

Gauge Invariance in Classical Electrodynamics¹

WOLFGANG ENGELHARDT²

Max-Planck-Institut für Plasmaphysik, D-85741 Garching, Germany

RÉSUMÉ. Le concept de l'invariance de jauge dans l'électrodynamique classique suppose tacitement que les équations de Maxwell possèdent des solutions uniques. Mais en calculant le champ électromagnétique d'une particule en mouvement faisant usage de la jauge de Lorenz ainsi que de la jauge de Coulomb, et résolvant directement les équations des champs nous obtenons des solutions contradictoires. Nous concluons donc que l'hypothèse tacite de l'unicité de la solution n'est pas justifiée. La raison pour cette difficulté peut être attribuée aux équations d'onde inhomogènes qui connectent simultanément les champs propageants et leurs sources.

ABSTRACT. The concept of gauge invariance in classical electrodynamics assumes tacitly that Maxwell's equations have unique solutions. By calculating the electromagnetic field of a moving particle both in Lorenz and in Coulomb gauge and directly from the field equations we obtain, however, contradicting solutions. We conclude that the tacit assumption of uniqueness is not justified. The reason for this failure is traced back to the inhomogeneous wave equations which connect the propagating fields and their sources at the same time.

P.A.C.S.: 03.50.De; 11.15.-q; 41.20.-q; 41.60.-m

¹This paper is written by W. Engelhardt in his private capacity. No official support by the Max-Planck-Institut für Plasmaphysik is intended or should be inferred.

²Home address: Fasaneriestrasse 8, D-80636 München, Germany
E-mail address: wolfgangw.engelhardt@t-online.de

1 Introduction

“The principle of gauge invariance plays a key role in the standard model, which describes electroweak and strong interactions of elementary particles.” This statement is quoted from an article by J. D. Jackson and L. B. Okun [1] which emphasizes the importance of the principle and delineates its historical evolution. The concept emerged from classical electrodynamics where the electromagnetic field expressed in terms of potentials: $\vec{E} = -\nabla\phi - \partial\vec{A}/c\partial t$, $\vec{B} = \text{rot } \vec{A}$ does not change under the transformation: $\vec{A} \rightarrow \vec{A} + \nabla\chi$, $\phi \rightarrow \phi - \partial\chi/c\partial t$. Since $\text{div } \vec{A} \rightarrow \text{div } \vec{A} + \Delta\chi$ and χ is an arbitrary function, the divergence of the vector potential can seemingly be chosen arbitrarily without influencing the fields. This feature was exploited to de-couple Maxwell’s inhomogeneous equations by either choosing $\text{div } \vec{A} + \partial\phi/c\partial t = 0$ (Lorenz³ gauge) or $\text{div } \vec{A} = 0$ (Coulomb gauge). The solution for the fields should be entirely independent of this choice.

There is, however, a tacit assumption behind the formalism of electrodynamic gauge invariance: Maxwell’s equations must have unique solutions, otherwise it is meaningless to talk about potentials from which fields may be derived. In reference [1] it is said: “It took almost a century to formulate this nonuniqueness of potentials that exists despite the uniqueness of the electromagnetic fields.” To our knowledge it was never attempted to prove that the electromagnetic field resulting from a solution of Maxwell’s equations is actually unique, it was just taken for granted. If there were no unique solution of Maxwell’s linear system of first order field equations, gauge transformations on (nonexisting) potentials would be irrelevant.

In this paper we start with the usual assumption that unique solutions of Maxwell’s equations do exist and try to calculate them both with the help of potentials (Sections 2 - 4) and directly from the field equations (Section 5). For the electromagnetic field of a moving particle we find, however, contradicting solutions depending on the method used. In Section 6 we show that the standard Liénard-Wiechert fields, which satisfy Maxwell’s source-free equations, cannot be considered as a unique solution of Maxwell’s inhomogeneous equations. Thus, we infer that the tacit assumption concerning the existence of unique solutions is not justified in general.

³In [1] it is pointed out that Ludwig Valentin Lorenz published more than 25 years before Hendrik Antoon Lorentz what is commonly known as the “Lorentz condition”. To give proper credit to Lorenz we use in this paper the term “Lorenz gauge”.

The reason for this failure is discussed in Section 8 where we come to the conclusion that the mixture of elliptic and hyperbolic equations, as formulated by Maxwell, does not permit a physical solution for moving point sources.

2 The electromagnetic field of a moving particle calculated in Lorenz gauge

The electromagnetic field produced by a moving particle is calculated [2] from Maxwell's equations:

$$\operatorname{div} \vec{E} = 4\pi\rho \quad (1)$$

$$\operatorname{rot} \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad (2)$$

$$\operatorname{div} \vec{B} = 0 \quad (3)$$

$$\operatorname{rot} \vec{B} = \frac{4\pi}{c} \rho \vec{v} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \quad (4)$$

where ρ is the charge density located in a narrow region round a center moving with velocity \vec{v} . The total charge of the particle is the integral over all space:

$$\iiint \rho(\vec{x}', t) d^3x' = e \quad (5)$$

The usual method to solve equations (1-4) is to adopt the potential ansatz:

$$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \nabla\phi, \quad \vec{B} = \operatorname{rot} \vec{A} \quad (6)$$

which satisfies automatically equations (2) and (3) and leads to two second order differential equations:

$$\Delta\phi + \frac{1}{c} \operatorname{div} \frac{\partial \vec{A}}{\partial t} = -4\pi\rho \quad (7)$$

$$\Delta \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c} \rho \vec{v} + \nabla \left(\operatorname{div} \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \right) \quad (8)$$

The freedom in the choice of the divergence of the vector potential is exploited to de-couple equations (7) and (8). Adopting the Lorenz condition:

$$\text{div } \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0 \quad (9)$$

one obtains two wave equations of the same structure:

$$\Delta \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -4\pi \rho \quad (10)$$

$$\Delta \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c} \rho \vec{v} \quad (11)$$

The solution of (10) subject to the boundary condition that the scalar potential vanishes at infinity is:

$$\phi(\vec{x}, t) = \iiint \frac{\rho(\vec{x}', t - \frac{R}{c})}{R} d^3x' \quad , \quad R = |\vec{x} - \vec{x}'| \quad (12)$$

where the position of the charge density is to be taken at the retarded time:

$$t' = t - \frac{R}{c} \quad (13)$$

The advanced solution with $t' = t + R/c$ is excluded on physical grounds.

The integral (12) may be carried out by employing the δ -function formalism:

$$\rho\left(\vec{x}', t - \frac{R}{c}\right) = \int_{-\infty}^{+\infty} \rho(\vec{x}', t') \delta\left(t' - t + \frac{R}{c}\right) dt' \quad (14)$$

Substituting this into (12) yields:

$$\phi(\vec{x}, t) = \int_{-\infty}^{+\infty} \iiint \frac{\rho(\vec{x}', t')}{R} \delta\left(t' - t + \frac{R}{c}\right) d^3x' dt' \quad (15)$$

The integration over all space for a point-like particle results with (5) in:

$$\phi(\vec{x}, t) = \int_{-\infty}^{+\infty} \frac{e}{R} \delta\left(t' - t + \frac{R}{c}\right) dt' \quad (16)$$

where R expresses now the distance between the field point and the position of the charge $\vec{x}'(t')$ at the retarded time. Changing to the variable

$$u = t' - t + \frac{R(\vec{x}, \vec{x}'(t'))}{c}, \quad \frac{du}{dt'} = 1 - \frac{\vec{R} \cdot d\vec{x}'(t')}{cR}, \quad \frac{d\vec{x}'(t')}{dt'} = \vec{v}(t') \quad (17)$$

we may integrate (16) and obtain the result:

$$\phi(\vec{x}, t) = \left[\frac{e}{R \left(1 - \frac{\vec{R} \cdot \vec{v}}{Rc}\right)} \right]_{t'=t-\frac{R}{c}} \quad (18)$$

Similarly, we find from (11):

$$\vec{A}(\vec{x}, t) = \left[\frac{e\vec{v}}{cR \left(1 - \frac{\vec{R} \cdot \vec{v}}{Rc}\right)} \right]_{t'=t-\frac{R}{c}} \quad (19)$$

Solutions (18) and (19) are the well-known retarded Liénard-Wiechert potentials. With the differentiation rules resulting from (13):

$$\frac{\partial}{\partial t} = \frac{1}{1 - \frac{\vec{R} \cdot \vec{v}}{Rc}} \frac{\partial}{\partial t'} = \frac{1}{\lambda} \frac{\partial}{\partial t'}, \quad \frac{\partial}{\partial \vec{x}} (f(t')) = -\frac{df}{dt'} \frac{1}{c} \frac{\partial R}{\partial \vec{x}}, \quad \frac{\partial R}{\partial \vec{x}} = \frac{\vec{R}}{\lambda R} \quad (20)$$

one obtains with (6) for the fields:

$$\begin{aligned} \vec{E}(\vec{x}, t) &= e \left[\frac{1}{\lambda^3} \left(\frac{\vec{R}}{R^3} - \frac{\vec{v}}{cR^2} \right) \left(1 - \frac{v^2}{c^2} + \frac{1}{c^2} \vec{R} \cdot \frac{d\vec{v}}{dt'} \right) - \frac{1}{c^2 R \lambda^2} \frac{d\vec{v}}{dt'} \right] \\ \vec{B}(\vec{x}, t) &= -e \left[\frac{\vec{R} \times \vec{v}}{cR^3 \lambda^3} \left(1 - \frac{v^2}{c^2} + \frac{1}{c^2} \vec{R} \cdot \frac{d\vec{v}}{dt'} \right) + \frac{\vec{R}}{c^2 R^2 \lambda^2} \times \frac{d\vec{v}}{dt'} \right] \end{aligned} \quad (21)$$

where $t' = t - \frac{R}{c}$.

3 Solution in Coulomb gauge

The fields as given by (21) must be the same when they are calculated in Coulomb gauge by adopting the condition:

$$\text{div } \vec{A} = 0 \quad (22)$$

Equations (7) and (8) become:

$$\Delta\phi = -4\pi\rho \quad (23)$$

$$\Delta\vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c} \rho \vec{v} + \frac{1}{c} \frac{\partial}{\partial t} \nabla\phi \quad (24)$$

Solution of (23) yields the instantaneous Coulomb potential:

$$\phi_C(\vec{x}, t) = \frac{e}{r}, \quad r = |\vec{x} - \vec{x}'(t)| \quad (25)$$

which substituted into (24) results in a wave equation for the 'Coulomb' vector potential:

$$\Delta\vec{A}_C - \frac{1}{c^2} \frac{\partial^2 \vec{A}_C}{\partial t^2} = -\frac{4\pi}{c} \rho \vec{v} - \frac{e}{c} \frac{\partial}{\partial t} \left(\frac{\vec{r}}{r^3} \right) \quad (26)$$

The vector $\vec{r} = \vec{x} - \vec{x}'(t)$ denotes now the simultaneous distance between charge and field point. The first term on the right-hand-side of (26) yields a contribution to the vector potential which is identical with (19):

$$\vec{A}_{C1}(\vec{x}, t) = \left[\frac{e\vec{v}}{cR \left(1 - \frac{\vec{R}\cdot\vec{v}}{Rc}\right)} \right]_{t'=t-\frac{R}{c}} \quad (27)$$

The solution of the wave equation for the second part of the vector potential may be written in the form:

$$\begin{aligned} \vec{A}_{C2}(\vec{x}, t) &= \frac{e}{4\pi c} \int_{-\infty}^{+\infty} \iiint \frac{\partial}{\partial t'} \left(\frac{\vec{r}}{r^3} \right) \delta\left(t' - t + \frac{R}{c}\right) \frac{d^3s}{R} dt' \\ \vec{r} &= \vec{s} - \vec{x}'(t'), \quad R = |\vec{s} - \vec{x}| \end{aligned} \quad (28)$$

when we employ a δ -function as in the previous Section. Here we have used the integration variable \vec{s} in distinction of the position $\vec{x}'(t)$ of the charge. The contribution of \vec{A}_{C2} to the electric field is with (6):

$$\vec{E}_2(\vec{x}, t) = -\frac{e}{4\pi c^2} \iiint \left[\frac{\partial^2}{\partial t'^2} \left(\frac{\vec{r}}{r^3} \right) \right]_{t'=t-\frac{R}{c}} \frac{d^3s}{R} \quad (29)$$

and the contribution to the magnetic field may be written as:

$$\begin{aligned}\vec{B}_2(\vec{x}, t) &= \frac{e}{4\pi c} \int_{-\infty}^{+\infty} \iiint \nabla \left(\frac{\delta}{R} \right) \times \frac{\partial}{\partial t'} \left(\frac{\vec{r}}{r^3} \right) d^3s dt' \\ &= \frac{e}{4\pi c} \int_{-\infty}^{+\infty} \iiint \left(\frac{\delta'}{cR^2} - \frac{\delta}{R^3} \right) \vec{R} \times \frac{\partial}{\partial t'} \left(\frac{\vec{r}}{r^3} \right) d^3s dt' \quad (30)\end{aligned}$$

Performing a partial integration over t' we obtain the solution in the form:

$$\vec{B}_2(\vec{x}, t) = -\frac{e}{4\pi c} \iiint \left[\frac{\vec{R}}{R^3} \times \left(\frac{\partial}{\partial t'} \left(\frac{\vec{r}}{r^3} \right) + \frac{R}{c} \frac{\partial^2}{\partial t'^2} \left(\frac{\vec{r}}{r^3} \right) \right) \right]_{t'=t-\frac{R}{c}} d^3s \quad (31)$$

Since (27) yields already the magnetic field as given by (21), the contribution (31) must vanish which is not likely to occur: The finite cross-product is to be integrated with different weight, so that both terms in (31) cannot vanish simultaneously. Furthermore, the condition for identical electric fields in Coulomb and in Lorenz gauge:

$$\nabla\phi_{LW} = \nabla\phi_C + \frac{1}{c} \frac{\partial \vec{A}_{C2}}{\partial t} \quad (32)$$

where ϕ_{LW} denotes the potential (18), cannot be satisfied, if (28) is not an irrotational field. Even if the second term in (32) could be written as a gradient: $\partial \vec{A}_{C2}/c \partial t = \nabla\phi_2$, the condition: $\phi_{LW} = \phi_C + \phi_2 + const$ would still be violated, since $\partial \vec{A}_{C2}/\partial t$, and thereby ϕ_2 , would depend on the acceleration according to (28), which is not the case for ϕ_{LW} nor ϕ_C .

In order to quantify these qualitative considerations the integral (28):

$$\begin{aligned}\vec{A}_{C2}(\vec{x}, t) &= -\frac{e}{4\pi c} \iiint \left[\frac{\vec{v}}{r^3} - \frac{3\vec{r}(\vec{r} \cdot \vec{v})}{r^5} \right]_{t'=t-\frac{R}{c}} \frac{d^3s}{R} \\ \vec{r} &= \vec{s} - \vec{x}'(t') \quad , \quad \vec{R} = \vec{s} - \vec{x}\end{aligned} \quad (33)$$

may be evaluated analytically for the case of a constant velocity of the charge:

$$\vec{x}'(t') = \vec{x}_0 + \vec{v}_0 t' = \vec{x}_0 + \vec{v}_0 \left(t - \frac{R}{c} \right) \quad (34)$$

The integration variable \vec{s} may be replaced by \vec{R} so that the vector \vec{r} may be written as:

$$\vec{r} = \vec{R} + \vec{x} - (\vec{x}_0 + \vec{v}_0 t) + \frac{R}{c} \vec{v}_0 \quad (35)$$

We assume that the charge moves along the z-axis of a coordinate system having its origin at $\vec{R} = 0$. The z-component of the vector potential evaluated on the z-axis becomes then:

$$A_{C2z} = -\frac{e v_z}{4 \pi c} \iiint \left[\frac{1}{r^3} - \frac{3 (R_z + \beta R + z)^2}{r^5} \right] \frac{d^3 R}{R}, \quad \beta = \frac{v_z}{c}$$

$$r^2 = R^2 (1 + \beta^2) + 2 \beta R R_z + 2 z (\beta R + R_z) + z^2 \quad (36)$$

where $z(t)$ denotes the distance between the field point and the position of the charge at time t . Using spherical coordinates:

$$\vec{R} = R \sin \theta \cos \varphi \vec{i} + R \sin \theta \sin \varphi \vec{j} + R \cos \theta \vec{k}, \quad d^3 R = R^2 \sin \theta dR d\theta d\varphi \quad (37)$$

expression (36) becomes:

$$A_{C2z} = -\frac{e v_z}{4 \pi c} \int_0^{2\pi} \int_0^\pi \int_0^\infty \left[\frac{1}{r^3} - \frac{3 (R (\beta + \cos \theta) + z)^2}{r^5} \right] R \sin \theta dR d\theta d\varphi$$

$$r^2 = R^2 (1 + \beta^2 + 2 \beta \cos \theta) + 2 R z (\beta + \cos \theta) + z^2 \quad (38)$$

The integration over φ yields a factor of 2π , since the integrand is independent of φ . Upon indefinite integration over R and θ one obtains:

$$A_{C2z} =$$

$$-\frac{e v_z}{2 c z} \frac{R^2 (R + (\beta R + z) \cos \theta)}{(\beta R + z)^2 \sqrt{R^2 (1 + \beta^2 + 2 \beta \cos \theta) + 2 R z (\beta + \cos \theta) + z^2}} \quad (39)$$

The integral vanishes both at $R = 0$ and at $R = \infty$. It is singular at $R = z/(1 - \beta)$, $\theta = \pi$. Close to the singularity we expand it by substituting $R = z/(1 - \beta) - z \epsilon_1$, $\theta = \pi - \epsilon_2$ and obtain in lowest order:

$$A_{C2z} = \frac{e v_z}{2 c z} (1 - \beta)^2 \left(\frac{\epsilon_1}{\epsilon_2} - \frac{(1 - \beta) (5 - 9 \beta)}{4} \frac{\epsilon_1^2}{\epsilon_2} \right) \quad (40)$$

Obviously, the integral assumes no definite value when we go to the limits $\epsilon_1 = \epsilon_2 = 0$, as it does not converge absolutely.

If we perform the same calculation on (29) we obtain in addition to undefined terms a diverging contribution:

$$E_{2z} = -\frac{e v_z^2}{2 c^2 z^2} \frac{1 - \beta}{\epsilon_1} + \frac{0}{0} \quad (41)$$

which is also encountered when we calculate (31).

From these results we must conclude that Maxwell's equations do not yield a physical solution for the fields of a moving particle in Coulomb gauge. Furthermore, the undefined fields as derived from (25), (27) and (28) by using (6) disagree definitely with the fields as given by (21) in Lorenz gauge which are well defined. Similar conclusions were reached by Onoochin [3] without evaluating the integral (33) explicitly.

Jackson has attempted [4] to find an approximate 'quasistatic' solution in Coulomb gauge which should be valid for velocities $v \ll c$ in a region very close to the particle where retardation may be neglected. We discuss this attempt in Appendix A and show that it also leads to an inconsistency.

In a recent paper [6] Hnizdo has given a solution of (26) which is based on the gauge function as defined in equation (66) below in Section 7, and on the formal solution (67) for the gauge function. The second formal solution (70) is ignored in this consideration. Hnizdo arrives at a similar result as we found in (40), but he establishes uniqueness by applying a regularization procedure which can hardly be justified from a mathematical point of view. We discuss his approach in Appendix B.

4 Helmholtz's ansatz

Having obtained contradicting solutions in Lorenz gauge and in Coulomb gauge we infer that Maxwell's equations contain an inconsistency which does not permit to find a unique solution for the fields. In order to trace this problem, we employ Helmholtz's theorem which states that any vector field may be expressed as the sum of a rotational and an irrotational field. This was already used in the ansatz (6). Now we apply it to the electric field:

$$\vec{E} = \text{rot } \vec{U} - \nabla \phi, \quad \text{div } \vec{U} = 0 \quad (42)$$

Substituting this into (2) we obtain:

$$\Delta \vec{U} = \frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad (43)$$

Taking the rotation of (4) and inserting (42) yields a second Poisson equation:

$$\Delta \left(\vec{B} - \frac{1}{c} \frac{\partial \vec{U}}{\partial t} \right) = -\frac{4\pi}{c} \nabla \rho \times \vec{v} \quad (44)$$

Its solution for a point charge is:

$$\vec{B} = \frac{1}{c} \frac{\partial \vec{U}}{\partial t} + \frac{e}{c} \left(\frac{\vec{v} \times \vec{r}}{r^3} \right) \quad (45)$$

Substituting this into (43) yields a wave equation for the vector potential of the electric field:

$$\Delta \vec{U} - \frac{1}{c^2} \frac{\partial^2 \vec{U}}{\partial t^2} = \frac{e}{c^2} \frac{\partial}{\partial t} \left(\frac{\vec{v} \times \vec{r}}{r^3} \right) \quad (46)$$

which has the retarded solution:

$$\vec{U} = -\frac{e}{4\pi c^2} \iiint \left[\frac{\partial}{\partial t'} \left(\frac{\vec{v} \times \vec{r}}{r^3} \right) \right]_{t'=t-\frac{R}{c}} \frac{d^3 s}{R} \\ \vec{r} = \vec{s} - \vec{x}'(t') \quad , \quad \vec{R} = \vec{s} - \vec{x} \quad (47)$$

For the magnetic field we obtain with (45):

$$\vec{B} = -\frac{e}{4\pi c^3} \iiint \left[\frac{\partial^2}{\partial t'^2} \left(\frac{\vec{v} \times \vec{r}}{r^3} \right) \right]_{t'=t-\frac{R}{c}} \frac{d^3 s}{R} + \frac{e}{c} \left(\frac{\vec{v} \times \vec{r}}{r^3} \right) \quad (48)$$

and the electric field as derived from (47) with (42) becomes:

$$\vec{E} = -\frac{e}{4\pi c^2} \iiint \left[\frac{\vec{R}}{R^3} \times \left(\frac{R}{c} \frac{\partial^2}{\partial t'^2} \left(\frac{\vec{v} \times \vec{r}}{r^3} \right) + \frac{\partial}{\partial t'} \left(\frac{\vec{v} \times \vec{r}}{r^3} \right) \right) \right]_{t'=t-\frac{R}{c}} d^3 s \\ + \frac{e \vec{r}}{r^3} \quad (49)$$

where we have added the Coulomb field which results from insertion of (42) into (1). We note that neither of the expressions (48) and (49) agrees with the fields as calculated in Sections 2 and 3, because the fields derived from the Helmholtz ansatz (42) depend on the second time derivative of the velocity. Assuming a constant velocity of the particle one could also show that the integrals (48) and (49) actually diverge.

5 Direct solution of the field equations

The two types of potential ansatz (6) and (42) resulted in different solutions for the fields. We, therefore, want to calculate the fields directly from (1 - 4) without using any potential ansatz. By elimination of the electric and the magnetic field, respectively, we find the wave equations:

$$\Delta \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = \frac{4\pi}{c} \nabla \rho \times \vec{v} \quad (50)$$

$$\vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \frac{4\pi}{c^2} \frac{\partial}{\partial t} (\rho \vec{v}) + 4\pi \nabla \rho \quad (51)$$

By applying the standard method of solving this kind of wave equation, as it was described in Section 2, it can be shown that the ensuing solution of (50) and (51) yields exactly the fields as given by (21). However, by deriving the hyperbolic equations (50) and (51) we have ignored the fact that Maxwell's equations are actually a mixture of hyperbolic and elliptic equations which became very obvious in the previous Section. In order to take this into account we split the electric field into its irrotational and its rotational part:

$$\vec{E} = \vec{E}_g + \vec{E}_r \quad (52)$$

The rotational part does not enter equation (1). The irrotational part is just the quasistatic Coulomb field which does not propagate:

$$\vec{E}_g = \frac{e \vec{r}}{r^3}, \quad \vec{r} = \vec{x} - \vec{x}'(t) \quad (53)$$

The rotational part obeys the wave equation:

$$\Delta \vec{E}_r - \frac{1}{c^2} \frac{\partial^2 \vec{E}_r}{\partial t^2} = \frac{4\pi}{c^2} \frac{\partial}{\partial t} (\rho \vec{v}) + \frac{e}{c^2} \frac{\partial^2}{\partial t^2} \left(\frac{\vec{r}}{r^3} \right) \quad (54)$$

which has the retarded solution:

$$\begin{aligned} \vec{E}_r(\vec{x}, t) = & \quad (55) \\ & - \iiint \left[\frac{1}{c^2} \frac{\partial}{\partial t} (\rho(\vec{s}, t')) \vec{v}(t') + \frac{e}{4\pi c^2} \frac{\partial^2}{\partial t'^2} \left(\frac{\vec{r}}{r^3} \right) \right]_{t'=t-\frac{|\vec{x}-\vec{s}|}{c}} \frac{d^3 s}{|\vec{x}-\vec{s}|} \end{aligned}$$

Adding (53) one obtains the electric field as it was derived in Section 3 in Coulomb gauge, whereas the retarded solution of (50) yields the magnetic field as it was derived in Section 2 in Lorenz gauge. It was

shown in Section 3 that the integral (55) diverges. Hence, it does not represent a physical solution for the rotational part of the electric field.

This analysis shows that the inconsistency inherent to Maxwell's equations is not an artefact produced by employing a potential ansatz. It seems to result from the mixture of hyperbolic and elliptic differential equations for the fields, as they were formulated by Maxwell. Only in Lorenz gauge the elliptic equations are completely removed so that there is seemingly agreement between the solutions of the hyperbolic equations (50) and (51) and the hyperbolic potential equations (10) and (11).

6 The inhomogeneous wave equations for a moving point source

The discrepancies encountered in Sections 3 - 5 are apparently related to the fact that Maxwell's set of equations mixes hyperbolic and elliptic structures so that unique solutions are not possible. In view of this finding it is somewhat surprising that in Lorenz gauge the elliptic features seem to be removed altogether so that equations (10) and (11) yield the unique solutions (18) and (19), provided the advanced solutions are suppressed on physical grounds. A closer look at the inhomogeneous wave equation (10) reveals, however, that the elliptic character is still there, but concealed in the inhomogeneity. If it is brought out, it turns out that the solution (18) cannot be considered as unique.

In order to see this we employ a different method of solution than that used in Section 2. Due to the linearity of (10) one may split the potential into two contributions:

$$\phi = \phi_0 + \phi_1 \quad (56)$$

The wave equation may then be split into a Poisson equation:

$$\Delta\phi_0 = -4\pi\rho \quad (57)$$

and into another wave equation:

$$\Delta\phi_1 - \frac{1}{c^2} \frac{\partial^2\phi_1}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2\phi_0}{\partial t^2} \quad (58)$$

where the solution of the elliptic equation (57) enters as an extended

source⁴ Adding (57) and (58) the wave equation (10) is recovered. The retarded solution of (58) is:

$$\phi_1(\vec{x}, t) = -\frac{e}{4\pi c^2} \iiint \left[\frac{\partial^2}{\partial t'^2} \left(\frac{1}{r} \right) \right]_{t'=t-\frac{R}{c}} \frac{d^3s}{R}$$

$$\vec{r} = \vec{s} - \vec{x}'(t') \quad , \quad \vec{R} = \vec{s} - \vec{x} \quad (59)$$

where we have substituted the instantaneous Coulomb potential (25) resulting from a solution of (57). Carrying out the differentiation in (59) we have:

$$\phi_1(\vec{x}, t) = \frac{e}{4\pi c^2} \iiint \left[\frac{1}{r^3} \left(v^2 - \vec{r} \cdot \frac{d\vec{v}}{dt'} \right) - \frac{3(\vec{r} \cdot \vec{v})^2}{r^5} \right]_{t'=t-\frac{R}{c}} \frac{d^3s}{R} \quad (60)$$

This integral depends on the acceleration which is not the case for the potential (18). At constant velocity the integral is very similar to (33) and we know from the calculation in Section 3 that it has no defined limiting value according to (40).

Similar considerations apply to the inhomogeneous wave equation (11), the solution of which may formally be written as:

$$\vec{A}(\vec{x}, t) = \frac{e\vec{v}}{cr} - \frac{e}{4\pi c^3} \iiint \left[\frac{\partial^2}{\partial t'^2} \left(\frac{\vec{v}}{r} \right) \right]_{t'=t-\frac{R}{c}} \frac{d^3s}{R} \quad (61)$$

by applying the same method of splitting the vector potential into two parts. If result (61) is substituted into (6), the fields would depend on the third derivative of the velocity, which is not the case according to (21), so that (61) is incompatible with (19).

We must conclude then that the potentials (56) and (61) disagree with the Liénard-Wiechert potentials (18) and (19) which turn out not to be a unique solution of the inhomogeneous wave equations (10) and (11), even if the advanced solutions are suppressed. As a matter of fact, equations (10) and (11) have no physical solution judged from our results (60) and (61) which do not admit a defined limiting value.

There is a direct way of showing that the Liénard-Wiechert result, which leads to the fields (21), satisfies only Maxwell's homogeneous equations, but the inhomogeneities are not taken into account properly. Let

⁴The ansatz (56) was also used in [2], but it was erroneously assumed that $\partial^2\phi_0/\partial t^2$ in (58) may be neglected.

us consider Green's first identity:

$$\iiint \phi \nabla \phi \cdot d^2 \vec{x} = \iiint (\phi \Delta \phi + |\nabla \phi|^2) d^3 x \quad (62)$$

The surface integral on the left-hand-side vanishes at infinity both for the Coulomb and for the Lorenz potential so that the volume integral over all space on the right-hand-side must vanish as well. Substituting the Coulomb potential (25) together with (23) one obtains an integral equation which must be satisfied by the charge density:

$$4 \pi e \int_0^\infty \left(-\frac{4 \pi}{r} \rho(r) + \frac{e}{r^4} \right) r^2 dr = 0 \quad (63)$$

where spherical coordinates centered around the position of the charge were used. If one inserts the Liénard-Wiechert potential (18) together with (10) into (62), one obtains:

$$2 \pi \int_0^\infty \int_0^\pi \left(-4 \pi \phi_{LW} \rho(r) + \frac{\phi_{LW}}{c^2} \frac{\partial^2 \phi_{LW}}{\partial t^2} + |\nabla \phi_{LW}|^2 \right) \sin \theta d\theta r^2 dr = 0 \quad (64)$$

This integral equation depends now on the velocity which may be easily verified by choosing a constant velocity so that (18) yields:

$$\phi_{LW} = \frac{e}{\left[(\vec{v} \cdot \vec{r}/c)^2 + (1 - v^2/c^2) r^2 \right]^{\frac{1}{2}}}, \quad \vec{r} = \vec{x} - (\vec{x}_0 + \vec{v} t) \quad (65)$$

Both integral equations (63) and (64) cannot be satisfied by the same function $\rho(r)$, unless the 'shape' of the point charge would depend on the velocity, as suggested by Onoochin in Reference [3].

7 Transformation of the Lorenz potentials into Coulomb potentials

From the results obtained in Sections 2 - 6 it should be evident by now that a unique gauge transformation, which transforms the Lorenz potentials of a point source into the corresponding Coulomb potentials, cannot

exist. We finally want to show this explicitly. The gauge transformation is effected by a generating function χ :

$$\phi_C = \phi_L - \frac{1}{c} \frac{\partial \chi(\vec{x}, t)}{\partial t} \quad , \quad \vec{A}_C = \vec{A}_L + \nabla \chi(\vec{x}, t) \quad (66)$$

By integrating the first relation over time one obtains immediately:

$$\chi(\vec{x}, t) = \int_{t_0}^t c (\phi_L - \phi_C) dt + \chi_0(\vec{x}) \quad (67)$$

where the Lorenz potential is given by (18) and the Coulomb potential by (25). The gauge function must also satisfy the Poisson equation:

$$\Delta \chi(\vec{x}, t) = \text{div } \vec{A}_C - \text{div } \vec{A}_L \quad (68)$$

which follows from the second relation in (66). Equation (68) has the instantaneous solution:

$$\chi(\vec{x}, t) = -\frac{1}{4\pi c} \iiint \frac{\partial \phi_L(\vec{s}, t)}{\partial t} \frac{d^3 s}{|\vec{s} - \vec{x}|} \quad (69)$$

where we have substituted the conditions (9) and (22) into (68). If we insert the Lorenz potential as given by (18) and apply the first differentiation rule in (20), we find:

$$\chi(\vec{x}, t) = -\frac{e}{4\pi c} \iiint \frac{1}{\lambda^3 R^3} \left[\vec{R} \cdot \vec{v} + \frac{R}{c} \left(\vec{R} \cdot \frac{d\vec{v}}{dt'} - v^2 \right) \right] \frac{d^3 s}{|\vec{s} - \vec{x}|}$$

$$\vec{R} = \vec{s} - \vec{x}'(t') \quad , \quad t' = t - \frac{|\vec{s} - \vec{x}|}{c} \quad (70)$$

As this expression depends not only on the velocity, but also on the acceleration, it is not compatible with expression (67). Furthermore, the integral (70) has no unique limiting value, but depends on the chosen integration variables. In order to see this we assume a constant velocity. The Liénard-Wiechert potential (18) becomes in this case:

$$\phi_L(\vec{x}, t) = \frac{e}{\left[(\vec{v} \cdot \vec{r}/c)^2 + (1 - v^2/c^2) r^2 \right]^{\frac{1}{2}}} \quad , \quad \vec{r} = \vec{x} - \vec{x}'(t) \quad (71)$$

Substitution into (69) yields:

$$\chi(\vec{x}, t) = -\frac{e}{4\pi c} \iiint \frac{\vec{v} \cdot \vec{s}}{\left[(\vec{v} \cdot \vec{s}/c)^2 + (1 - v^2/c^2) s^2 \right]^{\frac{3}{2}} |\vec{r} - \vec{s}|} d^3s \quad (72)$$

where we have chosen a coordinate system with its origin at the position $\vec{x}'(t)$ of the charge. If we change to the integration variable $\vec{s}' = \vec{s} - \vec{r}$, which is equivalent to shifting the origin of the coordinate system to the field point \vec{x} , we obtain instead:

$$\chi(\vec{x}, t) = -\frac{e}{4\pi c} \iiint \frac{\vec{v} \cdot (\vec{s}' + \vec{r})}{\left[(\vec{v} \cdot (\vec{s}' + \vec{r})/c)^2 + (1 - v^2/c^2) (s'^2 + r^2 - 2\vec{s}' \cdot \vec{r}) \right]^{\frac{3}{2}} s'} d^3s' \quad (73)$$

Apart from a common logarithmic singularity at infinity the integrals (72) and (73) are conditionally convergent and assume different limiting values which was verified by calculating them in spherical coordinates. Numerical calculations of (72) and (73) in cylindrical coordinates yield still different limiting values. None of these results agrees with (67).

In accordance with the previous conclusions reached in this paper we infer that no unique function χ exists which would transform the Lorenz potentials of a point source into the corresponding Coulomb potentials. Hence, the principle of 'gauge invariance' is not applicable to classical electrodynamics.

8 Discussion

The nature of the inconsistencies encountered in Sections 2 - 5 is apparently connected to the feature of Maxwell's equations of mixing elliptic and hyperbolic structures. Even if the equations are reduced to wave equations like (10) and (11) or (50) and (51), the elliptic character is still there in form of the inhomogeneity and may be made visible by the method of solution employed in Section 6. There we were compelled to conclude that the inhomogeneous wave equation does not have, as a matter of fact, a unique solution, or even leads to unphysical diverging solutions, at least in case of a moving point source. In principle, it is well known that the inhomogeneous wave equation has an infinite manifold

of solutions, but it is generally believed that suppression of the advanced solutions reduces it to a physical solution, the properties of which are uniquely determined by the behaviour of the source. According to our analysis in Section 6 we must maintain, however, that the inhomogeneous wave equations do not correctly describe the physical process of wave production by a moving point source.

Our result is not too surprising, if we realize that the inhomogeneous wave equations (50) and (51) relate the measurable fields at a certain location with temporal changes in a remote source at the *same* time. Although both the fields and the sources in equations (1 - 4) were differentiated at time t when, e.g., the electric field was eliminated to obtain (50), the retarded solutions (21) require that the differentiation of the sources is dated back to the earlier time $t - R/c$. However, the source may be an extinguished star the light of which we see only now at time t . It makes no sense to differentiate a source not existing any more which has no influence whatsoever on the light we see after a billion years. When we differentiate Maxwell's equations now to obtain the wave equations for the fields, we treat the temporal changes in the sources as if they would happen now at time t . In the retarded solutions, however, we date back the change in the sources to a remote past. This procedure is inconsistent, but inescapable due to the structure of Maxwell's equations.

If the same procedure would be applied to acoustic waves, one would encounter similar inconsistencies. Instead, from the hydrodynamic equations one derives linearized homogeneous wave equations for the pressure and the fluid velocity. These are solved by imposing suitable boundary conditions which are determined, e.g., by the oscillating membrane of a loudspeaker. Maxwell's equations, however, lead to inhomogeneous wave equations which connect the *travelling* fields with the source at the *same* time, a *contradictio in adjecto*. This became quite obvious in Section 5 where equation (51) predicts that the *total* electric field is produced in a point-like region and travels within a finite time to the field point where an observer may be placed. On the other hand, equation (53) predicts that *part* of the field has already arrived there instantaneously, as soon as any change in the source occurred. This inconsistency cannot be resolved without altering equations (1 - 4).

It is well possible that Maxwell was fully aware of this problem, because in his 'Treatise' [7] he did *not* derive an inhomogeneous wave equation. He used the Coulomb gauge and derived equation (24). Then

he argued that in the ‘ether’ there does not exist any current or free charge. This way he was left with:

$$\Delta \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = \frac{1}{c} \frac{\partial}{\partial t} \nabla \phi \quad , \quad \Delta \phi = 0$$

Now he committed a formal inconsistency by concluding that the vanishing of the Laplacian of the scalar potential justifies to omit also the gradient of the potential. This is, of course, not true close to a charged body. In other words, he omitted part of the ‘displacement current’, which was invented by him in the first place, in order to ‘derive’ a homogeneous wave equation for the vector potential. At last he suggested that this equation should be solved by imposing Cauchy-type boundary conditions on \vec{A} and $\partial \vec{A} / \partial t$. The result would be a travelling vector wave from which the measurable fields could be derived with (6) at any place and time where the wave has arrived. In his last Chapter XXIII (Article 862) Maxwell discusses Riemann’s inhomogeneous wave equation which is formally the same as Lorenz’s equation (10). Taking reference to Clausius he states that Riemann’s formula is not in agreement with “the known laws of electrodynamics”. Maxwell’s method of using only a homogeneous wave equation to describe electromagnetic waves is still in practical use, when the radiation emitted by an antenna is calculated. Only the homogeneous wave equation is used together with plausible boundary conditions resulting from a physics which goes beyond Maxwell’s equations.

In parentheses we remark that Einstein gave his famous paper of 1905 the title: “Zur Elektrodynamik bewegter Körper”, but the ‘moving bodies’ which carry charges and currents are not treated in his analysis. He deals only with Maxwell’s *homogeneous* equations which do not lead to contradictions. The *instantaneous* Coulomb potential is left out of the consideration.

For slowly varying fields Maxwell’s equations do make sense. They describe correctly the phenomenon of induction in transformers where only the instantaneous fields come into play and where the displacement current is negligible. When a condenser is charged up, the displacement current must be allowed for, but then it is only the instantaneous Coulomb field which matters in practice. A quasistatic ‘near field’ theory can be carried through satisfactorily, but amalgamating it with wave phenomena, in the way Maxwell has tried it, leads to the contradictions

which we have demonstrated. These were also recognized by Dunn [8], but not worked out in detail.

Appendix A

The wave equation (26) for the second part of the ‘Coulomb’ vector potential:

$$\Delta \vec{A}_{C2} - \frac{1}{c^2} \frac{\partial^2 \vec{A}_{C2}}{\partial t^2} = \frac{e}{c} \left(\frac{\vec{v}}{r^3} - \frac{3 \vec{r} (\vec{v} \cdot \vec{r})}{r^5} \right) \quad (\text{A.1})$$

may be taken as a Poisson equation which has the formal solution:

$$\begin{aligned} \vec{A}_{C2} = & -\frac{e}{4\pi c} \iiint \left(\frac{\vec{v}}{r^3} - \frac{3 \vec{r} (\vec{v} \cdot \vec{r})}{r^5} \right) \frac{d^3 s}{|\vec{s} - \vec{x}|} \\ & - \frac{1}{4\pi c^2} \iiint \frac{\partial^2 \vec{A}_{C2}}{\partial t^2} \frac{d^3 s}{|\vec{s} - \vec{x}|}, \quad \vec{r} = \vec{s} - \vec{x}'(t) \end{aligned} \quad (\text{A.2})$$

Close to the charge and at small velocity $v \ll c$ the second term may be expected to be negligibly small so that the first integral could be considered as an approximate solution of (A1) with limited applicability.

This attempt to obtain a ‘quasistatic’ solution was pursued by Jackson [4] in order to find the interaction Lagrangian between two particles moving at nonrelativistic velocities. He chose a coordinate system centered at $\vec{x}' = 0$ and performed a partial integration:

$$\begin{aligned} -\frac{4\pi c}{e} \vec{A}_{C2} &= \iiint \frac{\partial}{\partial \vec{s}} \left(\frac{\vec{v} \cdot \vec{s}}{s^3} \right) \frac{d^3 s}{|\vec{s} - \vec{x}|} \\ &= -\iiint \left(\frac{\vec{v} \cdot \vec{s}}{s^3} \right) \frac{\partial}{\partial \vec{s}} \left(\frac{1}{|\vec{s} - \vec{x}|} \right) d^3 s = \frac{\partial}{\partial \vec{x}} \iiint \frac{\vec{v} \cdot \vec{s}}{s^3} \frac{d^3 s}{|\vec{s} - \vec{x}|} \end{aligned} \quad (\text{A.3})$$

Now the integration was straightforward and yielded:

$$\vec{A}_{C2} = -\frac{e}{c} \frac{\partial}{\partial \vec{x}} \left(\frac{\vec{v} \cdot \vec{x}}{2 |\vec{x}|} \right) = \frac{e}{2c} \left(-\frac{\vec{v}}{|\vec{x}|} + \frac{\vec{x} (\vec{v} \cdot \vec{x})}{|\vec{x}|^3} \right) \quad (\text{A.4})$$

In order to obtain the total vector potential, the unretarded contribution from expression (27) must be added:

$$\vec{A}_C = \frac{e}{2c} \left(\frac{\vec{v}}{|\vec{x}|} + \frac{\vec{x} (\vec{v} \cdot \vec{x})}{|\vec{x}|^3} \right) \quad (\text{A.5})$$

It turns out, however, that the solution (A.4) does not satisfy the Poisson equation from which it was calculated. Substituting (A.4) into the left-hand-side of (A.1) and ignoring the second time derivative yields:

$$-\frac{e\vec{v}}{2c}\Delta\left(\frac{1}{|\vec{x}|}\right)+\frac{e}{c}\left(\frac{\vec{v}}{|\vec{x}|^3}-\frac{3\vec{x}(\vec{v}\cdot\vec{x})}{|\vec{x}|^5}\right) \quad (\text{A.6})$$

The distance vector \vec{x} pointing from the origin to the field point may be identified with \vec{r} , since the position of the charge was assumed at the origin. The second term of (A.6) equals the right-hand-side of (A.1), but the first term yields a δ - function:

$$-\frac{e\vec{v}}{2c}\Delta\left(\frac{1}{|\vec{x}|}\right)=\frac{2\pi e\vec{v}}{c}\delta(\vec{x})=\frac{2\pi\rho(\vec{x})\vec{v}}{c} \quad (\text{A.7})$$

which remains unaccounted for in (A.1).

The reason for the discrepancy is that the first integral in (A.2) does not converge absolutely, as we have shown in Section 3. Consequently, the operations of partial integration as well as interchanging the sequence of differentiation and integration in (A.3) are not permitted and lead to an incorrect result. The first integral in (A.2) has, in fact, no defined limiting value as is obvious from expression (40).

In a private communication Professor Jackson explained how he could obtain (A5) from an expansion procedure applied on the Liénard-Wiechert potential - which is based on the Lorenz gauge - and that this was actually being done by Darwin who derived (A.5) in 1920 [5].

Appendix B

Hnizdo's article *Potentials of a uniformly moving point charge in the Coulomb gauge* [6] was apparently motivated by Onoochin's objections against 'mainstream' electrodynamics as published in Ref. [3]. Onoochin reached conclusions which are similar to those arrived at in the present paper, namely that the electrodynamic field depends on the choice of the gauge. He speculates that the 'shape' of the electron could depend on its velocity, an idea which was already pursued (unsuccessfully) by Lorentz. Hnizdo tries to resolve the problem by using a certain regularization procedure when the integral (33) is evaluated.

Both authors do not emphasize that this type of integral is conditionally convergent. This property may lead to the known fact that the value of the integral depends on the sequence of integration, on the chosen coordinate system, or on the position of the origin. A typical example is given in Bronstein-Semendjajew's *Taschenbuch der Mathematik*, Verlag Harri Deutsch, Frankfurt, on page 347:

$$\int_0^1 \int_0^1 \frac{y^2 - x^2}{(x^2 + y^2)^2} dx dy = \frac{\pi}{4} \quad \text{or:} \quad = -\frac{\pi}{4} \quad (\text{B.1})$$

The result depends on whether one integrates first over x and then over y , or vice versa. Although one obtains a finite result in both cases, its value is not unique.

Hnizdo uses the formal solution (67) to obtain his formulas (16 - 18) for the difference between the vector potentials in Coulomb and in Lorenz gauge. He claims that these formulae do not have singularities, but this is actually not true. The y -component for the difference is, e.g.:

$$A_{Cy} - A_{Ly} = -\frac{c}{v} \frac{y(x - vt)}{y^2 + z^2} (V_C - V_L) \quad (\text{B.2})$$

If this expression is expanded around the point $(x = vt + \epsilon_1, y = 0 + \epsilon_2, z = 0 + \epsilon_3)$, one obtains:

$$A_{Cy} - A_{Ly} = -\frac{c}{v} \frac{\epsilon_2 \epsilon_1}{\epsilon_2^2 + \epsilon_3^2} (V_C - V_L) \quad (\text{B.3})$$

This result is similar to our result (40) where we concluded that no limiting value exists. In fact, (B.3) can assume any value between zero and infinity depending on the way how one approaches the limits $\epsilon_1 \rightarrow 0, \epsilon_2 \rightarrow 0, \epsilon_3 \rightarrow 0$.

Hnizdo avoids the ambiguity by using a regularization procedure as defined in his equation (32). It amounts to using spherical coordinates centered around the charge point. This prescription is, however, arbitrary and lacks a rigorous justification. If one deviates from it, one finds non-uniqueness as demonstrated in our comparison of equations (72) and (73). It should be noted that Hnizdo's regularization would not work, when applied to the expression (70) for the gauge function which is incompatible with (67). This demonstrates again that Maxwell's equations do not have a unique solution (for moving point charges) representing a measurable well defined field.

Acknowledgments

The author is deeply indebted to Dr. O. Kardaun for extensive and stimulating discussions of the subject. In particular, he pointed out that the vector potential in Coulomb gauge leads to a conditionally convergent integral. He also pointed to the mixed character of Maxwell's equations involving elliptic and hyperbolic equations.

A communication with Professor J. D. Jackson in October 2001 was very useful for clarifying the issue dealt with in Appendix A.

Critical previous comments by Professor D. Pfirsch and Dr. R. Gruber helped to formulate the paper more concisely than originally conceived. Professor Pfirsch's written comments, in particular, alluded to the possibility that a gauge transformation between Lorenz and Coulomb potentials might not exist, as discussed in Section 7.

References

- [1] J. D. Jackson, L. B. Okun, *Reviews of Modern Physics*, **73**, 663, (2001).
- [2] Our calculation follows the methods applied in:
R. Becker, F. Sauter, *Theorie der Elektrizität*, B. G. Teubner Verlagsgesellschaft, Stuttgart (1962), §§ 66, 69.
- [3] V. V. Onoichin, *Annales de la Fondation Louis de Broglie*, **27**, 163, (2002).
- [4] D. Jackson, *Classical Electrodynamics*, Third Edition, John Wiley & Sons, Inc., New York (1999), Chapter 12.6: derivation of equation (12.80).
- [5] C. G. Darwin, *Phil. Mag. ser. 6*, **39**, 537, (1920).
- [6] V. Hnizdo, *European Journal of Physics*, **25**, 351, (2004).
- [7] J. C. Maxwell, *A Treatise on Electricity and Magnetism*, Dover Publications, Inc., New York (1954), Vol. 2, Articles 783, 784.
- [8] D. A. Dunn, *Models of Particles and Moving Media*, Academic Press, New York and London (1971), page 118ff.

(Manuscrit reçu le 30 août 2004)