de Sitter group [20, 34], scale invariance gets spontaneously broken by a pseudo-Goldstone type 'radius vector' [40], odd under CP, in order to recover the Hilbert-Einstein action plus the Euler term.

From the work of Stelle [37] we know that the curvature squared gravity in Riemannian spacetime is perturbatively *renormalizable* but plagued with ghost [19]. However, by absorbing the quadratic Weyl curvature part of (1) in the Wess-Zumino action, these negative-metric states can be removed dynamically [8] and unitarity restored.

Thus it seems that there is still room for a quantization program based on Yang's theory of gravity, departing, in a gauge covariant approach, from the nilpotency of the corresponding BRST charges [29] or from superconnections [32].

Acknowledgments

We would like to thank Yuval Ne'eman and Dmitri Vassiliev for valuable comments, as well as Humberto H. Peralta for help with the Latex. Moreover, (E.W.M.) acknowledges the support of the SNI and thanks Noelia, Markus Gérard Erik, and Miryam Sophie Naomi for encouragement.

Appendix A: A letter of Robert Mills

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July 25, 1989

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Dear Dr. Mielke:

I was very please to get your letter of July 5, which reached me here a few days ago. I appreciate very much your comments on my paper on gauge fields in AJP.

The paper was intended, as I'm sure you can tell, more as an introduction to the subject than as a history and, as you can also tell, I'm not much of an historian. I learn my history by reading other peoples' books, and not by tracing back to original sources! It is therefore a pleasure to me to get some of the benefit of your knowledge of the history, specially from the European perspective.

I was well aware of the fact that Hilbert was Noether's mentor and advocate at Göttingen, but I had not appreciated the interactions between them that must have played an important role in the development of the famous theorem, nor the nature and degree of Klein's influence. I was in particular unaware of Klein's "Erlanger Programm", and would like to know more about it. My German is almost nonexistent – is "Erlanger" related to "erlangen", or is it a proper noun?

I have struggled through a couple of the relevant passages of your paper with Hehl that you kindly sent me, and what I could uncover looks very interesting. I haven't been able to understand Schrödinger's 1923 idea. It sounds from what you say as if he simply observed that the quantization of action is equivalent to a quantization of a $\log \Omega$ in units of $hc\gamma/e$. Does he then go on to speculate on γ being imaginary and Ω being complex? In this pre-Schrödinger equation context I should think that would be very radical – the most noteworthy thing about the idea.

I am particularly intrigued by your comments about Schrödinger's 1932 paper, which you mentioned briefly in the paper. I shall look it up when I get back to civilization in September. Was Schrödinger thinking in the context of quantum field theory, or was he working from the analogy of general relativity? I gather from your letter that he was thinking of a curvature tensor of some sort rather than a physical field analogous to the EM field. In 1954, when Yang and I were thinking about this (with me following in his dust) we were not thinking at all of generalized curvatures. If Frank saw the similarity to the Riemann tensor he kept it to himself; the whole thrust was to extend the electromagnetic case, with the nonlinear terms as an exciting complication

When the gravitational case began to be discussed it is my impression that the gravitational field was seen by physicists as an example of a gauge field without the converse idea, of other gauge fields as connections on a curved manifold, being thought of at all until a later time when physicists became aware of fiber bundle theory. Frank's excitement when that idea did surface among physicists, makes it seem pretty clear to me that he hadn't thought of gauge theory in a geometrical light previously.

I must confess that I didn't mention Oskar Klein's model in my paper because I felt, very possibly incorrectly, that while the equations looked somewhat the same it wasn't really the same idea. My recollection (and I didn't study it in depth) is that he has a nonlinear field equation, but without any relationship to a local invariance, the central idea in gauge theory. I would be happy to have your comments on this perception for future reference.

I am still very much puzzled by gauge fields. Nothing seems to fit together quite as smoothly as a good theory should, as I mention at the end of the AJP article. I mentioned there some of my uneasiness about the awkwardness of quantizing gauge fields, and the suggestion that perhaps we don't really understand quantum theory. I'm disturbed also by the difficulty of defining and performing gauge transformations on the quantized field operators. The only way I know of to get from one gauge to another (except for trivial c-number transformations) is to go back to the classical theory and quantize again in the other gauge. Is there a theorem that says that all the theories arrived at in this way are equivalent? (I made a sort of effort in this direction in a paper on gauge transformations within the Feynman graph prescription, in the early 1970's. I don't have the reference with me, but it's in Phys. Rev., and is something about "propagator gauge transformations" as I recall.)

Another cause of uneasiness is the lack of any real principle for determining the Lagrangian. For the straightforward cases of internal symmetries it's pretty obvious that the only reasonable choice is the standard $\mathbf{F}_{\mu\nu} \cdot \mathbf{F}^{\mu\nu}$. When you get to general relativity, though, the choice seems much less obvious, and the choice nature has made is apparently not what a gauge theorist would have chosen. Is gravity really a gauge field, or is it rather a symptom of the inadequacy of the gauge field concept? If it's a gauge field, what are the gauge potentials? The connection in GR is not a fundamental field as it is in a true gauge field; and I can't tell whether to regard the connection or the curvature tensor as the proper analog of $\mathbf{F}_{\mu\nu}$. These all may be questions to which the professional GR people have clear answers by now, and I'd be glad to know how things now stand. (Also, if gravity is a gauge field, then the symmetry group has to be the Poincaré group, which means torsion has to be included. Is it the present understanding that this is a consistent and unambiguous program, at least at the classical level?)

My deepest question is the one I pose at the end of he article: Is it possible for the full unitary invariance of quantum theory to be a local symmetry? The associated gauge field is then an operator potential A_μ, which looks for all the world like the EM potential field except that the symmetry group is very non-Abelian. This now satisfies my wish for an operator gauge transformation, though the Lie algebra doesn't seem to reduce to finite representations (and I'm not sure I'd want it to). If the infinitesimal transformations are generated by an infinitesimal operator field $\theta(x)$, then the covariant derivative of any field ψ is

$$D_\mu \psi = \partial_\mu \psi - i\,[A_\mu, \psi] \; .$$

The potentials transform according to

$$\delta A_\mu = D_\mu \theta \, ,$$

and the operator field $F_{\mu\nu}$ has the nonlinear term $[A_\mu, A_\nu]$. You can write down obvious field equations for the interactions with a simple Dirac field, but I don't have a Hamiltonian or any kind of commutation relations. Even the equal time commutators can't be nice because the operator field θ makes a mess of them when you make a gauge transformation.

I think one the more appealing things about this idea is the hope of a local formulation of quantum theory. The state vector itself would be defined locally, and would be subject to parallel displacement and the effects of curvature, whatever these might mean physically. I have hunted around for some conceivable system of physical interpretation for such an algebra, but without success.

Anyway, if the idea has merit maybe someone will see how to exploit it. If not, then there has to be something wrong with the dream that gives rise to it. Comments would be appreciated.

Again, many thanks for your interest, and for the things you sent me. I shall look up your 1987 paper when I return to Columbus.

With best wishes,

Sincerely yours,
Bob Mills
Robert Mills

Appendix B: Geometry of a Riemann–Cartan spacetime

Our geometrical arena consists of a four–dimensional manifold equipped with a local metric of Lorentz signature $(o_{\alpha\beta}) = \mathrm{diag}(-1, 1, 1, 1)$. For the representation of spinors in a curved spacetime, it is necessary to have the anholonomic formalism available on par. Therefore, we introduce an orthonormal local frame and coframe field

$$e_\alpha = e^i{}_\alpha\, \partial_i\,, \qquad \vartheta^\alpha = e_j{}^\alpha\, dx^j \tag{17}$$

of dimension [1/length] and [length], respectively. According to our conventions, $\alpha, \beta, ... = 0, 1, ..., 3$ are anholonomic frame indices, $i, j, k, ... = 0, 1, ..., 3$ are holonomic or world indices, and $\wedge$ denotes the exterior product. The *coframe* field of basis one–forms are reciprocal to the frame e_α with respect to the *interior product* $\rfloor$, i.e., $e_\alpha \rfloor \vartheta^\beta = e^i{}_\alpha\, e_i{}^\beta = \delta^\beta_\alpha$.

In a Yang–Mills type gauge theory of gravity, the coframe ϑ^α of dimension [length] and the dimensionless connection one-form $\Gamma_\alpha{}^\beta = \Gamma_{i\alpha}{}^\beta\, dx^i$ are regarded as gauge potentials of non–linearly realized *local translations* and *local linear transformations*, respectively, cf. Ref. [40]. The corresponding translational field strength is the *torsion* two–form

$$T^\alpha := D\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta = \frac{1}{2}\, T_{ij}{}^\alpha\, dx^i \wedge dx^j, \tag{18}$$

of dimension [length] and the dimensionless Riemann–Cartan (RC) *curvature* two–form [11]

$$R_\alpha{}^\beta := d\Gamma_\alpha{}^\beta - \Gamma_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta = \frac{1}{2}\, R_{ij\alpha}{}^\beta\, dx^i \wedge dx^j. \tag{19}$$

These field strengths obey the *first* and *second Bianchi identities*

$$DT^\alpha \equiv R_\gamma{}^\alpha \wedge \vartheta^\gamma, \qquad \text{and} \qquad DR^{\alpha\beta} \equiv 0\,. \tag{20}$$

The corresponding Lagrangians [9] are the Chern–Simons type boundary terms

$$dC_{\rm TT} := \frac{1}{2\ell^2} d\left(\vartheta^\alpha \wedge T_\alpha\right) = \frac{1}{2\ell^2}\left(T^\alpha \wedge T_\alpha + R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta\right) =: V_{\rm NY}\,, \tag{21}$$

$$dC_{\rm RR} := \frac{1}{2} d\big(\Gamma_{\alpha\beta} \wedge R^{\alpha\beta} - \frac{1}{3}\Gamma_\alpha{}^\beta \wedge \Gamma_\beta{}^\gamma \wedge \Gamma_\gamma{}^\alpha\big) = \frac{1}{2} R_{\alpha\beta} \wedge R^{\alpha\beta} =: V_{\rm Pontr}\,, \tag{22}$$

where ℓ is a fundamental length. Up to normalizations, they are also known as Nieh–Yan four-form [33] and Pontrjagin term, respectively. Observe that both are violating parity P, see, e.g., Ref. [25], where also a more condensed notation in 'Clifforms' is used.

The Riemannian content of our geometrical framework can be brought out by splitting the RC connection according to $\Gamma^{\alpha\beta} = \Gamma^{\{\}\,\alpha\beta} - K^{\alpha\beta}$ into the unique Levi–Civita connection $\Gamma^{\{\}\,\alpha\beta}$ of Riemannian geometry and into the *contortion*

$$K_{\alpha\beta} = -K^{\beta\alpha} = e_{[\alpha} \rfloor T_{\beta]} - \frac{1}{2}(e_\alpha \rfloor e_\beta \rfloor T_\gamma)\vartheta^\gamma\,. \tag{23}$$

It follows from (18) that the latter is implicitly related to torsion via $T^\alpha = K^\alpha{}_\beta \wedge \vartheta^\beta$. In turn, the RC curvature two–form (19) decomposes as follows

$$R^{\alpha\beta} = R^{\{\}\,\alpha\beta} + D^{\{\}}\, K^{\alpha\beta} + K^\alpha{}_\mu \wedge K^{\mu\beta}. \tag{24}$$

Appendix C: Dual forms

On an n–dimensional manifold with metric index s, the Hodge dual of p–forms is almost involutive, i.e. : ${}^{**}\alpha = (-1)^{p(n-p)+s}\alpha$. For spacetimes where $s = 1$ holds, it induces an *almost complex structure*, cf. [2]. In four dimensions, the *Hodge dual* applied to two–forms is *conformally invariant* [1].

Our *Hodge dual* * of exterior forms is defined such that the normalization

$${}^*(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta) = \eta^{\alpha\beta\gamma\delta}, \text{ where } \quad \eta_{\alpha\beta\gamma\delta} := +\delta^{0123}_{\alpha\beta\gamma\delta} \tag{25}$$

holds.

From the volume four–form $\eta = \frac{1}{4!}\eta_{\alpha\beta\gamma\delta}\,\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta$, the so-called η- or dual basis $\{\eta,\,\eta_\alpha,\,\eta_{\alpha\beta},\,\eta_{\alpha\beta\gamma}\,\eta_{\alpha\beta\gamma\delta}\}$ of exterior forms can be generated by consecutive interior products: $\eta_\alpha := e_\alpha\rfloor\eta = {}^*\vartheta_\alpha$, $\eta_{\alpha\beta} := e_\beta\rfloor\eta_\alpha = \eta_{\alpha\beta\gamma}\,\vartheta^\gamma = e_\beta\rfloor e_\alpha\rfloor\eta = {}^*(\vartheta_\alpha \wedge \vartheta_\beta) = \frac{1}{2}\eta_{\alpha\beta\gamma\delta}\,\vartheta^\gamma \wedge \vartheta^\delta$, and $\eta_{\alpha\beta\gamma} := e_\gamma\rfloor\eta_{\alpha\beta} = {}^*(\vartheta_\alpha \wedge \vartheta_\beta \wedge \vartheta_\gamma)$. Anholonomic indices are lowered by $o_{\alpha\beta} = e^i{}_\alpha\, e^j{}_\beta\, g_{ij}$, where $(o_{\alpha\beta}) = \mathrm{diag}(-1,\,1,\,1,\,1)$ denotes the signature of spacetime.

The *Lie dual* of Lorentz algebra–valued forms such as contortion and curvature is defined by

$$K^{(\star)}_\alpha := \frac{1}{2}\eta_{\alpha\beta\gamma} \wedge K^{\beta\gamma}\,, \qquad R^{(\star)}_{\alpha\beta} := \frac{1}{2}\eta_{\alpha\beta\gamma\delta}R^{\gamma\delta}\,, \tag{26}$$

and satisfies

$$DR^{(\star)}_{\alpha\beta} \equiv 0 \tag{27}$$

due to the second Bianchi identity (20) and $D\eta_{\alpha\beta\gamma\delta} = 0$ for a vanishing Weyl covector.

In four dimensions, it is useful to consider also the self– or anti-selfdual torsion and curvature two-forms

$$\begin{aligned} T^{\pm}_\alpha &:= \frac{1}{2}\left(T_\alpha \pm {}^*T_\alpha\right)\,, \\ R^{\pm}_{\alpha\beta} &:= \frac{1}{2}\left(R_{\alpha\beta} \pm {}^*R_{\alpha\beta}\right)\,, \qquad R^{(\pm)}_{\alpha\beta} := \frac{1}{2}\left(R_{\alpha\beta} \pm R^{(\star)}_{\alpha\beta}\right)\,, \end{aligned} \tag{28}$$

defined in terms of the Hodge or Lie dual, respectively. In view of this, the teleparallel boundary term can be written as

$$dC_{\mathrm{TT}^*} := \frac{1}{2\ell^2}d(\vartheta^\alpha \wedge {}^*T_\alpha) = \frac{1}{2\ell^2}\left(T^\alpha \wedge {}^*T_\alpha - D\,{}^*T_\alpha\right)\,. \tag{29}$$

On the other hand, the topological Euler term

$$\begin{aligned} V_{\mathrm{Euler}} &:= \frac{(-1)^s}{2}\, d\Big(\Gamma_{\alpha\beta} \wedge R^{\alpha\beta(\star)} - \frac{1}{3}\Gamma_\alpha{}^{\beta(\star)} \wedge \Gamma_\beta{}^\gamma \wedge \Gamma_\gamma{}^\alpha\Big) \\ &= \frac{(-1)^s}{2}\, R^{\alpha\beta} \wedge R^{(\star)}_{\alpha\beta} \\ &\equiv \frac{1}{2}\, R_{\alpha\beta} \wedge {}^*R^{\alpha\beta} - 2Ric_{\alpha\beta} \wedge {}^*Ric^{\alpha\beta} + \frac{1}{2}Ric_\alpha{}^\alpha \wedge {}^*Ric_\beta{}^\beta \end{aligned} \tag{30}$$

for Riemann-Cartan spaces has, in view of the *Lanczos identity* [18], an equivalent representation in terms of Yang's Lagrangian V_{SKY} as well as a Ricci-squared and curvature scalar squared term, cf. Eq. (3.1) of Ref. [21].

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(Manuscrit reçu le 19 juillet 2004)