Hamiltonian Symmetry in Special Relativity : Relativity in Expanding Hyperbolic Space

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ABSTRACT. The source of the failure of covariance for Hamiltonian and center-of-mass methods in relativistic dynamics and of the absence of a covariant n-body Dirac equation is traced to the loss of a nonrelativistic kinematic symmetry between position and velocity in the transition to special relativity. The cure is found in a new, symmetric special relativity applicable to an expanding universe hyperbolic in both position and velocity spaces, with a double expansion in the Lorentz ratio v/c and a Hubble ratio $r/cH^{-1}(t)$. Its position-velocity symmetry has many consequences both local and cosmological. It perfects a Hamiltonian symmetry in relativity, connects the Hubble effect of distance on velocity with the Lorentz effect of velocity on length and time, cures the dislocation between relativistic and Hamiltonian dynamics and its consequences, solves the center-of-mass problem and allows the creation of covariant n-body Schrödinger and Dirac equations. The Hubble-Lorentz relationship has the alternating (symplectic) symmetry of Hamiltonian and quantum dynamics. This symmetry is displayed in a new hyperbolic Poincaré group, the symmetry group of relativistic kinematics in an expanding hyperbolic universe. Its symmetry and normalization require the time-dependence $\rho(t) = cH^{-1}(t)$ of the Hubble expansion to be shared with a time-varying light speed $c(t)$ in such a way that their product $c(t)\rho(t) = \sigma$ is constant while their quotient is the Hubble function $c(t)/\rho(t) = H(t)$.

I. Introduction

A. Special Relativity and the n-Body Dirac Equation

This paper initiates a revision of special relativistic physics to cure a defect responsible for the covariance and simultaneity difficulties that have impeded the use of Hamiltonian and center-of-mass methods in relativistic dynamics and the construction of a covariant n-body Dirac equation. The source of these difficulties is identified as the absence of a standard measure of scale in position space comparable to c in velocity space. The needed measure of scale is found in the time-dependent Hubble length $\rho(t)$ of an expanding hyperbolic universe. Its introduction in the structure of special relativity carries forward into the relativistic regime a kinematic symmetry between position and velocity spaces that is essential for the proper treatment of the dynamics of moving objects in spatially extended systems.

The development of special relativity accomplished the reconciliation of electromagnetic and optical theory with the kinematics of a moving body by introducing the scale magnitude c into the topology of velocity and momentum space, making these spaces hyperbolic. When the physical situation involves two or more bodies the additional variable of their distance of separation r_{ij} is an essential feature of the problem. This feature is one that special relativity in its standard form has not been able to treat with complete consistency because of the simultaneity problem. Its solution requires the introduction of a standard of length in position space comparable to c in velocity space. This standard of length is naturally identified with the time-dependent Hubble length $\rho(t) = cH^{-1}(t)$ in the expanding universe. Its incorporation in a new extended form of special relativity converts the flat position space of Einstein and Minkowski into a hyperbolic space isomorphic with relativistic velocity space. This establishes a kinematic symmetry between position and velocity that must now be recognized as an important structural feature of the physical universe.

The presence of the time-dependent Hubble standard of length in the new form of special relativity provides a natural solution to the simultaneity problem. All proper times in the relativistic system now have their common origin in the Big Bang singularity. In the doubly hyperbolic universe simultaneity in dynamics follows from the fact that this common proper time with its well-defined origin—expansion time—is the progress variable of dynamics. Simultaneity remains a frame-dependent concept in the relative times of clocks and electromagnetic signals, of intervals and of observers.

The new symmetric form of special relativity incorporates the Hubble expansion as well as the doubly hyperbolic kinematics of position and velocity spaces. The self-consistency of this structure is demonstrated by constructing the new hyperbolic Poincaré group. Its symmetry and normalization require recognizing that c as well as ρ can be cosmologically time-dependent, subject to the observed Hubble relationship $c(t)/\rho(t) = H(t)$ and a new product relationship $c(t)\rho(t) = \sigma$, where $\sigma = c_0^{\,2} H_0^{\,-1}$ is a new constant, the Hubble-Lorentz constant. The presence of this symmetry in a universe expanding from a singularity resolves the simultaneity problem and makes possible a new, self-consistent, covariant treatment of centers of mass and of two-body and n-body dynamics.

B. Quantum Mechanics and Relativity: Born's Principle of Reciprocity

One of the unfulfilled goals of physical theory is to integrate quantum mechanics and general relativity into a single consistent unit. Almost 70 years ago Max Born pointed out that a major obstacle to that union has been a deep difference in structural symmetry between the two component theories. A fundamental symmetry of quantum mechanics is the Hamiltonian reciprocity between the space-time coordinates x^ι and the momentum-energy components p_ι, with the symplectic sign-alternation of Hamilton's equations of dynamics $\dot{x}^\iota = \frac{\partial H}{\partial p_\iota}, \dot{p}_\iota = -\frac{\partial H}{\partial x^\iota}$ and of the quantal connections that take the form $p_\iota = \frac{\hbar}{i}\frac{\partial}{\partial x^\iota}, x^\iota = -\frac{\hbar}{i}\frac{\partial}{\partial p_\iota}$ in position and momentum representation respectively. This fundamental symmetry plays no part in the differential geometric structure of general relativity, which is based on introducing the line element $ds^2 = g_{\iota\kappa} x^\iota x^\kappa$ in position space alone and developing its consequences in an extension of classical dynamics.

The entire structure of nonrelativistic quantum mechanics with its commutators and uncertainty relations as it was developed from 1925 to 1927 found the preexisting Hamiltonian dynamical structure already perfectly adapted to the new physics—a profound indication that the

Hamiltonian symmetries express a fundamental pattern of the physical universe. Born, who had experienced that sudden development close to its center, always believed that the striking reciprocity between coordinates and momenta in Hamiltonian dynamics and in quantum mechanics reflected a deep and far-reaching symmetry in physics, important consequences of which were probably still waiting to be found [1,2]. He therefore proposed as a Principle of Reciprocity that position space and momentum space in relativity should be subject to geometrical laws with the same balanced structure. A consequence of this insight is that the Hamiltonian symmetry of quantum mechanics is fundamental, and must be preserved in a properly quantum mechanical relativistic theory.

Born's proposed version of General Relativity in the light of this Principle was unsuccessful. It failure is not surprising, because his application of the Principle was incomplete. Born did not notice that Special Relativity, the essential forerunner of General Relativity and its low-density limit, does not itself satisfy the Reciprocity Principle, because the limiting velocity c that controls the structure of velocity and momentum space is not matched by a counterbalancing limit in position space. The requirement of reciprocity in this direction provides the essential constraint that is needed to establish the structure of a revised special relativity. This can be achieved by using the time-dependent Hubble length $\rho(t)$ as a limiting length in position space, and accepting the possibility of a concurrent time-dependence to the limiting velocity $c(t)$. The Symmetric Special Relativity that results solves the problem of the relativistic center of mass and completes the Hamiltonian description of relativistic classical and quantum dynamics in two-body and n-body systems. It also provides the necessary gravitation-free limit for the extension of General Relativity that Born was seeking and leads to the possibility of a new solution of that problem.

C. Dynamics in the Relativistic and Non-Relativistic Regimes

The special relativistic modification of dynamics had the secondary effect of invalidating at high velocities the application of a number of important techniques and principles of the widest currency and usefulness in the treatment of dynamical problems in complex systems. The requirement of velocity covariance and issues of simultaneity in systems with spatially separated parts called into question the use of Hamiltonian methods and of formulations depending on the use of centers of mass in n-body systems. The desire to satisfy Minkowski's four-space symmetry connecting position space and time has had similar consequences. A striking gulf has developed separating the theoretical approaches that can be used with confidence in relativistic problems from those that are used with the greatest convenience and success in a vast range of problems in the nonrelativistic regime.

The development of quantum mechanics illustrated this anomaly from the beginning. The Schrödinger equation does not satisfy the space-time symmetry requirement and is entirely nonrelativistic. The Dirac equation crosses the relativistic frontier, but covariance requirements have limited it to the description of only a single moving particle. More generally, the need for explicit relativistic covariance led Feynman to reconstruct quantum mechanics on a purely Lagrangian basis, relinquishing to the sidelines the Hamiltonian symmetries on which it was founded. The relativistic center-of-mass problem remained unsolved. This appears most strikingly in the problem of the hydrogenic spectrum itself, where the Dirac solution is unable to predict the isotope shift in the Balmer spectrum, an effect of the two-body center of mass, because of the absence of a properly relativistic two-body Dirac equation. An ad hoc extension of the nonrelativistic expression for the center of mass, modified only by including the relativistic dependence of the participating masses on their relative velocities, has provided a successful algorithm that solves this problem in practical calculations of two-body and n-body problems, but it remains unjustified by a supporting general theory and its possible limits of validity are not known.

D. Minkowski's Four-Space Symmetry and Einstein's Simultaneity Issue

To reconcile Maxwell's electromagnetic field and its waves with the Galilean and Newtonian kinematics of moving bodies Lorentz, Einstein and Poincaré developed the principles of special relativity. The essential physics of the reconciliation is expressed by the Lorentz transformation. This does not require the underlying space to be Euclidean, it is equally compatible with the other homogeneous three-spaces of the FRW metric, a hyperbolic or an elliptic three-space. The common impression that the Lorentz transformation necessarily implies the Minkowski metric and its four-space symmetry of (ict, x, y, z) is no longer true if position space is allowed to be homogeneously curved in analogy to relativistic velocity space. Instead that symmetry becomes a useful approximation valid only locally in a limited region.

Einstein especially emphasized the conversion of simultaneity from an absolute to a relative concept in the case of events observed by witnesses in relative motion. When these considerations were applied to the prescription for the center of mass in systems of two or more particles in relative motion it has become the accepted opinion that the center of mass is not a properly definable concept for a relativistic system. Were this to be true, the requirement of continuity would demand a clear prescription for treating the transition region between the clearly understood expressions of the low energy region to some other expressions in the relativistic limit. Nothing of this kind has ever been propounded. Center of mass effects are seen experimentally under relativistic conditions in innumerable cases, and they have proved to be correctly calculable through a well-defined and reasonable algorithm. The anomaly is in the lack of a clear theoretical justification for this algorithm.

In the expanding hyperbolic universe the singularity at the origin of the expansion provides an absolute origin for a cosmological proper time variable that can also be identified as the progress variable of dynamics. Its absolute origin synchronizes the proper times of all components of any physical system and provides the basis for a unified treatment of a generalized center of mass and momentum for any n-body system. The simultaneity issue then provides no impediment to the development of an n-body relativistic dynamics and a generalized Dirac equation within whose context the current treatment of center of mass effects can be justified and confirmed.

E. The Nature of the Solution

All these circumstances suggest that even prior to quantization, special relativity has not made an adequate adjustment at the border connecting dynamics and relativity, and that a new examination of that border region is warranted. Such a reexamination must now be carried out in the light of the cosmological information of the Hubble expansion. What is needed is an approach to special relativity that (a) preserves the relativistic recognition of velocity and momentum space as hyperbolic, with a velocity scale measured by the speed of light, (b) extends throughout the relativistic regime the Hamiltonian and quantal symmetries between coordinates and momenta, (c) carries into the relativistic regime the parallelism between mass centers in position space and momentum centers in velocity space that is essential for the proper treatment of n-body problems, and (d) is compatible with one of the best established overall properties of cosmological structure, the Hubble expansion from an apparent singularity at a determinable past time.

These requirements can all be satisfied. Symmetry can be restored if the structure of position space contains a magnitude of scale ρ comparable to c in velocity space. In a relativistic system the position vector $\mathbf{r}$ is conjugate not to the nonrelativistic velocity $\mathbf{v}$ but to the relativistic velocity variable

$$\mathbf{u} = \mathbf{p}/m = \mathbf{v}\left(1 - v^2/c^2\right)^{-1/2}. \tag{1}$$

To match the role of c in the space of $\mathbf{u}$, the universe presents us with an appropriate scale magnitude in the space of $\mathbf{r}$, the Hubble length $\rho(t) = cH^{-1}(t)$, the distance to the cosmological horizon. It also presents us with an absolute origin for a universal proper time t. These features can be introduced to form a symmetric special relativity with hyperbolic spaces of position $\mathbf{r}$ and relativistic velocity $\mathbf{u}$. Astrophysical evidence provides information on the expansion function $H(t)$ but does not enforce the common assumption that c is constant. The time dependence of expansion will therefore be taken as shared between an increasing length scale $\rho(t)$ and a possibly time-dependent speed of light $c(t)$, with the constraint that their ratio is the astrophysically determined Hubble expansion function

$$H(t) = c(t)/\rho(t). \tag{2}$$

The resulting theory depends only on the functions and parameters of existing theory, supplemented by the Hubble function $H(t)$ and especially by its value H_0 at the present time. It will be shown to be a property of the position-velocity symmetry of the theory, as expressed in a new hyperbolic Poincaré group of the resulting geometric and kinematic system, that the product

$$c(t)\rho(t) = \sigma = c_0^{\,2} H_0^{-1} \tag{3}$$

is constant, the Hubble-Lorentz constant.

It is a remarkable feature of the symmetric special relativity of a doubly hyperbolic expanding universe with a universal absolute proper time that its extension to an n-body system makes possible a new, symmetric solution to the center-of-mass problem as well. In the doubly hyperbolic system the two centers in position and velocity space are replaced by a single mass hypercenter in a six-dimensional configuration space spanned by the combined hyperbolic coordinates of position and velocity. The proper time of the universal expansion is the progress time of dynamics for all particles. Observable time intervals and lifetimes remain relativistically frame-dependent. The Lorentz and Poincaré groups of special relativity in Minkowski space are extended to the double Lorentz group, a twelve-parameter group applicable to the six-dimensional phase space of hyperbolic position and velocity, and then to the hyperbolic Poincaré group, a thirteen-parameter group applicable to the seven dimensional kinematic space of position, velocity and time. All the important dynamical properties of center-of-mass transformations can then be extended unimpaired throughout the relativistic regime. The explicitly covariant relativistic Hamiltonian, the relativistic Schrödinger equation and the covariant n-body Dirac equation all follow.

The Lorentz transformation and special relativity solved the compatibility problem for electromagnetism and particle kinematics for purely local interactions of particle and field. It left the same compatibility problem unsolved for spatially extended systems in the high-velocity regime. That broader compatibility problem now is solved for both classical and quantal systems by recognizing the time-dependent Hubble length as the standard magnitude of scale in an expanding, hyperbolic position space. This solution brings to light a previously unrecognized kinematic symmetry between the spaces of position and relativistic velocity, a symmetry intimately connected with the symplectic symmetry of Hamiltonian dynamics. This symmetry and the new view of relativity, dynamics and quantum mechanics associated with it imply far-reaching changes and opportunities for development in directions stretching from microphysics to cosmology.

II. Hyperbolic Space and the Kinematic Symmetry of Position and Velocity

Several specific issues can be identified in special relativity that require reexamination in the further transition to a new, kinematically symmetric form of special relativity. A treatment of the coordinate representations of relativistic space-time that displays the smooth connection between an expanding, hyperbolic FRW four-space and its spatially flat Minkowski limit provides new insights into the contrasting roles of a cosmic proper time connected with the expansion and the frame-dependent observer times appropriate to a Minkowski-space description. In addition, the hyperbolic-to-flat connection shows quantitatively how the renowned four-space symmetry of space and time in special relativity is broken in the presence of a time-dependent universal expansion.

When the hyperbolic structure of position space is combined with the known hyperbolic structure of relativistic velocity space the resulting symmetry makes possible a new, symmetric treatment of the centers of mass and momentum in special relativity. When the doubly hyperbolic kinematics is extended to the case of two or more bodies, the separate three-space conditions in position and velocity in Galilean kinematics combine to a set of six conditions specifying a single center in a six-dimensional configuration space of combined hyperbolic position and velocity.

A. Position-Velocity Symmetry in Special Relativity

Special relativity reconciles electromagnetic theory with the kinematics of moving bodies by introducing the scale magnitude c in velocity space. The imperfection of the Dirac equation in dealing with the mass-dependence of the spectrum of even a two-body system shows that this reconciliation is incomplete. An important feature of its incompleteness is that it does not deal correctly with the kinematics and dynamics of systems with two or more spatially separated moving parts—systems depending on spatial intervals r_{ij}. A prominent symptom of this defect is the failure of covariance of the center of mass in standard relativistic theory. Nonrelativistically the center of mass and momentum expressions show a position-velocity symmetry . That symmetry is lost in ordinary special relativity.

This defect is cured by using the Hubble length $\rho(t) = cH^{-1}(t)$ as the standard of magnitude in position space. The four-dimensional space-time can then be described by the variables of the Friedmann, Robertson, Walker (FRW) metric for an open, expanding hyperbolic universe in the form:

$$ds^2 = c^2 dt^2 - \rho^2(t)\left[d\eta^2 + \sinh^2\eta\left(d\theta_\eta{}^2 + \sin^2\theta_\eta d\phi_\eta{}^2\right)\right]. \qquad (4)$$

This describes an expanding hyperbolic three-space whose Gaussian curvature is $\kappa_{\mathrm{r}} = -\rho^{-2}(t)$. The local three-vector

$$\mathbf{r} = \rho(t)\sinh\eta\,\hat{\boldsymbol{\eta}}(\theta_\eta,\phi_\eta) = (r,\theta_\eta,\phi_\eta), \tag{5}$$

where $r = \rho(t)\sinh\eta$, describes the coordinates of a point in that three-space from a local origin $\mathbf{r} = 0$. The hyperbolic arc η will be called the "separation". The complete four-space can be described by a four-vector of position

$$X(t;\boldsymbol{\eta}) = \rho(t)\begin{pmatrix} \cosh\eta \\ \sinh\eta\begin{pmatrix} \cos\theta_\eta \\ \sin\theta_\eta\cos\phi_\eta \\ \sin\theta_\eta\sin\phi_\eta \end{pmatrix} \end{pmatrix} = \rho(t)\begin{pmatrix} \cosh\eta \\ \sinh\eta\,\hat{\boldsymbol{\eta}}(\theta_\eta,\phi_\eta) \end{pmatrix} = \begin{pmatrix} \rho(t)\cosh\eta \\ \mathbf{r} \end{pmatrix}. \tag{6}$$

The hyperbolic geometry in this four-space is best parametrized by the curvature length $\rho(t)$ and the dimensionless hyperbolic vector $\boldsymbol{\eta} = (\eta,\theta_\eta,\phi_\eta)$, the separation vector.

The symmetry between this position space and relativistic velocity space is shown by the isomorphism between the position vector $\mathbf{r}$ and the relativistic velocity vector $\mathbf{u}$,

$$\mathbf{u} = \mathbf{p}/m = \mathbf{v}\gamma = \mathbf{v}(1 - v^2/c^2)^{-1/2}. \tag{7}$$

The curvature of hyperbolic velocity space is $\kappa_\mathbf{u} = -c^{-2}$. The hyperbolic geometry of this velocity space is best parametrized by making use of the rapidity ε and the directional angles $\theta_\varepsilon,\phi_\varepsilon$ of the rapidity vector $\boldsymbol{\varepsilon} = (\varepsilon.\theta_\varepsilon,\phi_\varepsilon)$, so that the relativistic velocity is

$$\mathbf{u} = c\sinh\varepsilon\,\hat{\boldsymbol{\varepsilon}}(\theta_\varepsilon,\phi_\varepsilon) = (u,\theta_\varepsilon,\phi_\varepsilon), \tag{8}$$

where $u = c\sinh\varepsilon$.

To complete the formal symmetry between the spaces of $\mathbf{r}$ and $\mathbf{u}$, we shall now take the light velocity as time-dependent, $c(t)$, identifying its present value specifically as $c(t_0) = c_0$. The velocity four-vector is then

$$U(t;\boldsymbol{\varepsilon}) = c(t)\begin{pmatrix} \cosh\varepsilon \\ \sinh\varepsilon\begin{pmatrix} \cos\theta_\varepsilon \\ \sin\theta_\varepsilon\cos\phi_\varepsilon \\ \sin\theta_\varepsilon\sin\phi_\varepsilon \end{pmatrix} \end{pmatrix} = c(t)\begin{pmatrix} \cosh\varepsilon \\ \sinh\varepsilon\,\hat{\boldsymbol{\varepsilon}}(\theta_\varepsilon,\phi_\varepsilon) \end{pmatrix} = \begin{pmatrix} c(t)\cosh\varepsilon \\ \mathbf{u} \end{pmatrix}. \tag{9}$$

The complete structural isomorphism between relativistic velocity space as described by Eq. (9) and expanding hyperbolic position space as described by Eq. (6) is a fundamental property of kinematics in symmetric special relativity.

B. Lorentz Invariance in the Hyperbolic System

It is well known that the Lorentz invariance group of velocity boosts is isomorphic with the invariance group of a hyperbolic three-dimensional geometry. In special relativity this invariance is expressed in two separate manifestations. In the original Lorentz transformation the operators of velocity boosts operate on operands of a physical nature different from velocity itself, the four-vectors of position and time, and the velocity correction to the original position vector is subtractive. In Einstein velocity addition the velocity boosts operate on the vectors of velocity itself, and the velocity correction to the velocity operand is necessarily additive. It is useful to think of these two effects as different, but related, manifestations of the Lorentz group. Velocity addition generates a geometric Lorentz group describing properties of velocity space itself. The effect of a velocity boost on observations in space and time is a kinematic effect, and generates what can be called a kinematic representation of the Lorentz group. In the physical universe we can contemplate describing the entire kinematic and geometric structure of position and velocity with the help of a larger kinematic group, within which these two representations of the Lorentz group will appear as separate subgroups.

In the expanding universe with a hyperbolic position space, translations within the position space are generalized rotations and will generate still a third representation of the Lorentz group, a second geometric representation. The strict isomorphism between the hyperbolic geometries of position and velocity spaces in this universe is reflected in isomorphic structure of the position and velocity four-vectors of Eqs. (6) and (9). It is natural now to look for a second analogue of the kinematic Lorentz subgroup of velocity boosts acting on position and time. Symmetry suggests this should appear in the effect that a shift in position space may have on velocity. Such an effect is well known in the physical universe. It is the Hubble effect, in which a shift in position from the observer to a remote location results in an additive contribution to the observed velocity. The symmetries of this process can be described by a fourth Lorentz subgroup. These four subgroups are part of a larger group structure that describes the symmetries of kinematics and geometry in the relativistic expanding hyperbolic universe. Eqs. (4) to (6) illustrate an important feature of the expanding hyperbolic universe: the separability of its expansion time coordinate t, a universal proper time common to the entire three-space of position, from the natural curvilinear coordinates $\boldsymbol{\eta} = (\eta, \theta_{\boldsymbol{\eta}}, \phi_{\boldsymbol{\eta}})$ of the curved position space. In the hyperbolic geometry the entire effect of a Lorentz velocity boost operating on a four-vector of position and time takes place in the space of $\boldsymbol{\eta}$ and the expansion time t is invariant. In the Minkowski space of the usual special relativity the separable coordinate system $(t; \eta, \theta_{\boldsymbol{\eta}}, \phi_{\boldsymbol{\eta}})$ is not available, and the Minkowski four-space can be parametrized by either the coordinate system $(c\tau, x^1, x^2, x^3)$ or $(c\tau; r, \theta_{\boldsymbol{\eta}}, \phi_{\boldsymbol{\eta}})$, neither of which is separable.

C. Position-Velocity Symmetry and Hamiltonian Symmetry

The kinematic symmetry $(\mathbf{r} \leftrightarrow \mathbf{u})$ or $(\boldsymbol{\eta} \leftrightarrow \boldsymbol{\varepsilon}, \rho \leftrightarrow c)$ shows a generic relationship with the position-momentum balance $(\mathbf{r} \leftrightarrow \mathbf{p})$ of Hamilton's equations and the quantum commutators, but the mass factor in $\mathbf{p} = m\mathbf{u}$ on the right hand side impairs the identification between the kinematic position-velocity symmetry and the Hamiltonian symmetry in its most common form. However, we can take advantage of the invariance of the Hamiltonian system and the quantum commutation relations under canonical transformations to introduce a mass-weighted set of generalized coordinates and momenta

$$\overline{\mathbf{q}}_i = m_i^{1/2}\mathbf{r}_i, \quad \overline{\mathbf{p}}_i = m_i^{-1/2}\mathbf{p}_i = m_i^{1/2}\mathbf{u}_i \tag{10}$$

which preserve the reciprocating symmetry

$$\overline{\mathbf{p}} \to \overline{\mathbf{q}}, \quad \overline{\mathbf{q}} \to -\overline{\mathbf{p}} \tag{11}$$

of the Hamiltonian system while simultaneously expressing the position-velocity balance $(\mathbf{r} \leftrightarrow \mathbf{u})$ of kinematic symmetry. This new system of kinematic coordinates and momenta has the doubly balanced pattern

$$\begin{pmatrix} \mathbf{r} \leftrightarrow \mathbf{u} \\ \overline{\mathbf{q}} = m^{1/2}\mathbf{r} \leftrightarrow m^{1/2}\mathbf{u} = \overline{\mathbf{p}} \end{pmatrix}. \tag{12}$$

We can now follow Born's conjecture and use the reciprocating, symplectic symmetry of Eq. (11) in the form of Eq. (12) as a guide in developing in the physics of the border between nonrelativistic dynamics and relativity. We shall apply Born's rule of reciprocity by requiring that the symmetries of Eqs. (11) and (12) survive unimpaired in the relativistic domain.

The use of the privileged mass-weighted coordinates and momenta of Eq. (10) brings the coordinate-momentum balance of the Hamiltonian equations and the quantum commutators into agreement with the position-velocity symmetry of the Galilean physics and its mass and momentum centers, and extends that isomorphism into the domain of relativistic velocities. In addition, we can recognize another aspect of these symmetric patterns in a reciprocal relationship between the Lorentz effect of a velocity boost on measurements of length and time and the Hubble effect of a shift in position on measurements of velocity. This Hubble-Lorentz reciprocity will be presented later in this paper. It reveals a new example of the alternating sign in the reciprocity relationship of Eq. (11), but appearing now as a mass-independent relationship in a purely position-velocity context. This new appearance of the fundamental Hamiltonian reciprocity is a striking confirmation of the importance of what can now be called Born's reciprocity principle.

III. Hyperbolic Space-Time and its Flat Space Limit

A. The Spaces and their Symmetries

The generating principle of special relativity is the requirement of compatibility between the laws of kinematics of moving particles and the properties of electromagnetic fields and waves described by Maxwell's equations. Its primary consequences are expressed in the Lorentz transformation of intervals of space and time under a velocity boost and the demand for Lorentz invariance of dynamical laws. This requirement of electromagnetic and kinematic compatibility can be satisfied in homogeneous and isotropic position three-spaces by geometries that are flat, negatively curved (hyperbolic) or positively curved (spherical or elliptical).

It is the coupling of the requirement of Lorentz invariance with the assumption of a flat position space that leads to the metric condition of ordinary special relativity for a space-time interval,

$$ds^2 = c^2 d\tau^2 - d\mathbf{r}^2 . \qquad (13)$$

This form for the metric is the source of Minkowski's requirement of formal four-space symmetry in the variables $(ic\tau, x, y, z)$ of special relativity. In general relativity Einstein adopted the metric of Eq. (13) as the boundary condition at infinity for the gravitational field around an isolated mass concentration. Fock proposed instead using a hyperbolic space for this boundary condition [3]. In that case the Minkowski-space metric of Eq. (13) is replaced, in the simplest approximation, by the metric of the time-dependent spatially hyperbolic space-time of Friedmann, Robertson and Walker, Eq. (4). In the global coordinates $(t, \eta, \theta_\eta, \phi_\eta)$ of Eq. (4) we can describe a point in space-time by the position four-vector of Eq. (6).

The time-dependent curvature length $\rho(t)$ in Eqs. (4) and (6) is connected with the Hubble expansion function

$$\rho(t) = cH^{-1}(t) . \qquad (14)$$

I shall assume that in the neighborhood of the present time t_0 this function is adequately represented by the linear approximation

$$\rho(t_0 + \delta t) = cH_0^{-1} + c\delta t . \qquad (15)$$

The present state of cosmological knowledge suggests a universe whose expansion is open and whose curvature is negative but exceedingly close to flat on the average. At the same time, the presence of gravitating masses insures that it is essentially nowhere locally truly flat. It is a standard result of gravitational theory that the cosmological density parameter $\Omega = 8\pi G d / 3H_0^2$, where G is the gravitational constant and d is the average density of matter, must exceed unity for the average curvature to be positive and the universe to be closed. A close to zero average curvature implies that positive curvature near massive matter in parts of the universe like our own must be compensated by negative curvature in the vast voids that are also seen in maps of the visible universe. This requires adopting Fock's boundary condition and the FRW metric of Eq. (4) in the gravitation-free limit. That is therefore an appropriate metric for an alternative version of special relativity. The hyperbolic geometry of position space which it implies will be shown to symmetrize the center of mass problem and provide it with a covariant solution.

Adopting the metric of Eq. (4) instead of Eq. (13) has the consequence that the familiar four-space symmetry of the variables $(ic\tau, x, y, z)$ of Minkowski space loses its universality and becomes a local and approximate symmetry appropriate to a region of space and time where $|\Delta r| / \rho(t_0) \equiv |\alpha_s| \cong |\Delta r| / c_0 H_0^{-1} << 1$ and $c_0 |\Delta t| / \rho(t_0) \equiv |\alpha_t| \cong |\Delta t| / H_0^{-1} << 1$. For local physics and microphysics this is not a significant limitation. For cosmological distances and times, or for issues of principle like the simultaneity issue, it becomes important to recognize that the Minkowski four-space symmetry of space and time is in fact not a requirement of relativity, but rather a convenient approximate rule, valid only locally, like two-dimensional Euclidean symmetry in local surveying on the earth's surface.

B. Hyperbolic and Minkowski Coordinates in Expanding Space-Time

1. The General Case

The expanding hyperbolic description of space-time in terms of the global coordinates $(\rho[t]; \eta, \theta_\eta, \phi_\eta)$, with their velocity-space conjugates $(c[t]; \varepsilon, \theta_\varepsilon, \phi_\varepsilon)$, provides the representation of choice to deal with general symmetry issues and questions of covariance. Depending as they do on a cosmologically remote origin in time these global coordinates are altogether unsuited for describing local or microscopic processes, where well-defined scales of length and time are needed, together with a local origin in time. For this purpose we need to convert to a local Galilean and Minkowski system $(\tau; \mathbf{r})$ of time and space coordinates suited to a region where the global parameters ρ and t are very large and imprecisely known. This is the requisite coordinate system with which to confront experimental reality on the local scale.

The important coordinate transformation between hyperbolic and Minkowski coordinates is a two-dimensional one, independent of the angular variables (θ_η, ϕ_η). It will be developed here for the simple case where all velocities are nonrelativistic. Assuming the relationship $\rho(t)$ is defined, the hyperbolic-to-Minkowski connection can be established in two steps:

Step 1: At any value of the expansion time t we define the new coordinates

$r = \rho(t)\sinh\eta, \qquad w = \rho(t)\cosh\eta.$(16)

This provides us with a length variable r which remains finite even in the limit $t \to \infty, \rho \to \infty, \eta \to 0$. Its companion variable w is the zeroth, timelike, component of a four-vector $X = (w,\mathbf{r}) = (w,x,y,z)$. The inverse connection is

$\rho^2(t) = w^2 - r^2, \qquad \tanh\eta = r/w.$(17)

In Step 2 the Minkowski time variable τ, a local time, is introduced by a simple translational shift in the variable w to a local origin $w_0 = \rho(t_0)$:

$$X^0 = w = \rho_0 + c_0\tau. \tag{18}$$

Eqs. (16) and (18) can be combined to give

$$\rho^2(t) = (\rho_0 + c_0\tau)^2 - r^2 \text{ and} \tag{19}$$

$$\tanh\eta = r/(\rho_0 + c_0\tau). \tag{20}$$

Both the global variable sets $(\rho[t];\boldsymbol{\eta})$ or $(w;\mathbf{r})$ and the purely local set $(\tau;\mathbf{r})$ are available to identify a four-space point in the expanding hyperbolic universe, and the descriptions are readily interconverted.

It is the practical need for a local origin of time in Galilean and Minkowski kinematics that makes it impossible to construct a global position four-vector like that of Eq. (6) in ordinary special relativity. The hyperbolic metric is compatible with the existence of the global position four-vector as well as the velocity four-vector, but the Minkowski one is not. The local construction does permit the covariant existence of the differential four-vector $dx^\iota = (cd\tau, d\mathbf{r})$ and the local validity of the four-space Minkowski symmetry associated with it. Globally in the wider hyperbolic universe that symmetry is no longer valid. Covariance symmetry under velocity boosts remains nonetheless. It can be expressed by a boost through a rapidity change $\Delta\boldsymbol{\varepsilon}$ or the equivalent velocity change $\Delta U(c,\Delta\boldsymbol{\varepsilon})$, carried out by the operation of the usual Lorentz matrix on position four-vector $X(\rho(t),\boldsymbol{\eta})$. This is supplemented in the hyperbolic geometry by a similar covariance under translations $\Delta\eta$ in hyperbolic position space, the space of the separation variables η.

2. The Effect of a Time-Varying Light Speed

The relationship between the hyperbolic and the Minkowski coordinate descriptions of four-space will depend on the assumption made about the time-dependence of the light speed. We consider two cases, the usual special relativity with constant $c = c_0$, and symmetric special relativity with a cosmologically time-varying light speed $c = c(t)$ subject to the symmetry condition of Eq. (3). To demonstrate the characteristic features of the situation it will suffice to assume the simplest plausible form for the Hubble expansion function. It is convenient to work with the reciprocal of the Hubble function, which has the dimensions of time. It will be denoted $h(t) \equiv H^{-1}(t)$. At least over a very large region in the neighborhood of the present time t_0 this relationship can be excellently approximated as linear one in the expansion time, and we can assume it has the form $H^{-1}(t) = h(t) = t$.

In case 1, ordinary special relativity, with constant light speed, the magnitude of the position four-vector $|X| = \rho(t) = c_0\left(H_0^{-1} + \delta t\right)$, and of its timelike component in Minkowski variables, Eq. (18), will be the same whenever the magnitude of the spatial components $\left|X^{1,2,3}\right| = \rho(t)\sinh\eta$ are negligible in comparison to X^0, i.e. whenever $\sinh\eta = r / c_0 H_0^{-1} << 1$. This condition is always satisfied on the human and even galactic scale, and the identity in magnitude of δt and τ can be relied upon in ordinary special relativity except at cosmological distances or under conditions involving relativistic velocity differences between one part of a system and another.

The case of symmetric special relativity, with the light speed and curvature length obeying Eqs. (2) and (3), brings in new and important features. It will be labeled Case II. In it the curvature length and light velocity take the form

$$\rho_{\text{II}}(t) = \sigma^{1/2} h^{1/2}(t) = \sigma^{1/2} t^{1/2}, \qquad c_{\text{II}}(t) = \sigma^{1/2} h^{-1/2}(t) = \sigma^{1/2} t^{-1/2}, \tag{21}$$

with $\sigma = c_0^{\,2} H_0^{\,-1}$. The left hand side of Eq. (17) is then $\rho_{\text{II}}^{\,2}(t) = \sigma h(t)$. Completing Eq. (17), we have

$$\rho_{\mathrm{II}}{}^{2}(t) = \sigma h(t) = \sigma t = w_{\mathrm{II}}{}^{2} - r^{2}. \tag{22}$$

It is now convenient to use $h = h(t) = H^{-1}(t)$ as a measure of the expansion time itself. Then

$$\rho_{\mathrm{II}}(t) = \sigma^{1/2} h^{1/2}, \quad c_{\mathrm{II}}(t) = \sigma^{1/2} h^{-1/2}, (23)$$

$$w_{\mathrm{II}} = \sigma^{1/2} h^{1/2} \cosh\eta = c_0 h_0{}^{1/2} h^{1/2} \cosh\eta. \tag{24}$$

We can introduce the Minkowski time τ near its local zero at t_0 by writing

$$w_{\mathrm{II}} = \rho_0 + a_0 c_0 \tau = c_0 \left(h_0 + a_0 \tau\right), \tag{25}$$

where a_0 is a coefficient which to be determined by the requirement that a time interval measured as Δh (or Δt) in expansion time and as $\Delta\tau$ in the local Minkowski time should be the identical in the neighborhood of t_0. This is equivalent to the condition on the derivatives

$$\left(\partial w / \partial h\right)_{t_0} = \left(\partial w / \partial \tau\right)_{t_0}. \tag{26}$$

Applying this to the expressions in Eq. (24) and (25) we find

$$a_0 = \frac{\cosh\eta}{2}. \tag{27}$$

We now combine Eqs. (24) and (25) using this value of a_0:

$$h_0{}^{1/2} h^{1/2} \cosh\eta = w_{\mathrm{II}} / c_0 = h_0 + \frac{\tau \cosh\eta}{2}. \tag{28}$$

Squaring both sides, we can put this into the alternative form

$$h = h_0 \left(\operatorname{sech}\eta + \frac{\tau}{2h_0} \right)^2. \tag{29}$$

We can now reexamine the two-parameter connection between the global coordinate system (h,η), which we can use as the equivalent of (t,η), and the local Galilean or Minkowski coordinate system (τ,r). The connection depends explicitly on the parameters (c_0, h_0). The equations are

$$\tau / h_0 = 2(h / h_0)^{1/2} - 2\,\mathrm{sech}\,\eta, \tag{30}$$

$$r / c_0 h_0 = (h / h_0)^{1/2} \sinh\eta. \tag{31}$$

In Eqs. (29) and (30), we note that $h = h(t) \approx t$ measures the expansion time from the remote Hubble singularity, while τ measures the Galilean and Minkowski time from a recent and local origin at $(h = h_0 = H_0^{-1}, \eta = 0)$. The transformation equations (30) and (31) are applicable unrestrictedly throughout space-time.

If we now write $h = h_0 + \delta t$ Eq. (29) becomes

$$\delta t = \tau + \left[\tau^2 / 4h_0 + h_0 \tanh^2\eta + (\mathrm{sech}\,\eta - 1)\tau\right]. \tag{32}$$

We can also express this with an explicit display of its dependence on the distance r instead of the separation η:

$$\tau = 2h_0\left[\left(1 + \delta t / h_0\right)^{1/2} - \left(1 + r^2 / c_0^2 h_0 h\right)^{-1/2}\right] = \delta t - \left[\delta t^2 / 4h_0 + r^2 / c_0^2 h_0 h \ldots\right] \tag{33}$$

Each of the bracketed terms on the right hand side of Eq. (32) and in the final form of Eq. (33) contains one of the two strong convergence factors $\alpha_{\text{space}} = r / cH_0^{-1}$ or $\alpha_{\text{time}} = \tau / H_0^{-1}$, and is fully negligible except at cosmological distances or times. This confirms that both measures of the flow of time, the cosmological time increment δt and the Galilean and Minkowski time τ, are identical in a very wide local region around the present time $\tau = 0$ and location $r \approx 0$, but they diverge elsewhere in regions cosmologically remote in either space or time.

The spatially local form of the above equations is especially simple. It prevails over a wide range of noncosmological distances where $|r| << c_0 H_0^{-1}$, in which case η in Eq. (32) essentially vanishes. In that case Eq. (32) becomes the quadratic equation

$$\delta t = \tau + \tau^2 / 4h_0 = \tau + \tau^2 / 4H_0^{-1}. \tag{34}$$

To avoid a spurious root, one may use Eq. (30) with $\eta = 0$ or (33) with $r = 0$.

For local measurements at times scales of human lifetimes or less, the quadratic term in this equation is negligible, and both time scales are equivalent, but for remote times past or future in the scale of terrestrial and cosmic history the divergence of the scales of t and of τ is important. The Minkowski time variable τ can thus be taken as measuring the time of an event in the past or future of the local point of observation—the historical time, while the time t is an astrophysically measured expansion time—the cosmological time.

In particular, the Hubble time looking back from the present to the beginning of the cosmic expansion, when $h = 0$, is different by a factor of two in the cosmological and the local historical time scales:

$$\delta t_{\text{Hubble}} = -H_0^{-1}, \qquad \tau_{\text{Hubble}} = -2H_0^{-1}. \tag{35}$$

This is not surprising if we remember that the cosmological time δt is measured by observing a signal which has been propagating through a medium of varying light speed, while the historical time to some past event represents a time interval measured locally and in the same velocity frame. We can describe the cosmological effect on long distance signal transmission as a foreshortening of the time.

The availability in the historical time scale of a time interval since the Big Bang twice as long as the cosmologically measured Hubble time for the evolution of stellar and galactic objects is of some significance in view of the well known astrophysical observations of stars of extraordinarily great age compared with the Hubble time.

C. The Roles of Time in an Open Expanding Universe

An open expanding universe is topologically equivalent to the expanding hyperbolic universe described most naturally by the coordinates $(\rho[t], \eta, \theta_\eta, \phi_\eta)$, but the extreme remoteness of the expansion origin of the time coordinate t and the imprecision of our knowledge of it and of the associated length measure $\rho(t)$ makes these coordinates impractical for the description of physical processes on a local, terrestrial or microscopic scale. For that purpose we must use the local coordinates $(\tau, \mathbf{r})$ of Galilean and special relativistic physics. As the development in the previous subsection has shown, the two time variables t and τ of the expanding hyperbolic universe represent distinct physical concepts: (a) the universal proper time t of the expanding length scale $\rho(t)$, and (b) the projection of t or $\rho(t)$ represented by the variable w and its increment $c\tau = w - w_0$ in the temporal direction of a four-dimensional space-time.

The time t is now a universal proper time, measured from the start of the cosmic expansion, an invariant under both velocity translations and displacements in hyperbolic position. Both t and $\rho(t)$ are invariants of the four-vector X. This proper time coordinate is the same for all particles of a system, and is the progress time of dynamics. Its synchronization follows from its absolute origin in the Hubble singularity. Multiple times are not required for dynamics, even in the description of n-particle systems. The time t of dynamics and of the universal expansion can be identified as "absolute proper time", "cosmological time" or "dynamical time".

The interval time or historical time τ, the time component of Minkowski four-space through the temporal variable $w(t,\eta)/c(t) = H_0^{-1} + \tau$ of which it is a part, represents only one frame-dependent component of a four-vector or tensor $X(t,\boldsymbol{\eta})$. Velocity boosts or translations in position will mix the temporal component w and the spatial components $\mathbf{r}$ of X, and thereby modify τ, but they leave t and δt unchanged. There is thus a profound physical difference between the cosmic time t or its proper time increment δt and the temporal component time w/c or its Minkowski time τ.

Special relativity has always contained two separate forms of time: proper time, an invariant property of a particle or a system of particles in its own home velocity frame, a scalar; and observer or clock time, a frame-dependent observable, the fourth component of covariant four-vectors of position. In symmetric special relativity this distinction becomes sharper and more emphatic, because the singularity at the beginning of the Hubble expansion is a unique origin common to the proper time of every point in the geometry. We now encounter on one hand the universal time t, a scalar invariant common to all particles, and on the other the observer- and particle-dependent time of clocks, τ_i or τ_{ij}, the time of intervals between events, of time delays and lifetimes. This interval time appears as an increment of the fourth component $w_i = w_{i,0} + c\tau_i$ or $w_{ij} = w_{ij,0} + c\tau_{ij}$ of a covariant position four-vector X_i or X_{ij} that is the peculiar property of a particular particle or object i or interval ij. In an n-body system it is these particle or subsystem local times that are multiple. Their variation with changes in the velocity frame in respect to which they are evaluated is a familiar relativistic property.

When position space is not flat their relationship with the universal proper time also depends on the distance between the events by which they are being measured, or the distance between an event and the observer. These local times always make their appearance as increments $\delta w_i / c = \tau_i$ or $\delta w_{ij} / c = \tau_{ij}$ of the time-like component of a four-vector or a four-by-four tensor of position. They can be called "local times", "clock times" or "observer times".

The development of the preceding section shows how this Minkowski variable can be extended smoothly throughout the whole range of time from the Big Bang to the indefinite future.

The time dimension of a four-space with an expanding hyperbolic position space can be envisaged as the analogue of the radial dimension in a spherical coordinate system. Its time coordinate is orthogonal to the hyperbolic spatial coordinates, and has its origin at the singularity of the expansion. It is an absolute proper time, and it is the same for all points in the hyperbolic three-space provided they are all stationary in the same velocity frame. When Hubble and Lorentz corrections are properly made, all points, moving or not, can be ascribed a unique absolute proper time.

The Minkowski limit can be brought into this picture by noting that it is reached when the hyperbolic pseudosphere expands so that a local region of it is carried to the infinite limit of a flat space. The local time coordinate is the height coordinate of a cylindrical system orthogonal to the hyperbolic three-space. At the same time the absolute origin of the height coordinate recedes to minus infinity, and only a local time with a nearby origin is definable.

These issues are clarified by the connection between the expanding hyperbolic coordinate system and the Minkowski one given in Eqs. (28) to (34). The time and separation variables $(t, \boldsymbol{\eta})$ of the hyperbolic world become impractical to use in a small local system and must be replaced by the variables $(\tau, \mathbf{r})$ that are valid in both the relativistic Minkowski world and the Galilean-Newtonian world. In the hyperbolic world t is an invariant, has an absolute origin in the remote past, is synchronized, and is common to all particles; particle positions and separations are expressed by the covariant variables $\boldsymbol{\eta}_i$ and $\boldsymbol{\eta}_{ij}$ orthogonal to t. In a local system neither t nor $\boldsymbol{\eta}$ are practically usable coordinates and the non-covariant variables τ and $\mathbf{r}$ are required. Locally, nonetheless, the set of variables $(c\tau, \mathbf{r})$ can be treated conditionally as a Minkowski four-vector in a region where corrections of the order $\alpha_r = \left(r / c_0 H_0^{-1}\right)$ and $\alpha_\tau = H_0 \tau$ are negligible. These relationships are completed by the development of the connection between the cosmological time t or δt and the historical time variable τ that is applicable over the entire span of time from the Big Bang to the present and into the far future.

D. The Effect of Spatial Curvature in the Hyperbolic-to-Minkowski Transition

It is informative to focus some attention on the transition regions in time and in space in which the practical identity between the cosmological and local time scales of δt and of τ gives way, first to the entry of the quadratic term in the time equation, (34), and then to spatial curvature terms.

Minkowski space-time can be thought of as the limit of expanding hyperbolic space-time in a region near the axis $\eta = 0$ when $\alpha_s = r / cH_0^{-1} \rightarrow 0$. This limit has the important property that r, τ and δt as well as the angles $(\theta_{\boldsymbol{\eta}}, \phi_{\boldsymbol{\eta}}) = (\theta_{\mathbf{r}}, \phi_{\mathbf{r}})$ remain well-behaved variables while $\eta = \sinh^{-1}(r / \rho[t_0 + \delta t]) \rightarrow 0$ vanishes in a singularity, and t_0 and $\rho(t_0)$ increase without limit.

The cosmic time t and its increment δt are invariants of the four-vector X, whereas τ, through the temporal variable $w / c = t_0 + \tau$ of which it is a part, represents only one frame-dependent component of X. Velocity boosts will mix the temporal component w and the spatial components $\mathbf{r}$ of X, and thereby modify τ, but they leave t and δt unchanged. There is thus a profound physical difference between the cosmic time t or its proper time increment δt and the temporal component time w / c or its Minkowski time increment τ. This physical distinction remains important even in the zero velocity frame, where the difference is

$$c\tau - c\delta t = \rho(t)(\cosh \eta - 1), \tag{36}$$

a quantity that vanishes only at the spatial origin $\mathbf{r} = 0, \boldsymbol{\eta} = 0$. It is important in principle, though exceedingly small in local applications. It is significant in magnitude at cosmological distances.

The Minkowski coordinates $(\tau, \mathbf{r}) = (\tau, r, \theta_{\mathbf{r}}, \phi_{\mathbf{r}})$ or (τ, x, y, z), measured from a local origin in space and time, are the natural tools for the description of systems on a terrestrial scale. The expanding hyperbolic coordinates $(t,\boldsymbol{\eta})$ are cosmologically based. They provide the framework for understanding the global symmetries and their consequences in a universe that is likely to be topologically close to the universe of our experience, but they are not useful for the detailed description of processes in a limited, local region of space and time. For that purpose the most appropriate description is through the familiar local time and space coordinates $(\tau,\mathbf{r})$; they are valid and complete to describe both nonrelativistic space and time and the flat Minkowski space-time of ordinary special relativity. They are equally applicable in the expanding hyperbolic universe provided they are being used in a region spatially small compared to the Hubble length and over a time period small compared to the Hubble time, but they need to be supplemented by additional parameters to specify the hyperbolic curvature and the time dependence of its expansion.

The transformations leading to Eqs. such as (32) and (33) make it possible to transform in either direction between the FRW hyperbolic coordinate system $(t,\eta,\theta_{\eta},\phi_{\eta})$ or $(\rho[t],\eta,\theta_{\eta},\phi_{\eta})$ and a Minkowski coordinate system $(c\tau,\mathbf{r})=(c\tau,x,y,z)$. This transformation requires the specification of additional parameters beyond those of the Minkowski coordinate system. We can choose these to be a local origin of incremental time $t_0 \cong H_0^{-1}$ and the curvature length at that time $\rho(t_0) \cong cH_0^{-1}$. We can now augment the zero-order translation between hyperbolic and Minkowski coordinate systems by an expansion making use of these constants. Because practical measurements will be made in the coordinates $(\tau,\mathbf{r})$ it will be most convenient to express the correction terms through the dimensionless coefficients $\alpha_r = r/\rho(t_0) \cong r/cH_0^{-1}$ and $\alpha_\tau = c\tau/\rho(t_0) \cong \tau/H_0^{-1}$ or $\alpha_{\delta t} = c\delta t/\rho(t_0) \cong \delta t/H_0^{-1}$.

The correction terms we seek can be identified with the expression of Eq. (28). It can be transformed by using the expansion

$$\rho(t)\cosh\eta = \rho(t)\left[1+\sinh^2\eta\right]^{1/2} = \left[\rho^2(t)+r^2\right]^{1/2} = \rho(t)\left[1+\frac{r^2}{2\rho^2(t)}-\frac{r^4}{8\rho^4(t)}\cdots\right] \quad (37)$$

In the neighborhood of $t_0 \cong H_0^{-1}$ the time-dependent Hubble length can be written

$$\rho(t) = \rho_0 + c\delta t + O\left(c^2\delta t^2/\rho_0\right) \cong cH_0^{-1} + c\delta t + O\left(cH_0\delta t^2\right). \quad (38)$$

The three spatial coordinates $\mathbf{r} = (r, \theta_\eta, \phi_\eta)$ are unaffected by this change. Their timelike companion w can now be measured from its value $w_0 = w(t_0, \eta_0) = w(t_0, 0) = \rho_0 \cong cH_0^{-1}$ at the local space-time origin $(t = t_0, r = 0)$ and expressed as a function of r and δt,

$$\delta w = c\tau = w - \rho_0 = \left[(\rho_0 + c\delta t)^2 + r^2\right]^{1/2} - \rho_0 = c\delta t + g(\rho_0, r, \delta t). \tag{39}$$

The function

$$\begin{aligned} g(\rho_0, r, \delta t) = c(\tau - \delta t) &= \frac{r^2}{2\rho_0} - \frac{r^2 c\delta t}{2\rho_0^{\,2}} - \frac{r^4 - 4r^2c^2\delta t^2}{8\rho_0^{\,3}} \cdots. \\ &= \rho_0\left[\frac{\alpha_r^{\,2}}{2} - \frac{\alpha_r^{\,2}\alpha_{\delta t}}{2} + \frac{\alpha_r^{\,2}\alpha_{\delta t}^{\,2}}{2} - \frac{\alpha_r^{\,4}}{8} \cdots\right] \end{aligned} \tag{40}$$

is a curvature correction that depends on the poorly known Hubble length ρ_0 only through terms in powers of the very small quantities $\alpha_r^{\,2} = (r / \rho_0)^2$ and $\alpha_{\delta t} = c_0\delta t / \rho_0$. It gives a measure of the error introduced by using the flat space approximation, and is ignorable in practical calculations except at cosmological distances and times.

Practical measurements are made in terms of τ, the local time variable of common experience, and not the incremental proper time δt. The curvature correction function g of Eq. (40) can then be expanded as a function of the observable local time τ, through the ratio $\alpha_\tau = \tau / H_0^{-1}$ instead of $\alpha_{\delta t} = \delta t / H_0^{-1}$; the difference appears only in the fourth-order correction:

$$c_0(\tau - \delta t) = \bar{g}(\rho_0, r, \tau) = \rho_0\left[\frac{\alpha_r^{\,2}}{2} - \frac{\alpha_r^{\,2}\alpha_\tau}{2} + \frac{\alpha_r^{\,2}\alpha_\tau^{\,2}}{2} + \frac{\alpha_r^{\,4}}{8} \cdots\right]. \tag{41}$$

In a hyperbolic expanding space-time we can now establish a tangential flat Minkowski space-time with the coordinates

$$\begin{aligned} &x^0 = X^0 - R_{\mathrm{H},0} = \delta w = c_0\tau = c_0\delta t + g(R_{\mathrm{H},0}, r, \tau), \\ &(x^i) = (X^i) = \mathbf{r}. \end{aligned} \tag{42}$$

The flat three-space of this Minkowski universe is tangent to the hyperbolic three-space at the local origin $r = 0$. The four components of the global four-vector X^κ have the covariance properties of a column four-vector in the hyperbolic four-space, but the Minkowski array x^κ does not. The differential of the Minkowski array, $dx^\kappa = dX^\kappa$, does not suffer from this defect, and can be treated as a four-vector.

For local applications, the only coordinates that will be needed in practice will be expressed as differences or differentials, describing an interval:

$$\begin{pmatrix} \Delta x_{12}{}^{0} \\ \left(\Delta x_{12}{}^{i}\right) \end{pmatrix} = \begin{pmatrix} \Delta X_{12}{}^{0} \\ \left(\Delta X_{12}{}^{i}\right) \end{pmatrix} = \begin{pmatrix} \Delta w_{12} \\ \Delta \mathbf{r}_{12} \end{pmatrix} = \begin{pmatrix} c_0 \Delta t_{12} + \Delta g_{12} \\ \Delta \mathbf{r}_{12} \end{pmatrix}. \tag{43}$$

Here, as above, the correction term Δg_{12} is ordinarily negligible except in the case of cosmological magnitudes.

E. The Independent Roles of Space and Proper Time

In the doubly hyperbolic universe one of the most profound physical symmetries is that between the vector three-spaces of position and velocity, with the one-dimensional manifold of universal proper time orthogonal to them all and invariant both to boosts in velocity space and to translational shifts in position space. A signature of the deep physical independence of this cosmic time variable is the difference in sign between the quadratic temporal and spatial terms in the metric expressions of Eqs. (4) and (13). Spatial intervals $\delta\mathbf{r}$ and temporal intervals $\delta\tau$ are always separated from one another by a rotation through $\pi/2$ in the complex plane. It is useful to adopt a notation that emphasizes this important distinction between the spatial and the time coordinates. For this reason it is convenient to employ the Minkowski real-and-imaginary notation for four-vectors and tensors. Position and velocity four-vectors can therefore be written in the form

$$X(t,\boldsymbol{\eta}) = \rho(t) \begin{pmatrix} \cosh\eta \\ i\sinh\eta\,\hat{\boldsymbol{\eta}}\left(\theta_{\eta},\phi_{\eta}\right) \end{pmatrix} = \begin{pmatrix} \rho(t)\cosh\eta \\ i\mathbf{r} \end{pmatrix} \text{ and} \tag{44}$$

$$U(t,\boldsymbol{\varepsilon}) = c(t) \begin{pmatrix} \cosh\varepsilon \\ i\sinh\varepsilon\,\hat{\boldsymbol{\varepsilon}}\left(\theta_{\varepsilon},\phi_{\varepsilon}\right) \end{pmatrix} = \begin{pmatrix} c(t)\cosh\varepsilon \\ i\mathbf{u} \end{pmatrix}. \tag{45}$$

IV. The Vectors of Position and Velocity in Hyperbolic Kinematics

The group theory of the geometry of a hyperbolic three-space is just that of the six-parameter Lorentz group. The addition of vectors in that geometry can be recognized as one of the primitive group operations. In preparation for studying kinematics in the pair of interrelated hyperbolic geometries of the spaces of position and velocity it will be useful first to review the case of a single hyperbolic geometry and establish the principles in a notation that can be carried over to the case of the paired curved geometries. The relationship between the intrinsic six-parameter structure of the Lorentz matrix and the three-parameter description of the location of a point in a hyperbolic three-space, as expressed in Eq. (6) above, exhibits important features of the geometry of a curved space that are not present in flat space. An appropriate formulation including these effects will provide an essential foundation for the extension to the doubly curved system.

A. Translations and their Lorentz Parameters

The symmetry group of translations and rotations in a hyperbolic three-space is just the six-parameter Lorentz group. The Lorentz matrices realizing this group can be expressed in a standard form (see, for instance, reference [4]). Its parameters fall into two sets of three, a vector $\boldsymbol{\eta} = (\eta, \theta_\eta, \phi_\eta)$ describing a pure hyperbolic translation through the geodesic arc η in the direction $\hat{\boldsymbol{\eta}}(\theta_\eta, \phi_\eta)$ and a rotational pseudovector $\boldsymbol{\omega} = (\omega, \kappa_\omega, \lambda_\omega)$ through the angle ω around the direction $\hat{\boldsymbol{\omega}}(\kappa_\omega, \lambda_\omega)$. We first define the displacement matrix $K(\boldsymbol{\eta})$ of a pure hyperbolic translation,

$$L(\boldsymbol{\eta}, 0) = K(\boldsymbol{\eta}) = \begin{pmatrix} \cosh\eta & -i\sinh\eta\,\hat{\boldsymbol{\eta}}^{\mathrm{T}}(\theta_\eta, \phi_\eta) \\ i\sinh\eta\,\hat{\boldsymbol{\eta}}(\theta_\eta, \phi_\eta) & \mathbf{I}_3 + (\cosh\eta - 1)\hat{\boldsymbol{\eta}}\hat{\boldsymbol{\eta}}^{\mathrm{T}} \end{pmatrix}, \tag{46}$$

and the Lorentz rotation matrix $R(\boldsymbol{\omega})$ of a pure three-space rotation

$$L(0, \boldsymbol{\omega}) = R(\boldsymbol{\omega}) = \begin{pmatrix} 1 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & \mathbf{R}_3(\boldsymbol{\omega}) \end{pmatrix}. \tag{47}$$

The matrices $K(\boldsymbol{\eta})$ and $R(\boldsymbol{\omega})$ have the useful property that

$$K^{-1}(\boldsymbol{\eta}) = K(-\boldsymbol{\eta}), \qquad R^{-1}(\boldsymbol{\omega}) = R(-\boldsymbol{\omega}). \tag{48}$$

The general Lorentz matrix $L(\boldsymbol{\eta},\boldsymbol{\omega})$ can be defined as the product

$$L(\boldsymbol{\eta},\boldsymbol{\omega}) := K(\boldsymbol{\eta})R(\boldsymbol{\omega}) = L(\boldsymbol{\eta},\mathbf{0})L(\mathbf{0},\boldsymbol{\omega}). \tag{49}$$

Because of noncommutativity, the order of the product must be respected.

The variables $\boldsymbol{\eta} = (\eta, \theta_{\boldsymbol{\eta}}, \phi_{\boldsymbol{\eta}})$ in the displacement matrix $K(\boldsymbol{\eta})$ are those of the position four-vector of Eq. (6). When that four-vector $X(t,\boldsymbol{\eta})$ is normalized it can be identified with the leading column (and row) of the displacement matrix or of the corresponding irrotational Lorentz matrix:

$$X^{\mu}(t,\boldsymbol{\eta}) / \rho(t) = K^{\mu}{}_{0}(\boldsymbol{\eta}) = L^{\mu}{}_{0}(\boldsymbol{\eta},\mathbf{0}). \tag{50}$$

The location of a point in hyperbolic space with respect to a local origin at $\boldsymbol{\eta}_0 = \mathbf{0}$ can be expressed either by the vector $\boldsymbol{\eta}$ or by the displacement matrix $K(\boldsymbol{\eta})$, i.e. by the irrotational Lorentz matrix $L(\boldsymbol{\eta},\mathbf{0})$.

The addition of vectors as the sum of sides in a triangle remains applicable in hyperbolic geometry, where it is carried out by the rules of hyperbolic trigonometry. Hyperbolic vector addition will be identified here by the special summation notation $\hat{+}$. In the triangle

$$\boldsymbol{\eta}_1 \hat{+} \boldsymbol{\eta}_2 \hat{+} \boldsymbol{\eta}_3 = 0 \tag{51}$$

the addition of sides follows the hyperbolic law of cosines

$$\cosh\eta_3 = \cosh\eta_1 \cosh\eta_2 + \sinh\eta_1 \sinh\eta_2 \cos\angle(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2). \tag{52}$$

This can be supplemented by the hyperbolic law of sines to obtain the necessary angles. In velocity space this vector addition in a hyperbolic triangle reproduces the results of Einstein vector addition of velocities.

B. The Rotations of Parallel Transport

As in spherical trigonometry, a sequence of non-collinear translations induces a rotation. In the product of Lorentz matrices

$$K(\boldsymbol{\eta}_1)K(\boldsymbol{\eta}_2) = L(\boldsymbol{\eta}_1,\mathbf{0})L(\boldsymbol{\eta}_2,\mathbf{0}) = L(\boldsymbol{\eta}_3,\boldsymbol{\omega}_{123}) = K(\boldsymbol{\eta}_3)R(\boldsymbol{\omega}_{123}) \tag{53}$$

the second order rotation $\boldsymbol{\omega}_{123}$ adjusts for the angular defect of the hyperbolic triangle, measured by its directed area. This area can be expressed as a hyperbolic vector product, for which we can use the symbol "$\hat{\times}$":

$$\boldsymbol{\omega}_{123} = [\boldsymbol{\eta}_1 \hat{\times} \boldsymbol{\eta}_2]/2 = [\boldsymbol{\eta}_2 \hat{\times} \boldsymbol{\eta}_3]/2 = [\boldsymbol{\eta}_3 \hat{\times} \boldsymbol{\eta}_1]/2. \tag{54}$$

This rotation arises in any curved space as a consequence of the parallel transport of a local vector describing the orientation of an infinitesimal test body carried over a nongeodesic path. Its exact magnitude ω_{123} is given by Euler's formula,

$$\cos(\omega_{123}/2) = \frac{1 + \cosh\eta_1 + \cosh\eta_2 + \cosh\eta_3}{4\cosh(\eta_1/2)\cosh(\eta_2/2)\cosh(\eta_3/2)}, \tag{55}$$

and equivalently by an expression that illustrates the connection with the area of the triangle and establishes the sign of $\boldsymbol{\omega}_{123}$:

$$\sin(\omega_{123}/2) = \frac{\sinh\eta_1 \sinh\eta_2 \sin\angle(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2)}{4\cosh(\eta_1/2)\cosh(\eta_2/2)\cosh(\eta_3/2)}. \tag{56}$$

In velocity space a similar rotation is responsible for the well-known Thomas precession correction in atomic hyperfine spectra. This is a physical consequence of the velocity-space rotation:

$$\delta\boldsymbol{\omega}_{\text{vel}} = \oint \boldsymbol{\varepsilon} \hat{\times} d\boldsymbol{\varepsilon}/2 \cong \oint \mathbf{v} \times d\mathbf{v}/(2c^2). \tag{57}$$

The rotation factor $R(\boldsymbol{\omega}_{123})$ in Eq. (53) appears in the important product and commutator relationship

$$K(\boldsymbol{\eta}_1)K(\boldsymbol{\eta}_2) = [K(\boldsymbol{\eta}_1), K(\boldsymbol{\eta}_2)]/2 = K(\boldsymbol{\eta}_1 \hat{+} \boldsymbol{\eta}_2)R(\boldsymbol{\eta}_1 \hat{\times} \boldsymbol{\eta}_2/2). \tag{58}$$

In practice, the induced rotation will be very small, because it contains the curvature-dependent factor $(\eta_1\eta_2 \cong r_1 r_2/\rho^2)$. Its role is nonetheless very important for an understanding of the physics of the system. In describing the effect of displacements in a hyperbolic position or velocity space we shall therefore make use of Lorentz matrices of the general form

$$L(\boldsymbol{\eta}, \boldsymbol{\omega}_{\eta}) = K(\boldsymbol{\eta})R(\boldsymbol{\omega}_{\eta}), \tag{59}$$

where the subscript on the rotation vector $\boldsymbol{\omega}_{\boldsymbol{\eta}}$ identifies its association with $\boldsymbol{\eta}$ as part of the six-parameter entity $(\boldsymbol{\eta},\boldsymbol{\omega}_{\boldsymbol{\eta}})$.

C. Position and Velocity as Four-by-Four Tensors

In Eq. (50) the position four-vector is seen to be identifiable with a column of the Lorentz displacement matrix $K(\boldsymbol{\eta})$. The addition of position vectors can be expressed through products of such matrices $K(\boldsymbol{\eta}_i)$. The occurrence of the induced rotations $R(\boldsymbol{\omega}_j)$ in these products leads to the conclusion that hyperbolic vector addition will be noncommutative. This can be accepted, but the associative law must still be fulfilled. This requires amplifying the parameter space of hyperbolic position variables from the three variables of $\boldsymbol{\eta}$ space to the six variables of a space of $(\boldsymbol{\eta},\boldsymbol{\omega})$ and representing position not by a four-vector $X^{\mu} = \rho(t)K^{\mu}{}_{0}(\boldsymbol{\eta})$ but by a Lorentz matrix which we shall usually take as

$$X^{\mu}{}_{\nu}(t,\boldsymbol{\eta}) = \rho(t)K^{\mu}{}_{\lambda}(\boldsymbol{\eta}) = \rho(t)L^{\mu}{}_{\nu}(\boldsymbol{\eta},\mathbf{0}), \tag{60}$$

but may occasionally write in the more extended form

$$X^{\mu}{}_{\nu}(t,\boldsymbol{\eta},\boldsymbol{\omega}) = \rho(t)K^{\mu}{}_{\lambda}(\boldsymbol{\eta})R^{\lambda}{}_{\nu}(\boldsymbol{\omega}) = \rho(t)L^{\mu}{}_{\nu}(\boldsymbol{\eta},\boldsymbol{\omega}). \tag{61}$$

This generalization to the four-by-four tensor form makes it possible to demonstrate the almost completely commutative nature of the addition of hyperbolic position and velocity vectors. This is supplemented only by an extremely small second-order anticommutative contribution, which is entirely of a rotational nature. It can be displayed if we write out the addition of two pure translations in the form

$$K^{\kappa\lambda}(\boldsymbol{\eta}_1)X^{\lambda\nu}(t,\boldsymbol{\eta}_2) = X^{\kappa\mu}(t,\boldsymbol{\eta}_1 \hat{+} \boldsymbol{\eta}_2)R^{\mu\nu}(\boldsymbol{\eta}_1 \times \boldsymbol{\eta}_2/2) \cong X^{\kappa\mu}(t,\boldsymbol{\eta}_1 \hat{+} \boldsymbol{\eta}_2). \tag{62}$$

The addition of velocities behaves in an identical fashion. Its rotation factor is of the same nature as the Thomas precession correction in atomic spectroscopy, and the resulting rotation through the angle $\boldsymbol{\omega}'_3 = \boldsymbol{\varepsilon}_1 \hat{\times} \boldsymbol{\varepsilon}_2/2 \cong \gamma_1\gamma_2\mathbf{v}_1 \times \mathbf{v}_2/c_0^{\,2}$ can be identified as the classical analogue of a spin rotation. The final velocity matrix then has an expression similar to Eq. (62)

$$K^{\kappa\lambda}(\boldsymbol{\varepsilon}_1)U^{\lambda\nu}(t,\boldsymbol{\varepsilon}_2) = U^{\kappa\mu}(t,\boldsymbol{\varepsilon}_1 \hat{+} \boldsymbol{\varepsilon}_2)R^{\mu\nu}(\boldsymbol{\varepsilon}_1 \times \boldsymbol{\varepsilon}_2/2) \cong U^{\kappa\nu}(t,\boldsymbol{\varepsilon}_1 \hat{+} \boldsymbol{\varepsilon}_2). \tag{63}$$

The first, commutative, term in this equation generates exactly the results of Einstein velocity addition, and the second term is the anticommutative correction connected with the Thomas factor. The second term vanishes identically in the addition of collinear velocities. Its effect is so small that it is not surprising that its practical consequences are recognized mostly in the quantum context of hyperfine spectra. In principle, however, it exists classically as well as quantally.

The second-rank tensor expressions for position and velocity, together with their rotational contributions, will be used further below. In addition to their involvement in terms like the Thomas precession, they play an important part in the structure of the generalization of the Poincaré group to hyperbolic space, the new hyperbolic Poincaré group. In this hyperbolic system position space and velocity-momentum space each carry three rotational dimensions of their own in the phase space of any particle.

D. The Operators and Matrices of Velocity Boosts and Positional Shifts

It is useful to recognize the addition of vectors in the hyperbolic spaces of position and velocity as the results of the geometric and kinematic operations of translation in those respective spaces. The operation of translation in velocity can be identified as a boost, and a translation in hyperbolic position space will be identified as a "shift". The rapidity $\boldsymbol{\varepsilon}$ is the geodesic vector of hyperbolic arc in velocity space. The dimensionless geodesic arc $\boldsymbol{\eta}$ of a hyperbolic vector in position space is the separation. We shall denote the operator of a boost through a rapidity interval $\boldsymbol{\varepsilon}_1$ by the notation $\hat{\mathbf{K}}_{\text{boost}}(\boldsymbol{\varepsilon}_1)$ and the corresponding operation of a translational shift in hyperbolic position through the separation $\boldsymbol{\eta}_1$ by $\hat{\mathbf{K}}_{\text{shift}}(\boldsymbol{\eta}_1)$. The result of each of these operations operating on a four-vector or four-by-four tensor in its appropriate space is to multiply that four-vector or tensor by a Lorentz matrix with the appropriate hyperbolic vector argument:

$$\begin{aligned}\hat{\mathbf{K}}_{\text{boost}}(\boldsymbol{\varepsilon}_1)U(t,\boldsymbol{\varepsilon}_2) &= K(\boldsymbol{\varepsilon}_1)U(t,\boldsymbol{\varepsilon}_2) = c(t)K(\boldsymbol{\varepsilon}_1)K(\boldsymbol{\varepsilon}_2)\\ &= c(t)K(\boldsymbol{\varepsilon}_1 \hat{+} \boldsymbol{\varepsilon}_2)R(\boldsymbol{\varepsilon}_1 \hat{\times} \boldsymbol{\varepsilon}_2/2) = U(t,\boldsymbol{\varepsilon}_1 \hat{+} \boldsymbol{\varepsilon}_2)R(\boldsymbol{\varepsilon}_1 \hat{\times} \boldsymbol{\varepsilon}_2/2),\end{aligned} \tag{64}$$

$$\begin{aligned}\hat{\mathbf{K}}_{\text{shift}}(\boldsymbol{\eta}_1)X(t,\boldsymbol{\eta}_2) &= K(\boldsymbol{\eta}_1)X(t,\boldsymbol{\eta}_2) = \rho(t)K(\boldsymbol{\eta}_1)K(\boldsymbol{\eta}_2)\\ &= \rho(t)K(\boldsymbol{\eta}_1 \hat{+} \boldsymbol{\eta}_2)R(\boldsymbol{\eta}_1 \hat{\times} \boldsymbol{\eta}_2/2) = X(t,\boldsymbol{\eta}_1 \hat{+} \boldsymbol{\eta}_2)R(\boldsymbol{\eta}_1 \hat{\times} \boldsymbol{\eta}_2/2).\end{aligned} \tag{65}$$

The principal consequence of these operations is the hyperbolic vector sum $\boldsymbol{\varepsilon}_1 \hat{+} \boldsymbol{\varepsilon}_2$ or $\boldsymbol{\eta}_1 \hat{+} \boldsymbol{\eta}_2$ respectively, and their secondary effect is the second order rotation $R(\boldsymbol{\varepsilon}_1 \hat{\times} \boldsymbol{\varepsilon}_2 / 2)$ or $R(\boldsymbol{\eta}_1 \hat{\times} \boldsymbol{\eta}_2 / 2)$.

The operation of a velocity boost on a position vector in hyperbolic space $\hat{\mathbf{K}}_{\text{boost}}(\boldsymbol{\varepsilon})X(t,\boldsymbol{\eta})$ is a straightforward generalization of the Lorentz transformation of special relativity, and will be presented in the next Section. The converse operation, that of a position shift on a velocity vector, $\hat{\mathbf{K}}_{\text{shift}}(\boldsymbol{\eta})U(\mathrm{t},\boldsymbol{\varepsilon})$, can be recognized as describing the Hubble effect, as will be quantitatively confirmed in the same Section.

V. The Lorentz Transformation and the Hubble Effect in a Hyperbolic Universe

In a hyperbolic geometry the Lorentz matrices and the Lorentz group provide the natural mechanism for dealing with issues of translational symmetry and the compounding of vectors in the curved space. In the preceding Section a notation has been presented for these purposes in the homogeneous situation of a velocity boosts operating in velocity space, and also of translational shifts operating in position space. The sign convention in the off-diagonal $\sinh\varepsilon$ elements in the Lorentz matrix of Eq. (46) and in the definition of four-vectors in Eqs. (44) and (45) is required to ensure that the addition of vectors in those spaces is correctly expressed.

The Lorentz transformation was initially developed in connection with the kinematics of an object in interaction with an electromagnetic signal, in the situation that can now be recognized as describing the effect of a velocity boost operating inhomogeneously on an observed position vector and time interval. In the next subsection, the familiar expressions of the Lorentz transformation of position vectors in Minkowski space under a velocity boost will be transcribed into the more general form applicable in hyperbolic position space. In this case, the kinematics of the Lorentz effect leads to the opposite choice of sign in the off-diagonal $\sinh\varepsilon$ elements from what is found in the homogeneous case of velocity addition. This is a characteristic and important feature in the structure of the kinematic system that will be described by the hyperbolic Poincaré group.

In addition to the homogeneous operations resulting in vector addition in the two spaces and the inhomogeneous operation of the Lorentz effect, the hyperbolic kinematic system must include a second inhomogeneous operation, that describing the effect of a shift in position space on a velocity vector. This can readily be identified with the Hubble effect. The transcription of the Hubble effect into the coordinates of the doubly hyperbolic universe will be given in Subsection B.

A. The Lorentz Effect of Velocity on Length and Time in the Hyperbolic World

The reconciliation of electromagnetics with particle kinematics achieved by the Lorentz transformation has two essential features. The feature most frequently stressed is the introduction of the electromagnetic upper bound c in velocity space. This electromagnetic condition makes velocity space hyperbolic, and it is responsible for the orthogonality requirement satisfied by the Lorentz transformation matrix. The electromagnetic condition is quadratic in form, as shown in the velocity requirement $\mathbf{v}^2 \leq c^2$ and in the metric conditions (13) and (4). When an orientation in space is fixed it offers two roots differing in sign. The choice of sign requires an additional condition. In the Lorentz transformation this choice is established by the linear kinematic condition connecting position, time and velocity between a stationary and a moving frame. That condition fixes the sign of the correction terms in the Lorentz transformation.

The essential features of the Lorentz effect in the Einstein-Minkowski world can be seen in the collinear case of the Lorentz transformation, where the direction of observation is collinear with both the velocity of motion $\mathbf{v}$ and the orientation of the vector being observed, $\Delta\mathbf{r}$. In this case, a local interval of length and time $(\Delta\tau, \Delta r)$, moving at a velocity v in the direction $\Delta\hat{\mathbf{r}}$, is seen by the observer as the interval $(\Delta\tau', \Delta r')$:

$$c_0\Delta\tau' = \gamma(c_0\Delta\tau - \Delta r v / c_0), \quad \Delta r' = \gamma(\Delta r - v\Delta\tau), \text{ where } \gamma = \left(1 - v^2 / c_0{}^2\right)^{-1/2}. \quad (66)$$

The electromagnetic condition is quadratic, and is responsible for the factor γ in Eq. (66). The kinematic condition is linear in form, and is responsible for the length correction $-\gamma\Delta r v / c_0$ in the temporal term $c_0\Delta\tau'$ and the temporal correction $-\gamma v\Delta\tau$ in $\Delta r'$. The negative sign of this correction term in each member of Eq. (66) is an essential signature of the physical process of the Lorentz velocity boost which that equation describes.

We can take advantage of the hyperbolic structure of velocity space and rewrite the equations (66) using the rapidity variable ε in the expressions $\gamma = \left(1 - v^2 / c_0^{\,2}\right)^{-1/2} = \cosh\varepsilon$ and $\gamma v / c_0 = \sinh\varepsilon$. Eq. (66) then becomes

$$c_0\Delta\tau' = c_0\Delta\tau\cosh\varepsilon - \Delta r\sinh\varepsilon, \qquad \Delta r' = \Delta r\cosh\varepsilon - c_0\Delta\tau\sinh\varepsilon. \tag{67}$$

We see that a boost through the velocity v or its equivalent rapidity ε results in an orthogonal linear transformation of the Minkowski space-time vector $\begin{pmatrix} c_0\Delta\tau \\ i\Delta r \end{pmatrix}$ by a Lorentz matrix:

$$\begin{pmatrix} c_0\Delta\tau' \\ i\Delta r' \end{pmatrix} = \begin{pmatrix} \gamma & i\gamma v / c_0 \\ -i\gamma v / c_0 & \gamma \end{pmatrix}\begin{pmatrix} c_0\Delta\tau \\ i\Delta r \end{pmatrix} = \begin{pmatrix} \cosh\varepsilon & i\sinh\varepsilon \\ -i\sinh\varepsilon & \cosh\varepsilon \end{pmatrix}\begin{pmatrix} c_0\Delta\tau \\ i\Delta r \end{pmatrix}. \tag{68}$$

The general definition given in Eq. (46) of the Lorentz matrix as a function of the hyperbolic parameters of position space is applicable unchanged to the rapidity parameters $\boldsymbol{\varepsilon}$ of velocity space. Its sign convention insures the correct behavior of the addition of vectors in their own space. When specialized to one-dimensional motion it reduces to

$$K(\varepsilon) = \begin{pmatrix} \cosh\varepsilon & -i\sinh\varepsilon \\ i\sinh\varepsilon & \cosh\varepsilon \end{pmatrix}. \tag{69}$$

The effect of a velocity boost operation on a Minkowski position vector, Eq. (68), is therefore expressed by the Lorentz matrix $K(-\varepsilon)$ where the argument has a negative sign,

$$\begin{pmatrix} c_0\Delta\tau' \\ i\Delta r' \end{pmatrix} = \begin{pmatrix} \cosh\varepsilon & i\sinh\varepsilon \\ -i\sinh\varepsilon & \cosh\varepsilon \end{pmatrix}\begin{pmatrix} c_0\Delta\tau \\ i\Delta r \end{pmatrix} = K(-\varepsilon)\begin{pmatrix} c_0\Delta\tau \\ i\Delta r \end{pmatrix}. \tag{70}$$

This change in sign characterizes the difference between the operation of a velocity boost on a position vector and its operation on a velocity or momentum vector, and is a systematic feature of kinematics in special relativity. Utilizing the operator notation of Eqs. (64) and (65), Eq. (70) implies

$$\begin{pmatrix} c_0\Delta\tau' \\ i\Delta r' \end{pmatrix} = \hat{\mathbf{K}}_{\text{boost}}(\varepsilon)\begin{pmatrix} c_0\Delta\tau \\ i\Delta r \end{pmatrix} = K(-\varepsilon)\begin{pmatrix} c_0\Delta\tau \\ i\Delta r \end{pmatrix}. \tag{71}$$

In Eqs. (66) to (71) the position and time variables are expressed in the local coordinates of a flat Minkowski space-time. They can be connected with the space and time coordinates of an expanding hyperbolic universe by the relationships of Section III.B above. The Minkowski vector can be recognized as a difference vector between two complete hyperbolic four-vectors with collinear spatial components, expressible therefore as hyperbolic space-time two-vectors:

$$\begin{aligned}\begin{pmatrix} c_0\Delta\tau \\ i\Delta r\end{pmatrix} &= \begin{pmatrix} \Delta w \\ i\Delta r\end{pmatrix} = \begin{pmatrix} w_2 \\ ir_2\end{pmatrix} - \begin{pmatrix} w_1 \\ ir_1\end{pmatrix} \\ &= \rho(t_2)\begin{pmatrix} \cosh\eta_2 \\ i\sinh\eta_2\end{pmatrix} - \rho(t_1)\begin{pmatrix} \cosh\eta_1 \\ i\sinh\eta_1\end{pmatrix}. \\ &= X(t_2,\eta_2) - X(t_1,\eta_1)\end{aligned} \tag{72}$$

The interval as seen by the observer can be described in the same way as the difference between two hyperbolic four-vectors in his velocity frame,

$$\begin{aligned}\begin{pmatrix} c_0\Delta\tau' \\ i\Delta r'\end{pmatrix} &= \begin{pmatrix} \Delta w' \\ i\Delta r'\end{pmatrix} = \begin{pmatrix} w_2' \\ ir_2'\end{pmatrix} - \begin{pmatrix} w_1' \\ ir_1'\end{pmatrix} = X(t_2',\eta_2') - X(t_1',\eta_1') \\ &= \rho(t_2')\begin{pmatrix} \cosh\eta_2' \\ i\sinh\eta_2'\end{pmatrix} - \rho(t_1')\begin{pmatrix} \cosh\eta_1' \\ i\sinh\eta_1'\end{pmatrix}.\end{aligned} \tag{73}$$

Combining Eqs. (68), (71) and (72) we also find

$$\begin{aligned}\begin{pmatrix} \Delta w' \\ i\Delta r'\end{pmatrix} &= \hat{\mathbf{K}}_{\text{boost}}(\varepsilon)\begin{pmatrix} \Delta w \\ i\Delta r\end{pmatrix} = K(-\varepsilon)\begin{pmatrix} \Delta w \\ i\Delta r\end{pmatrix} = K(-\varepsilon)X(t_2,\eta_2) - K(-\varepsilon)X(t_1,\eta_1) \\ &= \rho(t_2)\begin{pmatrix} \cosh(\eta_2-\varepsilon) \\ i\sinh(\eta_2-\varepsilon)\end{pmatrix} - \rho(t_1)\begin{pmatrix} \cosh(\eta_1-\varepsilon) \\ i\sinh(\eta_1-\varepsilon)\end{pmatrix}.\end{aligned} \tag{74}$$

This pair of equations are both satisfied provided that we have, for any value of i,

$$X(t_i',\eta_i') = \rho(t_i')\begin{pmatrix} \cosh\eta_i' \\ i\sinh\eta_i'\end{pmatrix} = \hat{\mathbf{K}}_{\text{boost}}(\varepsilon)X(t_i,\eta_i) = K(-\varepsilon)X(t_i,\eta_i) = \rho(t_i)\begin{pmatrix} \cosh(\eta_i-\varepsilon) \\ i\sinh(\eta_i-\varepsilon)\end{pmatrix}. \tag{75}$$

We can now generalize this relationship from the special case of one-dimensional motion to the full three-spaces of position and velocity. We can write this as the operation of a velocity boost on a hyperbolic position tensor, the natural generalization of the position four-vector:

$$\begin{aligned}\hat{\mathbf{K}}_{\text{boost}}(\mathtt{e}_1)X(t,\mathtt{h}_2) &= K(-\mathtt{e}_1)X(t,\mathtt{h}_2) = \rho(t)K(-\mathtt{e}_1)K(\mathtt{h}_2)\\ &= \rho(t)K(-\mathtt{e}_1\,\hat{+}\,\mathtt{h}_2)R(-\mathtt{e}_1\,\hat{\times}\,\mathtt{h}_2/2) = X(t,-\mathtt{e}_1\,\hat{+}\,\mathtt{h}_2)R(-\mathtt{e}_1\,\hat{\times}\,\mathtt{h}_2/2).\end{aligned} \tag{76}$$

We can expand the rotational matrix and take advantage of the extremely small magnitude of its angular argument to ignore that correction and get the simplified hyperbolic space version of the Lorentz transformation

$$\begin{aligned}\hat{\mathbf{K}}_{\text{boost}}(\mathtt{e}_1)X(t,\mathtt{h}_2) &= K(-\mathtt{e}_1)X(t,\mathtt{h}_2)\\ &= X(t,-\mathtt{e}_1\,\hat{+}\,\mathtt{h}_2)\left[1+O\left(\frac{-\gamma(v_1)\mathbf{v}_1\,\hat{\times}\,\mathbf{r}_2}{2c_0{}^2H_0{}^{-1}}\right)\right] \cong X(t,\mathtt{h}_2\,\hat{-}\,\mathtt{e}_1).\end{aligned} \tag{77}$$

B. The Hubble Effect in Hyperbolic Space

When the redshift of an astrophysical object is used as a measure of its velocity away from the observer, the Hubble effect is responsible for a component of velocity $\mathbf{v}_\text{H}$ representing a rate of expansion that increases linearly with the distance vector $\mathbf{r}_{0i}$ from the observer at 0 to the object i, $\mathbf{v}_\text{H} = H\mathbf{r}_{0i}$. In the doubly hyperbolic system we must replace $\mathbf{v}_\text{H}$ by the relativistic velocity $\mathbf{u}_\text{H}$ and introduce the time-dependent Hubble coefficient,

$$\mathbf{v}_\text{H} = H\mathbf{r}_{0i} \rightarrow \mathbf{u}_\text{H} = H(t)\mathbf{r}_{0i}\,. \tag{78}$$

When we describe $\mathbf{u}_\text{H}$ and $\mathbf{r}_{0i}$ by their appropriate hyperbolic coordinates of rapidity and separation this becomes

$$\begin{aligned}\mathbf{u}_\text{H} &= c(t)\sinh\varepsilon_\text{H}\,\hat{\boldsymbol{\varepsilon}}_\text{H}(\theta_{\boldsymbol{\varepsilon}},\phi_{\boldsymbol{\varepsilon}}) = H(t)\mathbf{r}_{0i} = H(t)\rho(t)\sinh\eta_{0i}\,\hat{\boldsymbol{\eta}}_{0i}(\theta_{\boldsymbol{\eta}},\phi_{\boldsymbol{\eta}})\\ &= c(t)\sinh\eta_{0i}\,\hat{\boldsymbol{\eta}}_{0i}(\theta_{\boldsymbol{\eta}},\phi_{\boldsymbol{\eta}}).\end{aligned} \tag{79}$$

In the natural curvilinear coordinate variables of the hyperbolic spaces of position and velocity the Hubble relationship can then be expressed in the particularly simple form

$$\boldsymbol{\varepsilon}_\text{H} = \boldsymbol{\eta}_{0i}\,. \tag{80}$$

i.e., the Hubble contribution $\boldsymbol{\varepsilon}_{\mathrm{H}}$ to the hyperbolic rapidity of the object with respect to the observer is identically equal to the vector distance between the two as expressed in the hyperbolic separation $\boldsymbol{\eta}_{0i}$.

The observed velocity $\mathbf{v}_{\mathrm{obs},i}$ of a given object can be treated as the sum of its Hubble velocity $\mathbf{v}_{\mathrm{H},i}$ and its peculiar velocity $\mathbf{v}_{\mathrm{pec},i}$, the velocity it would have had relative to the observer if it were transported back to that observer's local neighborhood. If the object i is a member of a group or cluster in a limited astronomical region its peculiar velocity can be treated as predominantly a local velocity with respect to some average background of the cluster itself. Astrophysical observations have confirmed the vector addition of a single Hubble velocity for the cluster to the peculiar velocities of individual members of a galactic or stellar population in a given region to give the observed velocity [5,6]:

$$\mathbf{v}_{\mathrm{obs},i} = \mathbf{v}_{\mathrm{pec},i} + \mathbf{v}_{\mathrm{H},C} = \mathbf{v}_{\mathrm{pec},i} + H\mathbf{r}_{0C}\,. \qquad (81)$$

We can then treat the peculiar velocity as predominantly a local velocity, $\mathbf{v}_{\mathrm{pec},i} \cong \mathbf{v}_{\mathrm{loc},C}$, and replace Eq. (81) by

$$\mathbf{v}_{\mathrm{obs},i} = \mathbf{v}_{\mathrm{loc},i} + \mathbf{v}_{\mathrm{H},C} = \mathbf{v}_{\mathrm{loc},i} + H\mathbf{r}_{0C}\,. \qquad (82)$$

We can now follow the pattern of Eqs. (78) to (80) and make the substitutions $\mathbf{v}_i \to c\boldsymbol{\varepsilon}_i, \mathbf{r}_i \to cH^{-1}(r)\boldsymbol{\eta}_i$, with the result

$$\boldsymbol{\varepsilon}_{\mathrm{obs},i} = \boldsymbol{\varepsilon}_{\mathrm{loc},i} \,\hat{+}\, \boldsymbol{\eta}_{0C}\,. \qquad (83)$$

This result can be expressed as describing a position shift through the separation vector $\boldsymbol{\eta}_{0C}$ operating on a local rapidity vector $\boldsymbol{\varepsilon}_{\mathrm{loc},i}$ and producing the resultant rapidity $\boldsymbol{\varepsilon}_{\mathrm{obs},i}$. It can be seen to justify a general relationship conjugate to Eq. (76),

$$\begin{aligned}\hat{\mathbf{K}}_{\mathrm{shift}}(\boldsymbol{\eta}_1)U(t,\boldsymbol{\varepsilon}_2) &= K(\boldsymbol{\eta}_1)U(t,\boldsymbol{\varepsilon}_2) = c(t)K(\boldsymbol{\eta}_1)K(\boldsymbol{\varepsilon}_2)\\ &= c(t)K(\boldsymbol{\eta}_1 \,\hat{+}\, \boldsymbol{\varepsilon}_2)R(\boldsymbol{\eta}_1 \,\hat{\times}\, \boldsymbol{\varepsilon}_2/2) = U(t,\boldsymbol{\eta}_1 \,\hat{+}\, \boldsymbol{\varepsilon}_2)R(\boldsymbol{\eta}_1 \,\hat{\times}\, \boldsymbol{\varepsilon}_2/2).\end{aligned} \qquad (84)$$

As usual, the very small rotational correction can be evaluated by expansion, leading to the simpler final result:

$$\hat{\mathbf{K}}_{\mathrm{shift}}(\boldsymbol{\eta}_1)U(t,\boldsymbol{\varepsilon}_2) = K(\boldsymbol{\eta}_1)U(t,\boldsymbol{\varepsilon}_2) = U(t,\boldsymbol{\eta}_1 \,\hat{+}\, \boldsymbol{\varepsilon}_2)\left[1 + O\left(\frac{\mathbf{r}_1 \times \mathbf{v}_2\gamma(v_2)}{2c_0{}^2H_0{}^{-1}}\right)\right] \cong U(t,\boldsymbol{\eta}_1 \,\hat{+}\, \boldsymbol{\varepsilon}_2). \qquad (85)$$

VI. The Hyperbolic Poincaré Group

The ordinary Lorentz and Poincaré groups embody the accepted symmetry principles of special relativistic kinematics and geometry in Minkowski space-time. When that space-time is replaced by the more general expanding hyperbolic space-time of a universe expressing the position-velocity and Hamiltonian symmetries that were lost in special relativity, notable changes in symmetry follow. They are expressed in the new hyperbolic Poincaré group.

In Sections IV and V we have developed on the basis of physical experience, including the Hubble effect as well as the Lorentz transformation, the properties of two sets of Lorentz operators, the operators of positional shifts $\hat{\mathbf{K}}_{\text{shift}}(\boldsymbol{\eta})$ and of velocity boosts $\hat{\mathbf{K}}_{\text{boost}}(\boldsymbol{\varepsilon})$, as each of them operates on the tensors of position $X(\boldsymbol{\eta})$ and of velocity $U(\boldsymbol{\varepsilon})$. These provide the fundamental information that must be embodied in a multiplication table for the group. We provide here the most important relationships and structural features of the group, especially the structure of its translational operations in position and velocity and the contrasting form of its time translation operator.

A. The Operators of Position Shifts and Velocity Boosts in the Hyperbolic Poincaré Group

We can assemble the principal results of Section IV, Eqs. (64) and (65), and Section V, Eqs. (76) and (84). They describe the effects of the operators of shifts and boosts, $\hat{\mathbf{K}}_{\text{shift}}(\boldsymbol{\eta})$ and $\hat{\mathbf{K}}_{\text{boost}}(\boldsymbol{\varepsilon})$, on position and velocity tensors, $X(t,\boldsymbol{\eta})$ and $U(t,\boldsymbol{\varepsilon})$:

$$\begin{aligned}
\hat{\mathbf{K}}_{\text{shift}}(\boldsymbol{\eta}_1)X(t,\boldsymbol{\eta}_2) &= K(\boldsymbol{\eta}_1)X(t,\boldsymbol{\eta}_2) = X(t,\boldsymbol{\eta}_1 \hat{+} \boldsymbol{\eta}_2)R(\boldsymbol{\eta}_1 \hat{\times} \boldsymbol{\eta}_2/2), \\
\hat{\mathbf{K}}_{\text{shift}}(\boldsymbol{\eta}_1)U(t,\boldsymbol{\varepsilon}_2) &= K(\boldsymbol{\eta}_1)U(t,\boldsymbol{\varepsilon}_2) = U(t,\boldsymbol{\eta}_1 \hat{+} \boldsymbol{\varepsilon}_2)R(\boldsymbol{\eta}_1 \hat{\times} \boldsymbol{\varepsilon}_2/2), \\
\hat{\mathbf{K}}_{\text{boost}}(\boldsymbol{\varepsilon}_1)X(t,\boldsymbol{\eta}_2) &= K(-\boldsymbol{\varepsilon}_1)X(t,\boldsymbol{\eta}_2) = X(t,\boldsymbol{\eta}_2 \hat{-} \boldsymbol{\varepsilon}_1)R(-\boldsymbol{\varepsilon}_1 \hat{\times} \boldsymbol{\eta}_2/2), \\
\hat{\mathbf{K}}_{\text{boost}}(\boldsymbol{\varepsilon}_1)U(t,\boldsymbol{\varepsilon}_2) &= K(\boldsymbol{\varepsilon}_1)U(t,\boldsymbol{\varepsilon}_2) = U(t,\boldsymbol{\varepsilon}_1 \hat{+} \boldsymbol{\varepsilon}_2)R(\boldsymbol{\varepsilon}_1 \hat{\times} \boldsymbol{\varepsilon}_2/2)
\end{aligned}, \qquad (86)$$

We now take advantage of the fact that the spaces of $\boldsymbol{\eta}$ and $\boldsymbol{\varepsilon}$ in which the shift and boost operators operate are orthogonal to the cosmological time variable t in the dimensional factors $\rho(t)$ and $c(t)$ in the position and velocity four-tensors $X(t,\boldsymbol{\eta})$ and $U(t,\boldsymbol{\varepsilon})$. Operators that depend only on the hyperbolic variables $\boldsymbol{\eta}$ and $\boldsymbol{\varepsilon}$ and their associated secondary rotation variables $\boldsymbol{\omega}_{\eta}$ and $\boldsymbol{\omega}_{\varepsilon}$ all commute with $\rho(t)$ and $c(t)$. We can thus multiply the Eqs. (86) by the appropriate factor $\rho^{-1}(t)$ or $c^{-1}(t)$ and use identities like

$$\rho^{-1}(t)\hat{\mathbf{K}}_{\text{shift}}(\boldsymbol{\eta}_1)X(t,\boldsymbol{\eta}_2)=\hat{\mathbf{K}}_{\text{shift}}(\boldsymbol{\eta}_1)\rho^{-1}(t)X(t,\boldsymbol{\eta}_2)=\hat{\mathbf{K}}_{\text{shift}}(\boldsymbol{\eta}_1)K(\boldsymbol{\eta}_2), \quad (87)$$

converting those equations into the set

$$\begin{aligned}
&\rho^{-1}(t)\hat{\mathbf{K}}_{\text{shift}}(\boldsymbol{\eta}_1)X(t,\boldsymbol{\eta}_2)=\hat{\mathbf{K}}_{\text{shift}}(\boldsymbol{\eta}_1)K(\boldsymbol{\eta}_2)=K(\boldsymbol{\eta}_1)K(\boldsymbol{\eta}_2)\\
&c^{-1}(t)\hat{\mathbf{K}}_{\text{shift}}(\boldsymbol{\eta}_1)U(t,\boldsymbol{\varepsilon}_2)=\hat{\mathbf{K}}_{\text{shift}}(\boldsymbol{\eta}_1)K(\boldsymbol{\varepsilon}_2)=K(\boldsymbol{\eta}_1)K(\boldsymbol{\varepsilon}_2),\\
&\rho^{-1}(t)\hat{\mathbf{K}}_{\text{boost}}(\boldsymbol{\varepsilon}_1)X(t,\boldsymbol{\eta}_2)=\hat{\mathbf{K}}_{\text{boost}}(\boldsymbol{\varepsilon}_1)K(\boldsymbol{\eta}_2)=K(-\boldsymbol{\varepsilon}_1)K(\boldsymbol{\eta}_2),\\
&c^{-1}(t)\hat{\mathbf{K}}_{\text{boost}}(\boldsymbol{\varepsilon}_1)U(t,\boldsymbol{\varepsilon}_2)=\hat{\mathbf{K}}_{\text{boost}}(\boldsymbol{\varepsilon}_1)K(\boldsymbol{\varepsilon}_2)=K(\boldsymbol{\varepsilon}_1)K(\boldsymbol{\varepsilon}_2).
\end{aligned} \quad (88)$$

It is clear from the sign change in the inhomogeneous elements of this array (the second and third equations) that the shift and boost operators cannot be represented by four-by-four Lorentz matrices. Instead, the operators $\hat{\mathbf{K}}_{\text{shift}}(\boldsymbol{\eta})$ and $\hat{\mathbf{K}}_{\text{boost}}(\boldsymbol{\varepsilon})$ must be represented by block-diagonal eight-by-eight matrices, and their position and velocity operands must be part of an eight-by-eight tensor. These will generate a larger group, the double Lorentz group.

We can represent the generating operators of this group by the shift and boost matrices

$$\mathbf{K}_{\text{shift}}(\boldsymbol{\eta}_{\text{shift}})=\begin{pmatrix}K(\boldsymbol{\eta}_{\text{shift}}) & 0\\ 0 & K(\boldsymbol{\eta}_{\text{shift}})\end{pmatrix} \text{ and } \mathbf{K}_{\text{boost}}(\boldsymbol{\varepsilon}_{\text{boost}})=\begin{pmatrix}K(-\boldsymbol{\varepsilon}_{\text{boost}}) & 0\\ 0 & K(\boldsymbol{\varepsilon}_{\text{boost}})\end{pmatrix}. \quad (89)$$

The equations of the array (86) or (88) are reproduced if these operators operate on an eight-by-eight double tensor of position and velocity,

$$\Xi(t;\boldsymbol{\eta}_{\text{sep}},\boldsymbol{\varepsilon}_{\text{rap}})=\begin{pmatrix}X(t,\boldsymbol{\eta}_{\text{sep}}) & 0\\ 0 & U(t,\boldsymbol{\varepsilon}_{\text{rap}})\end{pmatrix}=\begin{pmatrix}\rho(t)K(\boldsymbol{\eta}_{\text{sep}}) & 0\\ 0 & c(t)K(\boldsymbol{\varepsilon}_{\text{rap}})\end{pmatrix}. \quad (90)$$

This in turn can be factored into a time-dependent part and a matrix with the structure of a member of the double Lorentz group itself,

$$\Xi\left(t;\boldsymbol{\eta}_{\text{sep}},\boldsymbol{\varepsilon}_{\text{rap}}\right)=\begin{pmatrix}\rho(t)\mathbf{1}_4 & \mathbf{0}_4\\ \mathbf{0}_4 & c(t)\mathbf{1}_4\end{pmatrix}\begin{pmatrix}K\left(\boldsymbol{\eta}_{\text{sep}}\right) & 0\\ 0 & K\left(\boldsymbol{\varepsilon}_{\text{rap}}\right)\end{pmatrix}. \tag{91}$$

The effect of the operators $\hat{\mathbf{K}}_{\text{shift}}(\boldsymbol{\eta})$ and $\hat{\mathbf{K}}_{\text{boost}}(\boldsymbol{\varepsilon})$ on the position-velocity double tensor is now expressed through multiplication by the matrices of Eq.(89):

$$\hat{\mathbf{K}}_{\text{shift}}(\boldsymbol{\eta}_1)\Xi\left(t;\boldsymbol{\eta}_2,\boldsymbol{\varepsilon}_2\right)=\mathbf{K}_{\text{shift}}(\boldsymbol{\eta}_1)\Xi\left(t;\boldsymbol{\eta}_2,\boldsymbol{\varepsilon}_2\right), \tag{92}$$

$$\hat{\mathbf{K}}_{\text{boost}}(\boldsymbol{\varepsilon}_1)\Xi\left(t;\boldsymbol{\eta}_2,\boldsymbol{\varepsilon}_2\right)=\mathbf{K}_{\text{boost}}(\boldsymbol{\varepsilon}_1)\Xi\left(t;\boldsymbol{\eta}_2,\boldsymbol{\varepsilon}_2\right). \tag{93}$$

The sign alternation that occurs between the heterogeneous second and third equations in (86) or (88) can be recognized as a new appearance in physics of the symplectic sign reciprocity that characterizes the Hamiltonian equations of dynamics and the quantum commutators. It is fully incorporated in the operations of the new hyperbolic Poincaré group as exhibited in Eq. (89). It gives additional support for the Reciprocity Principle advocated by Born in response to the extraordinary fruitfulness of the Hamiltonian symmetries in the development of quantum mechanics and justifies treating it as a structural principle of the physical universe.

Eq. (91) displays the separability of the position-velocity tensor into a one-parameter time-dependent part and a Lorentz double tensor depending explicitly on the parameters of a configuration space described by the rapidity and separation variables $(\boldsymbol{\eta},\boldsymbol{\varepsilon})$ supplemented by the three-dimensional rotation spaces associated with each of them. Its time-dependent factor can be looked upon as a matrix representing a time translation operator that generates a one-parameter normal subgroup of the hyperbolic Poincaré group. That group is therefore the direct product of the one-parameter time-translation subgroup and the twelve-parameter double Lorentz subgroup. We can discuss separately the structure of these two subgroups.

B. Time Dependence and its Subgroup

The time-dependence matrix, occurring as a factor in the position-velocity tensor $\Xi(t;\boldsymbol{\eta},\boldsymbol{\varepsilon})$, Eq. (91), can be taken as the product of a normalizing constant, which will be denoted $\sigma^{1/2}$, and an orthonormal matrix $\mathbf{Z}(t)$ that expresses the time-dependence of the position and velocity scaling factors $\rho(t)$ and $c(t)$:

$$\begin{pmatrix} \rho(t)\mathbf{1}_4 & \mathbf{0}_4 \\ \mathbf{0}_4 & c(t)\mathbf{1}_4 \end{pmatrix} = \sigma^{1/2}\mathbf{Z}(t) = \sigma^{1/2}\begin{pmatrix} \exp[\zeta(t)]\mathbf{1}_4 & \mathbf{0}_4 \\ \mathbf{0}_4 & \exp[-\zeta(t)]\mathbf{1}_4 \end{pmatrix}. \tag{94}$$

We find immediately that

$$\exp[2\zeta(t)] = \rho(t)/c(t) = H^{-1}(t), \text{ i.e., } \zeta(t) = [-\ln H(t)]/2 \tag{95}$$

and

$$\sigma = c(t)\rho(t) = c^2(t)H^{-1}(t) = c_0^{\,2}H_0^{\,-1}. \tag{96}$$

This fundamental constant can be evaluated, $\sigma \cong 4\cdot 10^{34}\,\mathrm{m}^2/\mathrm{s}$. We can now write the position-velocity tensor as

$$\begin{aligned} \Xi(t;\boldsymbol{\eta}_{\text{sep}},\boldsymbol{\varepsilon}_{\text{rap}}) &= \sigma^{1/2}\mathbf{Z}(t)\begin{pmatrix} K(\boldsymbol{\eta}_{\text{sep}}) & 0 \\ 0 & K(\boldsymbol{\varepsilon}_{\text{rap}}) \end{pmatrix} \\ &= \sigma^{1/2}\begin{pmatrix} H^{-1/2}(t)\mathbf{1}_4 & \mathbf{0} \\ \mathbf{0} & H^{1/2}(t)\mathbf{1}_4 \end{pmatrix}\begin{pmatrix} K(\boldsymbol{\eta}_{\text{sep}}) & 0 \\ 0 & K(\boldsymbol{\varepsilon}_{\text{rap}}) \end{pmatrix} \\ &= c_0 H_0^{-1/2}\begin{pmatrix} H^{-1/2}(t)\mathbf{1}_4 & \mathbf{0} \\ \mathbf{0} & H^{1/2}(t)\mathbf{1}_4 \end{pmatrix}\begin{pmatrix} K(\boldsymbol{\eta}_{\text{sep}}) & 0 \\ 0 & K(\boldsymbol{\varepsilon}_{\text{rap}}) \end{pmatrix}. \end{aligned} \tag{97}$$

As long as the function $H(t)$, and therefore $\zeta(t)$, is monotonic, we can take $\hat{\mathbf{Z}}(\Delta\zeta)$ as a time displacement operator operating through ζ as a transform of the time. This will provide a generalization of the time translation operation of the ordinary Poincaré group. The operator $\hat{\mathbf{Z}}(\Delta\zeta)$ and its matrix representation $\mathbf{Z}(t)$ in Eq. (94) generate a one-parameter translational group, the time-translation subgroup of the hyperbolic Poincaré group. This is a normal subgroup, and the hyperbolic Poincaré group is its direct product with the double Lorentz subgroup generated by the operators $\hat{\mathbf{K}}_{\text{shift}}(\boldsymbol{\eta})$ and $\hat{\mathbf{K}}_{\text{boost}}(\boldsymbol{\varepsilon})$.

A simple and illuminating example of the time-translation operator is encountered if we take the simplifying assumption that the linear time dependence of $H^{-1}(t)$ that dominates near the present time t_0 can be extrapolated indefinitely. In the expression of Eq. (94) we then write $\zeta(t) = t^{1/2}$. Then

$$\mathbf{Z}(t) = \begin{pmatrix} t^{1/2}\,\mathbf{1}_4 & \mathbf{0}_4 \\ \mathbf{0}_4 & t^{-1/2}\,\mathbf{1}_4 \end{pmatrix} \tag{98}$$

and the time displacement operator can be written as $\hat{\mathbf{Z}}(\Delta[\ln t])$, with the matrix representation

$$\mathbf{Z}(\Delta[\ln t]) = \begin{pmatrix} e^{\Delta(\ln t)/2}\,\mathbf{1}_4 & \mathbf{0}_4 \\ \mathbf{0}_4 & e^{-\Delta(\ln t)/2}\,\mathbf{1}_4 \end{pmatrix}. \tag{99}$$

C. The Translational Operators of the Double Lorentz Group

The double Lorentz matrix of Eq. (91) can be recognized a product of a shift matrix and a boost matrix:

$$\begin{pmatrix} K(\boldsymbol{\eta}_{\text{sep}}) & 0 \\ 0 & K(\boldsymbol{\varepsilon}_{\text{rap}}) \end{pmatrix} = \mathbf{K}_{\text{boost}}\left(-\frac{\boldsymbol{\eta}_{\text{sep}}}{2} \hat{+} \frac{\boldsymbol{\varepsilon}_{\text{rap}}}{2}\right)\mathbf{K}_{\text{shift}}\left(\frac{\boldsymbol{\eta}_{\text{sep}}}{2} \hat{+} \frac{\boldsymbol{\varepsilon}_{\text{rap}}}{2}\right). \tag{104}$$

It is a member of the principal subgroup of the hyperbolic Poincaré group, the double Lorentz group, a twelve-parameter group. This group can be generated by the shift and boost operators $\mathbf{K}_{\text{shift}}(\boldsymbol{\eta})$ and $\mathbf{K}_{\text{boost}}(\boldsymbol{\varepsilon})$ and their products. Each of them separately generates its own Lorentz subgroup of the double Lorentz group, the shift and the boost subgroups. Because these operators act not only homogeneously as translation operators within their respective fields, but interactively, with shifts acting on velocities in the Hubble effect and boosts acting on position space in the Lorentz transformation, their subgroups can be called the "interactive subgroups."

In addition to the interactive operators of shifts and boosts, we can form from them the translational operators of two other Lorentz subgroups, the geometric subgroups of position and of velocity, with the matrices:

$$\mathbf{K}_{\text{pos}}\left(\eta_{\text{sep}}\right)=\mathbf{K}_{\text{shift}}\left(-\eta_{\text{sep}}/2\right)\mathbf{K}_{\text{boost}}\left(\eta_{\text{sep}}/2\right)=\begin{pmatrix}K\left(\eta_{\text{sep}}\right) & 0\\ 0 & 1\end{pmatrix},$$
$$\mathbf{K}_{\text{vel}}\left(\varepsilon_{\text{rap}}\right)=\mathbf{K}_{\text{shift}}\left(\varepsilon_{\text{rap}}/2\right)\mathbf{K}_{\text{boost}}\left(\varepsilon_{\text{rap}}/2\right)=\begin{pmatrix}1 & 0\\ 0 & K\left(\varepsilon_{\text{rap}}\right)\end{pmatrix}. \quad (105)$$

The converse relationship is

$$\mathbf{K}_{\text{shift}}\left(\eta_{\text{shift}}\right)=\mathbf{K}_{\text{pos}}\left(\eta_{\text{shift}}\right)\mathbf{K}_{\text{vel}}\left(\eta_{\text{shift}}\right), \qquad \mathbf{K}_{\text{boost}}\left(\varepsilon_{\text{boost}}\right)=\mathbf{K}_{\text{pos}}\left(-\varepsilon_{\text{boost}}\right)\mathbf{K}_{\text{vel}}\left(\varepsilon_{\text{boost}}\right). \quad (106)$$

In each case these operators generate a corresponding set of rotational operators. The geometric operators $\mathbf{K}_{\text{pos}}(\eta)$ and $\mathbf{K}_{\text{vel}}(\varepsilon)$ and their subgroups commute with each other. The double Lorentz subgroup has in total four simple Lorentz subgroups, these two geometric subgroups and the two interactive subgroups generated by $\mathbf{K}_{\text{shift}}(\eta)$ and $\mathbf{K}_{\text{boost}}(\varepsilon)$ respectively.

In the geometric representation of the double Lorentz group the mutual commutation of is position and velocity subgroups makes it possible to identify a six dimensional manifold of pure double translations, represented by the matrices

$$\mathbf{K}(\eta,\varepsilon)=\begin{pmatrix}K(\eta) & 0\\ 0 & K(\varepsilon)\end{pmatrix}=\mathbf{K}_{\text{pos}}(\eta)\mathbf{K}_{\text{vel}}(\varepsilon)=\mathbf{K}_{\text{vel}}(\varepsilon)\mathbf{K}_{\text{pos}}(\eta). \quad (107)$$

With the help of Eq. (105) these matrices can also be described in the interaction representation as products of pure boost and pure shift operations, but the lack of commutativity prevents further simplification of the quadruple product that results.

D. Rotational Structure in the Double Lorentz Group

The shift and boost operators $\hat{\mathbf{K}}_{\text{shift}}\left(\eta_{\text{shift}}\right)$ and $\hat{\mathbf{K}}_{\text{boost}}\left(\varepsilon_{\text{boost}}\right)$ of the interactive family require introducing a comparable pair of double rotational matrices, constructed in the same way from the rotational matrices $R(\omega)$, Eq. (47), of the ordinary Lorentz group:

$$\mathbf{R}_{\text{int}}\left(\omega_{\text{int}}\right)=\begin{pmatrix}R\left(\omega_{\text{int}}\right) & 0\\ 0 & R\left(\omega_{\text{int}}\right)\end{pmatrix} \text{ and } \mathbf{Q}(\upsilon)=\begin{pmatrix}R(-\upsilon) & 0\\ 0 & R(\upsilon)\end{pmatrix}. \quad (108)$$

Their associated operators are $\hat{\mathbf{R}}_{\text{int}}(\boldsymbol{\omega}_{\text{int}})$ and $\hat{\mathbf{Q}}(\boldsymbol{\upsilon})$. Each of these has its own angular momentum operator. The operator $\hat{\mathbf{R}}_{\text{int}}(\boldsymbol{\omega}_{\text{int}})$ is the representative of the usual rotational and spin operator in the hyperbolic domain. In both of the interactive Lorentz subgroups the same rotational operator $\hat{\mathbf{R}}_{\text{int}}(\boldsymbol{\omega}_{\text{int}})$ is generated by products of shifts or boosts:

$$\begin{aligned}\mathbf{K}_{\text{shift}}(\boldsymbol{\eta}_1)\mathbf{K}_{\text{shift}}(\boldsymbol{\eta}_2) &= \mathbf{K}_{\text{shift}}(\boldsymbol{\eta}_1 \hat{+} \boldsymbol{\eta}_2)\mathbf{R}_{\text{int}}(\boldsymbol{\eta}_1 \hat{\times} \boldsymbol{\eta}_2 / 2), \\ \mathbf{K}_{\text{boost}}(\boldsymbol{\varepsilon}_1)\mathbf{K}_{\text{boost}}(\boldsymbol{\varepsilon}_2) &= \mathbf{K}_{\text{boost}}(\boldsymbol{\varepsilon}_1 \hat{+} \boldsymbol{\varepsilon}_2)\mathbf{R}_{\text{int}}(\boldsymbol{\varepsilon}_1 \hat{\times} \boldsymbol{\varepsilon}_2 / 2).\end{aligned} \tag{109}$$

The geometric operators $\hat{\mathbf{K}}_{\text{pos}}(\boldsymbol{\eta}_{\text{sep}})$ and $\hat{\mathbf{K}}_{\text{vel}}(\boldsymbol{\varepsilon}_{\text{rap}})$ similarly have their own rotational operators

$$\mathbf{R}_{\text{pos}}(\boldsymbol{\omega}_{\text{pos}}) = \begin{pmatrix} R(\boldsymbol{\omega}_{\text{pos}}) & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \mathbf{R}_{\text{vel}}(\boldsymbol{\omega}_{\text{vel}}) = \begin{pmatrix} 1 & 0 \\ 0 & R(\boldsymbol{\omega}_{\text{vel}}) \end{pmatrix}. \tag{110}$$

They are connected with the interactive rotation operators by the equations

$$\mathbf{R}_{\text{int}}(\boldsymbol{\omega}_{\text{int}}) = \mathbf{R}_{\text{pos}}(\boldsymbol{\omega}_{\text{int}})\mathbf{R}_{\text{vel}}(\boldsymbol{\omega}_{\text{int}}), \qquad \mathbf{Q}(\boldsymbol{\upsilon}) = \mathbf{R}_{\text{pos}}(\boldsymbol{\upsilon})\mathbf{R}_{\text{vel}}(\boldsymbol{\upsilon}), \tag{111}$$

with their inverse,

$$\mathbf{R}_{\text{pos}}(\mathrm{w}_{\text{pos}}) = \mathbf{R}_{\text{int}}(\mathrm{w}_{\text{pos}}/2)\mathbf{Q}(-\mathrm{w}_{\text{pos}}/2), \qquad \mathbf{R}_{\text{vel}}(\mathrm{w}_{\text{vel}}) = \mathbf{R}_{\text{int}}(\mathrm{w}_{\text{pos}}/2)\mathbf{Q}(\mathrm{w}_{\text{pos}}/2). \tag{112}$$

The rotation operator $\hat{\mathbf{Q}}(\boldsymbol{\upsilon})$ is new, representing effects that are undetectable when either position or velocity space is strictly flat. It has the novel property of generating rotations in opposing senses in the spaces of position and velocity. It will be called the "contrarotation" operator. Like ordinary rotations, this contrarotation will be quantized, and it may make a contribution to particle physics.

The contrarotation operator $\hat{\mathbf{Q}}(\boldsymbol{\upsilon})$ is generated only by the inhomogeneous product of a shift and a boost. In such a product we also encounter the double translation matrix of Eq. (107). The cross product of shifts and boosts is then

$$\begin{aligned}\mathrm{K}_{\text{shift}}(\boldsymbol{\eta}_{\text{shift}})\mathrm{K}_{\text{boost}}(\boldsymbol{\varepsilon}_{\text{boost}}) &= -\mathrm{K}_{\text{boost}}(\boldsymbol{\varepsilon}_{\text{boost}})\mathrm{K}_{\text{shift}}(\boldsymbol{\eta}_{\text{shift}}) \\ &= \mathrm{K}\left([\boldsymbol{\eta}_{\text{shift}} \hat{-} \boldsymbol{\varepsilon}_{\text{boost}}], [\boldsymbol{\eta}_{\text{shift}} \hat{+} \boldsymbol{\varepsilon}_{\text{boost}}]\right) \mathrm{Q}\left(\boldsymbol{\eta}_{\text{shift}} \hat{\times} \boldsymbol{\varepsilon}_{\text{boost}}/2\right).\end{aligned} \tag{113}$$

In a similar way, we can form products of the rotation operators $\mathbf{R}_{\text{int}}$ and $\mathbf{Q}$,

$$\mathbf{R}_{\text{int}}(\boldsymbol{\omega}_{\text{int}})\mathbf{Q}(\boldsymbol{\upsilon}) = -\mathbf{Q}(\boldsymbol{\upsilon})\mathbf{R}_{\text{int}}(\boldsymbol{\omega}_{\text{int}}) = \mathbf{F}(\boldsymbol{\omega}_{\text{int}},\boldsymbol{\upsilon})\mathbf{Q}(-\boldsymbol{\omega}\mathbin{\hat{\times}}\boldsymbol{\upsilon}/2), \tag{114}$$

where

$$\mathbf{F}(\boldsymbol{\omega}_{\text{int}},\boldsymbol{\upsilon}) := \mathbf{R}_{\text{pos}}(\boldsymbol{\omega}_{\text{int}}\mathbin{\hat{-}}\boldsymbol{\upsilon})\mathbf{R}_{\text{vel}}(\boldsymbol{\omega}_{\text{int}}\mathbin{\hat{+}}\boldsymbol{\upsilon}). \tag{115}$$

The full double Lorentz group in the doubly hyperbolic three-spaces of position and velocity has twelve independent parameters. In the interactive representation they are best taken as $(\boldsymbol{\eta}_{\text{shift}},\boldsymbol{\varepsilon}_{\text{boost}},\boldsymbol{\omega}_{\text{int}},\boldsymbol{\upsilon})$. In the geometric representation they fall into two sets corresponding to the normal subgroups of position and velocity, $(\boldsymbol{\eta}_{\text{sep}},\boldsymbol{\omega}_{\text{pos}})$ and $(\boldsymbol{\varepsilon}_{\text{rap}},\boldsymbol{\omega}_{\text{vel}})$.

E. The Connection between the Geometric and Interactive Representations

The full double Lorentz group in the doubly hyperbolic three-spaces of position and velocity has twelve independent parameters. In the interactive representation they are best taken as $(\boldsymbol{\eta}_{\text{shift}},\boldsymbol{\varepsilon}_{\text{boost}},\boldsymbol{\omega}_{\text{int}},\boldsymbol{\upsilon})$. In the geometric representation they fall into two sets corresponding to the normal subgroups of position and velocity, $(\boldsymbol{\eta}_{\text{sep}},\boldsymbol{\omega}_{\text{pos}})$ and $(\boldsymbol{\varepsilon}_{\text{rap}},\boldsymbol{\omega}_{\text{vel}})$. Any operation in the double Lorentz group can be expressed as a product of the four primary operators of either the interactive or the geometric representation. These can be put in the form, respectively, of

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_{\text{int}}(\boldsymbol{\eta}_{\text{shift}},\boldsymbol{\varepsilon}_{\text{boost}},\boldsymbol{\omega}_{\text{int}},\boldsymbol{\upsilon}) = \mathbf{K}_{\text{shift}}(\boldsymbol{\eta}_{\text{shift}})\mathbf{K}_{\text{boost}}(\boldsymbol{\varepsilon}_{\text{boost}})\mathbf{R}(\boldsymbol{\omega}_{\text{int}})\mathbf{Q}(\boldsymbol{\upsilon}) \tag{116}$$

and

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_{\text{geom}}(\boldsymbol{\eta}_{\text{sep}},\boldsymbol{\omega}_{\text{pos}},\boldsymbol{\varepsilon}_{\text{rap}},\boldsymbol{\omega}_{\text{vel}}) = \mathbf{K}_{\text{pos}}(\boldsymbol{\eta}_{\text{sep}})\mathbf{K}_{\text{vel}}(\boldsymbol{\varepsilon}_{\text{rap}})\mathbf{R}_{\text{pos}}(\boldsymbol{\omega}_{\text{pos}})\mathbf{R}_{\text{vel}}(\boldsymbol{\omega}_{\text{vel}}). \tag{117}$$

It is important to find the relationship between these two representations, so as to express the variables of either one in terms of the other. This can be done as follows.

Using the identities of Eqs. (106) and (111), taking advantage of the commutation properties of the geometric operators, and defining the new rotation vector

$$\boldsymbol{\beta} = \left(\boldsymbol{\eta}_{\text{sep}} \hat{\times} \boldsymbol{\varepsilon}_{\text{rap}} / 2\right), \tag{118}$$

we can convert the translational factors on the right hand side of Eq. (116) into the expression

$$\mathbf{K}_{\text{shift}}(\boldsymbol{\eta}_{\text{shift}})\mathbf{K}_{\text{boost}}(\boldsymbol{\varepsilon}_{\text{boost}}) = \mathbf{K}_{\text{pos}}(\boldsymbol{\eta}_{\text{shift}} \hat{-} \boldsymbol{\varepsilon}_{\text{boost}})\mathbf{K}_{\text{vel}}(\boldsymbol{\eta}_{\text{shift}} \hat{+} \boldsymbol{\varepsilon}_{\text{boost}})\mathbf{R}_{\text{pos}}(-\boldsymbol{\beta})\mathbf{R}_{\text{vel}}(\boldsymbol{\beta}) \tag{119}$$

and the rotational factors into

$$\mathbf{R}_{\text{int}}(\boldsymbol{\omega}_{\text{int}})\mathbf{Q}(\boldsymbol{\upsilon}) = \mathbf{R}_{\text{pos}}(\boldsymbol{\omega}_{\text{int}} \hat{-} \boldsymbol{\upsilon})\mathbf{R}_{\text{vel}}(\boldsymbol{\omega}_{\text{int}} \hat{+} \boldsymbol{\upsilon})\mathbf{R}_{\text{pos}}(-\boldsymbol{\omega}_{\text{int}} \hat{\times} \boldsymbol{\upsilon}/2)\mathbf{R}_{\text{vel}}(\boldsymbol{\omega}_{\text{int}} \hat{\times} \boldsymbol{\upsilon}/2). \tag{120}$$

If we use the commutation properties again we can write Eq. (116) as

$$\begin{aligned}\boldsymbol{\Lambda}_{\text{int}}(\boldsymbol{\eta}_{\text{shift}},\boldsymbol{\varepsilon}_{\text{boost}},\boldsymbol{\omega}_{\text{int}},\boldsymbol{\upsilon}) &= \mathbf{K}_{\text{pos}}(\boldsymbol{\eta}_{\text{shift}} \hat{-} \boldsymbol{\varepsilon}_{\text{boost}})\mathbf{K}_{\text{vel}}(\boldsymbol{\eta}_{\text{shift}} \hat{+} \boldsymbol{\varepsilon}_{\text{boost}}) \\ &\times\left[\mathbf{R}_{\text{pos}}(-\boldsymbol{\beta})\mathbf{R}_{\text{pos}}(\boldsymbol{\omega}_{\text{int}})\mathbf{R}_{\text{pos}}(-\boldsymbol{\upsilon})\right]\left[\mathbf{R}_{\text{vel}}(\boldsymbol{\beta})\mathbf{R}_{\text{vel}}(\boldsymbol{\omega}_{\text{int}})\mathbf{R}_{\text{vel}}(\boldsymbol{\upsilon})\right].\end{aligned} \tag{121}$$

We can now identify the expressions of Eqs. (117) and (121) factor by factor. The parameters of hyperbolic arc in the two representations are connected by the simple expressions

$$\boldsymbol{\eta}_{\text{sep}} = \boldsymbol{\eta}_{\text{shift}} \hat{-} \boldsymbol{\varepsilon}_{\text{boost}}, \quad \boldsymbol{\varepsilon}_{\text{rap}} = \boldsymbol{\varepsilon}_{\text{boost}} \hat{+} \boldsymbol{\eta}_{\text{shift}}. \tag{122}$$

Their inverse is

$$\boldsymbol{\eta}_{\text{shift}} = \left(\boldsymbol{\eta}_{\text{sep}} \hat{+} \boldsymbol{\varepsilon}_{\text{rap}}\right)/2, \quad \boldsymbol{\varepsilon}_{\text{boost}} = \left(\boldsymbol{\varepsilon}_{\text{rap}} \hat{-} \boldsymbol{\eta}_{\text{sep}}\right)/2. \tag{123}$$

The angular coordinates have the connection formulas

$$\begin{aligned}\mathbf{R}_{\text{pos}}(\boldsymbol{\omega}_{\text{pos}}) &= \mathbf{R}_{\text{pos}}(-\boldsymbol{\beta})\mathbf{R}_{\text{pos}}(\boldsymbol{\omega}_{\text{int}})\mathbf{R}_{\text{pos}}(-\boldsymbol{\upsilon}), \\ \mathbf{R}_{\text{vel}}(\boldsymbol{\omega}_{\text{vel}}) &= \mathbf{R}_{\text{vel}}(\boldsymbol{\beta})\mathbf{R}_{\text{vel}}(\boldsymbol{\omega}_{\text{int}})\mathbf{R}_{\text{vel}}(\boldsymbol{\upsilon}).\end{aligned} \tag{124}$$

so that we can write their relationships as

$$\boldsymbol{\omega}_{\text{pos}} = \boldsymbol{\omega}_{\text{int}} \hat{-} \boldsymbol{\zeta}, \quad \boldsymbol{\omega}_{\text{vel}} = \boldsymbol{\omega}_{\text{int}} \hat{+} \boldsymbol{\zeta}, \text{ where} \tag{125}$$

$$\left(\boldsymbol{\omega}_{\text{vel}} \hat{-} \boldsymbol{\omega}_{\text{pos}}\right)/2 = \boldsymbol{\zeta} = \boldsymbol{\upsilon} \hat{+} \boldsymbol{\beta} \hat{+} \left(\boldsymbol{\omega}_{\text{int}} \hat{\times} [\boldsymbol{\upsilon} \hat{-} \boldsymbol{\beta}]/2\right)\ldots, \quad \text{and} \tag{126}$$

$$\left(\boldsymbol{\omega}_{\text{pos}} \hat{+} \boldsymbol{\omega}_{\text{vel}}\right)/2 = \boldsymbol{\omega}_{\text{int}}\,. \tag{127}$$

The relationship of Eqs. (122) and (123) connecting the hyperbolic arcs of position and velocity with those of shift and boost is independent of the angular coordinates. It is the hyperbolic equivalent of a linear combination to form the sum and difference of two vectors. It can be used to rewrite the product relationship of Eq. (119) in the useful form:

$$\mathbf{K}_{\text{shift}}\left(\boldsymbol{\eta}_{\text{shift}}\right)\mathbf{K}_{\text{boost}}\left(\boldsymbol{\varepsilon}_{\text{boost}}\right) = \mathbf{K}_{\text{pos}}\left(\boldsymbol{\eta}_{\text{sep}}\right)\mathbf{K}_{\text{vel}}\left(\boldsymbol{\varepsilon}_{\text{rap}}\right)\mathbf{Q}(\boldsymbol{\beta}), \tag{128}$$

where $\boldsymbol{\beta}$ can now be expressed in either of two forms,

$$\boldsymbol{\beta} = \left(\boldsymbol{\eta}_{\text{sep}} \hat{\times} \boldsymbol{\varepsilon}_{\text{rap}}/2\right) = \left(\boldsymbol{\eta}_{\text{shift}} \hat{\times} \boldsymbol{\varepsilon}_{\text{boost}}\right). \tag{64}$$

The connections of the angular coordinates are given by Eq. (124) depend not only on the angular variables $\boldsymbol{\omega}_{\text{int}}$ and $\boldsymbol{\upsilon}$ but also on the hyperbolic arcs through their product $\boldsymbol{\beta}$. Because $\boldsymbol{\beta}$ and $\boldsymbol{\upsilon}$ are both usually very small it is particularly convenient to use them as the parameters of the expansion, Eq. (126).

F. Finite Symmetries in the Hyperbolic Poincaré Group

In the Lorentz group the finite subgroup D_2 can be based on the four-by-four matrices of parity and time-reversal as generators:

$$P = \begin{pmatrix} 1 & \mathbf{0}^{\mathbf{T}} \\ 0 & -\mathbf{I}_3 \end{pmatrix}, \quad T = \begin{pmatrix} -1 & \mathbf{0}^{\mathbf{T}} \\ 0 & \mathbf{I}_3 \end{pmatrix}. \tag{130}$$

The corresponding finite subgroup in the double Lorentz group has the structure $D_2 \otimes D_2$. Two of its four generators are operators that formally resemble P and T. They can be realized by the matrices

$$\mathbf{P}_{\text{int}} = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}, \quad \mathbf{T}_{\text{int}} = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}. \tag{131}$$

These obey the usual relationships

$$\mathbf{P}_{\text{int}}{}^{2} = \mathbf{T}_{\text{int}}{}^{2} = \mathbf{I}_{8}, \quad \mathbf{P}_{\text{int}}\mathbf{T}_{\text{int}} = \mathbf{T}_{\text{int}}\mathbf{P}_{\text{int}} = -\mathbf{I}_{8} = \mathbf{C}. \tag{132}$$

To these operators we can add as a third generator the operator of kinematic symmetry

$$\mathbf{K} = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix} = \mathbf{Ch} = \mathbf{hC}, \tag{133}$$

where $\mathbf{h}$ is the operator

$$\mathbf{h} = \begin{Bmatrix} -\mathbf{1}_4 & 0 \\ 0 & \mathbf{1}_4 \end{Bmatrix}. \tag{134}$$

A fourth generator of $D_2 \otimes D_2$ is still needed. It can be taken as any one of the four operators

$$\begin{aligned} \mathbf{P}_{\text{vel}} &= \begin{pmatrix} I_4 & 0 \\ 0 & P \end{pmatrix}, \qquad \mathbf{T}_{\text{vel}} = \mathbf{KP}_{\text{vel}} = \mathbf{P}_{\text{vel}}\mathbf{K} = \begin{pmatrix} I_4 & 0 \\ 0 & T \end{pmatrix}, \\ \mathbf{P}_{\text{pos}} &= \mathbf{P}_{\text{II}}\mathbf{P}_{\text{vel}} = \mathbf{P}_{\text{vel}}\mathbf{P}_{\text{II}} = \begin{pmatrix} P & 0 \\ 0 & I_4 \end{pmatrix}, \qquad \mathbf{T}_{\text{pos}} = \mathbf{hP}_{\text{pos}} = \mathbf{P}_{\text{pos}}\mathbf{h} = \begin{pmatrix} T & 0 \\ 0 & I_4 \end{pmatrix}. \end{aligned} \tag{135}$$

These four are just the parity and time-reversal operators of the respective Lorentz geometric subgroups of position and velocity.

The operator known as "time-reversal" appears in the twelve-parameter double Lorentz group whose domain of operation is position and velocity space but not the space of time itself. Its role in the reversal of time is seen more fully in the full hyperbolic Poincaré group and in the Poincaré group of Minkowski space as the vehicle for the local description of hyperbolic space in the asymptotic limit. It operates on the local, differential variables of observer time τ, and not on the absolute proper time □ itself.

VII. The Time-Dependent Light Speed and its Consequences

In the doubly hyperbolic kinematic system the usual Poincaré group must be replaced by the larger hyperbolic Poincaré group. The larger group must connect with the more familiar one by an appropriate limiting process that leads to the flat space Minkowski limit. An important feature that must be maintained in the matrices of both groups is the orthonormal property. In the hyperbolic Poincaré group this property must also subsist in the matrices $\mathbf{Z}(t)$ of the time-translation subgroup. It is this condition that enforces the constancy of the product $c(t)\rho(t)=\sigma$, Eq. (96). From that in turn, combined with the Hubble condition $c(t)/\rho(t)=H(t)$, Eq. (95), the cosmological time-dependence of c necessarily follows.

By Noether's theorem, the constancy of mass can be taken as connected with the time-translation symmetry of the system. The invariance of angular momenta is similarly associated with the spatial homogeneity and isotropy of the system. The dimensionality of the Hubble-Lorentz constant σ, $\left[l^2t^{-1}\right]$, show that it is a coefficient of proportionality between mass and angular momentum or action. Since these are both conserved quantities, σ is appropriately constant. It follows from this proportionality that quantized angular momentum is associated with its own rest mass,

$$m_{\hbar/2}=\hbar/2\sigma\cong 2\cdot 10^{-76}\,\mathrm{g}\cong 2\cdot 10^{-49}m_e\,. \tag{136}$$

Photons will then carry a minute rest mass associated with the angular momentum of the transition generating them.

In symmetric special relativity we can maintain the assumption that mass is constant. Energy then is not. Interaction energies are better presented as interaction masses. The electromagnetic mass of the Coulomb interaction can be written

$$\Delta M_{\text{e.m.}}=\frac{e^2}{r_{12}c^2}=\frac{e^2}{\rho(t)c^2(t)\sinh\eta_{12}}=\frac{e^2}{c(t)\sigma\sinh\eta_{12}}, \tag{137}$$

from which it follows that $e^2/c(t)$ must be constant and that e must be decreasing cosmologically as $t^{-1/2}$. The electromagnetic fine-structure constant $\alpha_{\text{e.m.}}=e^2/\hbar c(t)$ remains constant.

Similarly the interaction mass of the gravitational interaction can be presented as

$$\Delta M_{\text{grav}}=-\frac{m_1m_2G}{r_{12}c^2}=\frac{-m_1m_2G}{c(t)\sigma\sinh\eta_{12}}. \tag{138}$$

Here, if masses are constant $G/c(t)$ must be constant and G must be decreasing cosmologically as t^{-1}. It follows also that the Planck length

$l_{\text{Planck}} = \left(\hbar G / c^3\right)^{1/2}$ is not constant but expanding as $t^{1/2}$. The cosmologically time-invariant Planck parameters are the Planck time $t_{\text{Planck}} = l_{\text{Planck}} / c = \left(\hbar G / c^5\right)^{1/2} = 5.37 \times 10^{-44}\,\text{sec}$ and the Planck mass $m_{\text{Planck}} = (\hbar c / G)^{1/2} = 1.22 \times 10^{19}\,\text{GeV}$.

In the expanding hyperbolic system the geometry of the light cone is usefully presented in terms of the hyperbolic separation variable η, invariant to the changing length scale $\rho(t)$. The relevant differential equation is $\rho(t)d\eta = c(t)dt$. Its integral

$$\Delta\eta = \int c(t)\rho^{-1}(t)dt = \int H(t)\,dt \tag{139}$$

does not require separate knowledge of the time dependence of $c(t)$. It can be evaluated if we use the simple approximation $H^{-1}(t) = t$ of the linear Hubble expansion:

$$\eta(t_2) - \eta(t_1) = \int_{t_1}^{t_2} t^{-1}dt = \ln(t_2 / t_1)\,. \tag{140}$$

As has long been known, a cosmologically decreasing speed of light can provide an alternative answer to the horizon problem of cosmology.

VIII. The Structure of the Hyperbolic Poincaré Group

The hyperbolic Poincaré group is the direct product of a one-dimensional time-translation subgroup $\mathcal{T}_{\text{time}}(1)$, whose parameter is the cosmological proper time, and the twelve-parameter double Lorentz subgroup $\mathcal{L}^2$.

$$\mathcal{P}_{\text{H}} = \mathcal{T}_{\text{time}}(1) \otimes \mathcal{L}^2\,. \tag{141}$$

The double Lorentz subgroup itself can be written as a direct product of the two ordinary Lorentz subgroups of the geometric representation, $\mathcal{L}_{\text{vel}}$ in hyperbolic velocity space and $\mathcal{L}_{\text{pos}}$ in hyperbolic position space:

$$\mathcal{L}^2 = \mathcal{L}_{\text{pos}} \otimes \mathcal{L}_{\text{vel}} = \otimes\mathcal{O}(3,1)_{\text{pos}} \otimes \mathcal{O}(3,1)_{\text{vel}}\,. \tag{142}$$

The full hyperbolic Poincaré group can then be expressed as the triple direct product

$$\mathcal{P}_{\mathrm{H}} = \mathcal{T}_{\mathrm{time}}(1) \otimes \mathcal{L}^2 = \mathcal{T}_{\mathrm{time}}(1) \otimes \mathcal{O}(3,1)_{\mathrm{pos}} \otimes \mathcal{O}(3,1)_{\mathrm{vel}}. \tag{143}$$

The Lie algebra of the double Lorentz group $\mathcal{L}^2$ is most simply developed in this geometric representation. However, the operators of two additional Lorentz subgroups, the boost and shift subgroups $\mathcal{L}_{\mathrm{boost}}$ and $\mathcal{L}_{\mathrm{shift}}$ of the interactive representation, are of great physical importance, being required for both the Lorentz transformation and the Hubble effect.

Each of the geometric Lorentz groups is itself the semidirect product of its own rotational subgroup $\mathcal{R}_{\mathrm{pos}}(3)$ or $\mathcal{R}_{\mathrm{vel}}(3)$, which I will now express as $\mathcal{J}_{\mathrm{pos}}$ and $\mathcal{J}_{\mathrm{vel}}$, and its translational manifold of hyperbolic three-space, $\breve{\mathcal{K}}_{\mathrm{pos}}(3)$ or $\breve{\mathcal{K}}_{\mathrm{vel}}(3)$, a subset but not a group:

$$\mathcal{L}_{\mathrm{pos}} = \breve{\mathcal{K}}_{\mathrm{pos}} \wedge \mathcal{J}_{\mathrm{pos}}; \qquad \mathcal{L}_{\mathrm{vel}} = \breve{\mathcal{K}}_{\mathrm{vel}} \wedge \mathcal{J}_{\mathrm{vel}}. \tag{144}$$

We can now reexpress the double Lorentz group as the semidirect product

$$\mathcal{L}^2{}_{\mathrm{geom}} = \mathcal{L}_{\mathrm{pos}} \otimes \mathcal{L}_{\mathrm{vel}} = \breve{\mathcal{K}}^2{}_{\mathrm{geom}} \wedge \mathcal{R}^2{}_{\mathrm{geom}} \tag{145}$$

of a six-parameter translational double manifold

$$\breve{\mathcal{K}}^2{}_{\mathrm{geom}} = \breve{\mathcal{K}}_{\mathrm{pos}}(3) \wedge \breve{\mathcal{K}}_{\mathrm{vel}}(3) \tag{146}$$

and a double rotation subgroup

$$\mathcal{R}^2{}_{\mathrm{geom}} = \mathcal{J}_{\mathrm{pos}} \otimes \mathcal{J}_{\mathrm{vel}}. \tag{147}$$

In the geometric representation not only are there six dimensions of translation, three in position and three in velocity, but we must also recognize three rotational degrees of freedom in each of these three-spaces. When position and velocity spaces are both curved the subgroups $\mathcal{J}_{\mathrm{pos}}$ and $\mathcal{J}_{\mathrm{vel}}$ are clearly distinguished, but as one or both of them approaches the pure flat space limit the difference between them becomes tenuous and they both appear to approach a single common limit $\mathcal{J}$. This is the reason for the reduction of the thirteen-parameter Poincaré group in curved position space to the ten-parameter ordinary Poincaré group when position space is flat.

The interaction representation of the double Lorentz group is formed by choosing a new set of generating operators for the group in the form of the products

$$\begin{aligned}&\hat{\mathbf{K}}_{\text{shift}}(\mathtt{h}_{\text{shift}}) := \hat{\mathbf{K}}_{\text{pos}}(\mathtt{h}_{\text{shift}})\hat{\mathbf{K}}_{\text{vel}}(\mathtt{h}_{\text{shift}}), \quad &&\hat{\mathbf{K}}_{\text{boost}}(\mathtt{e}_{\text{boost}}) := \hat{\mathbf{K}}_{\text{pos}}(-\mathtt{e}_{\text{boost}})\hat{\mathbf{K}}_{\text{vel}}(\mathtt{e}_{\text{boost}}),\\ &\hat{\mathbf{J}}(\mathtt{w}) := \hat{\mathbf{J}}_{\text{pos}}(\mathtt{w})\hat{\mathbf{J}}_{\text{vel}}(\mathtt{w}), &&\hat{\mathbf{Q}}(\mathtt{u}) := \hat{\mathbf{J}}_{\text{pos}}(-\mathtt{u})\hat{\mathbf{J}}_{\text{vel}}(\mathtt{u}).\end{aligned} \tag{148}$$

These are the operators of the interactional rotational subgroup $\mathcal{R}_{\text{int}}(3) = \mathcal{J}$, the contrarotational submanifold $\breve{\mathcal{Q}}(3)$ and the two translational submanifolds $\breve{\mathcal{K}}_{\text{shift}}(3), \breve{\mathcal{K}}_{\text{boost}}(3)$. From these we can form as semidirect products the double rotational group

$$\mathcal{R}^2{}_{\text{int}} = \mathcal{J} \wedge \breve{\mathcal{Q}} \tag{149}$$

and the double translational manifold

$$\breve{\mathcal{K}}^2{}_{\text{int}} = \breve{\mathcal{K}}_{\text{shift}} \wedge \breve{\mathcal{K}}_{\text{boost}}. \tag{150}$$

The interactive form of the double Lorentz group is then

$$\mathcal{L}^2{}_{\text{int}} = \breve{\mathcal{K}}^2{}_{\text{int}} \wedge \mathcal{R}^2{}_{\text{int}} = \breve{\mathcal{K}}_{\text{shift}} \wedge \breve{\mathcal{K}}_{\text{boost}} \wedge \mathcal{J} \wedge \breve{\mathcal{Q}}. \tag{151}$$

This explicitly displays the presence of the two interactive Lorentz subgroups

$$\mathcal{L}_{\text{shift}} = \breve{\mathcal{K}}_{\text{shift}} \wedge \mathcal{J}, \qquad \mathcal{L}_{\text{boost}} = \breve{\mathcal{K}}_{\text{boost}} \wedge \mathcal{J}, \tag{152}$$

which share the same rotational subgroup $\mathcal{J}$.

Unlike the geometric form of the double Lorentz group, the interactive form is not a direct product of its two Lorentz subgroup. Their overlapping product

$$\mathcal{L}_{\text{shift}} \circ \mathcal{L}_{\text{boost}} = \breve{\mathcal{K}}_{\text{shift}} \wedge \breve{\mathcal{K}}_{\text{boost}} \wedge \mathcal{J} \tag{153}$$

must be supplemented by the contrarotational manifold $\breve{\mathcal{Q}}$ to make up the entire group $\mathcal{L}^2$. This can now be presented as a product of its Lorentz subgroups in the two forms

$$\mathcal{L}^2 = \mathcal{L}_{\text{pos}} \otimes \mathcal{L}_{\text{vel}} = (\mathcal{L}_{\text{shift}} \circ \mathcal{L}_{\text{boost}}) \wedge \breve{\mathcal{Q}}. \tag{154}$$

The hyperbolic Poincaré group is the direct product of $\mathcal{L}^2$ with the one-dimensional translation group in cosmological time,

$$\mathcal{P}_{\mathrm{H}} = \mathcal{T}_{\mathrm{time}}(1) \otimes \mathcal{L}^2 . \tag{155}$$

To examine the transition between the full hyperbolic Poincaré group and the degenerate Poincaré group of ordinary use in Minkowski space we can write $\mathcal{P}_{\mathrm{H}}$ in the less symmetric form

$$\mathcal{P}_{\mathrm{H}} = \mathcal{T}_{\mathrm{time}}(1) \otimes \mathcal{L}^2 = \mathcal{T}_{\mathrm{time}}(1) \otimes \left(\mathcal{L}_{\mathrm{boost}} \wedge \breve{\mathcal{K}}_{\mathrm{shift}}(3) \wedge \breve{\mathcal{Q}} \right). \tag{156}$$

In the flat space limit the translational manifold becomes a subgroup, $\breve{\mathcal{K}}_{\mathrm{shift}}(3) \to \mathcal{T}_{\mathrm{pos}}(3)$, while the contrarotational manifold degenerates to the identity, $\breve{\mathcal{Q}} \to \mathbf{1}$, and can be omitted:

$$\mathcal{P}_{\mathrm{H}} \to \mathcal{T}_{\mathrm{time}}(1) \otimes \left(\mathcal{L}_{\mathrm{boost}} \otimes \mathcal{T}_{\mathrm{pos}}(3) \right) = \mathcal{P} . \tag{157}$$

In the description of the ordinary Poincaré group the four operations of translation can be represented in a combined subgroup, but the real-and-imaginary character of Minkowski four-space must be recognized by this as $\mathcal{T}(3,1)$, so that

$$\mathcal{P} = \mathcal{T}(3,1) \otimes \mathcal{O}(3,1). \tag{159}$$

IX. The Lie Algebra of the Hyperbolic Poincaré Group

1. The Lie Algebra of the Operators of the Geometric Representation

The Lorentz matrices of rank 4, from which we shall build the operators of the eight-by-eight matrices of the double Lorentz group, can be represented in a standard notation (see [7,8,9], for example). They will be parametrized here by the hyperbolic translation vectors of rapidity $\boldsymbol{\varepsilon} = (\varepsilon, \theta_{\boldsymbol{\varepsilon}}, \phi_{\boldsymbol{\varepsilon}})$ or separation $\boldsymbol{\eta} = (\eta, \theta_{\boldsymbol{\eta}}, \phi_{\boldsymbol{\eta}})$ with their associated rotations of parallel transport $\boldsymbol{\omega}_{\boldsymbol{\varepsilon}} = (\omega_{\boldsymbol{\varepsilon}}; \kappa_{\boldsymbol{\varepsilon}}, \lambda_{\boldsymbol{\varepsilon}})$ or $\boldsymbol{\omega}_{\boldsymbol{\eta}} = (\omega_{\boldsymbol{\eta}}; \kappa_{\boldsymbol{\eta}}, \lambda_{\boldsymbol{\eta}})$, expressed as vectors in the angle and axis notation. Infinitesimally these can be can be expressed in rectangular coordinates, $\delta\boldsymbol{\varepsilon} = \{\delta\varepsilon^j\}$, $\delta\boldsymbol{\eta} = \{\delta\eta^j\}$ and $\delta\boldsymbol{\omega}_{\boldsymbol{\varepsilon}} = \{\delta\omega_{\boldsymbol{\varepsilon}}{}^j\}$, $\delta\boldsymbol{\omega}_{\boldsymbol{\eta}} = \{\delta\omega_{\boldsymbol{\eta}}{}^j\}$. Their Lie algebra is based on differential expressions of the form

$$\hat{L}(\delta\boldsymbol{\varepsilon},\delta\boldsymbol{\omega}_{\boldsymbol{\varepsilon}}) = \mathbf{1} - i\,\delta\varepsilon^{j}\hat{K}_{j} - i\delta\omega_{\boldsymbol{\varepsilon}}{}^{j}\hat{J}_{j}, \qquad (159)$$

where the translational three-vectors $\{\hat{K}_m\}$ and $\{\hat{J}_m\}$ are composed of four-by-four matrix operators. These matrices are the generators of the homogeneous Lorentz group:

$$\hat{K}_1 = -i\begin{bmatrix} 0 & \begin{pmatrix}1 & 0 & 0\end{pmatrix} \\ \begin{pmatrix}1\\0\\0\end{pmatrix} & \mathbf{0}\end{bmatrix},\quad \hat{K}_2 = -i\begin{bmatrix} 0 & \begin{pmatrix}0 & 1 & 0\end{pmatrix} \\ \begin{pmatrix}0\\1\\0\end{pmatrix} & \mathbf{0}\end{bmatrix},\quad \hat{K}_3 = -i\begin{bmatrix} 0 & \begin{pmatrix}0 & 0 & 1\end{pmatrix} \\ \begin{pmatrix}0\\0\\1\end{pmatrix} & \mathbf{0}\end{bmatrix}; \qquad (160)$$

$$\hat{J}_1 = -i\begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \begin{pmatrix}0 & 0 & 0\\0 & 0 & -1\\0 & 1 & 0\end{pmatrix}\end{bmatrix},\quad \hat{J}_2 = -i\begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \begin{pmatrix}0 & 0 & 1\\0 & 0 & 0\\-1 & 0 & 0\end{pmatrix}\end{bmatrix},\quad \hat{J}_3 = -i\begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \begin{pmatrix}0 & -1 & 0\\1 & 0 & 0\\0 & 0 & 0\end{pmatrix}\end{bmatrix}.$$

From these expressions we can obtain the usual commutators of $\hat{K}_m$ and $\hat{J}_m$,

$$\left[\hat{K}_m,\hat{K}_n\right] = -i\,\varepsilon^{mnl}\hat{J}_l,\quad \left[\hat{K}_m,\hat{J}_n\right] = i\,\varepsilon^{mnl}\hat{K}_l,\quad \left[\hat{J}_m,\hat{J}_n\right] = i\varepsilon^{mnl}\hat{J}_l. \qquad (161)$$

In the double Lorentz group we must deal with matrices of rank eight. In position-velocity representation the generators are the eight-by-eight matrices

$$\hat{\mathbf{K}}_{\mathrm{pos},m} = \begin{Bmatrix}\hat{K}_m & \mathbf{0}\\ \mathbf{0} & I_4\end{Bmatrix},\qquad \hat{\mathbf{J}}_{\mathrm{pos},m} = \begin{Bmatrix}\hat{J}_m & \mathbf{0}\\ \mathbf{0} & I_4\end{Bmatrix},$$
$$\hat{\mathbf{K}}_{\mathrm{vel},m} = \begin{Bmatrix}I_4 & \mathbf{0}\\ \mathbf{0} & \hat{K}_m\end{Bmatrix},\qquad \hat{\mathbf{J}}_{\mathrm{vel},m} = \begin{Bmatrix}I_4 & \mathbf{0}\\ \mathbf{0} & \hat{J}_m\end{Bmatrix}. \qquad (162)$$

The infinitesimal operations that generate these subgroups can be expressed as

$$\mathbf{L}_{\mathrm{pos}}(\delta\boldsymbol{\eta}_{\mathrm{sep}},\delta\boldsymbol{\omega}_{\mathrm{pos}}) = \exp\left(-i\delta\eta_{\mathrm{sep}}{}^{j}\hat{\mathbf{K}}_{\mathrm{pos},j} - i\,\delta\omega_{\mathrm{pos}}{}^{j}\hat{\mathbf{J}}_{\mathrm{pos},j}\right),$$
$$\mathbf{L}_{\mathrm{vel}}(\delta\boldsymbol{\varepsilon}_{\mathrm{rap}},\delta\boldsymbol{\omega}_{\mathrm{vel}}) = \exp\left(-i\,\delta\varepsilon_{\mathrm{rap}}{}^{j}\hat{\mathbf{K}}_{\mathrm{vel},j} - i\,\delta\omega_{\mathrm{vel}}{}^{j}\hat{\mathbf{J}}_{\mathrm{vel},j}\right). \qquad (163)$$

The Lie algebra of each of the two subgroups is just that of the elementary Lorentz group, and the operators of the position subgroup commute with those of the velocity subgroup. In this geometric representation almost all the relationships of the Lie algebra can be obtained by inspection immediately from those of the elementary Lorentz group.

2. The Lie Algebra of the Shift and Boost Operators

We can now establish the Lie algebra of the double Lorentz group $\mathcal{L}^2$, in the physically useful interactive representation, using the eighth-rank matrix form of the operators $\hat{\mathbf{L}}_{\text{shift}}$ and $\hat{\mathbf{L}}_{\text{boost}}$.

In this representation of the double Lorentz group the translational matrices of rank eight are expressed as

$$\hat{\mathbf{K}}_{\text{shift}}\left(\delta\mathrm{h}_{\text{shift}}\right)=\begin{Bmatrix} K\left(\delta\mathrm{h}_{\text{shift}}\right) & \mathbf{0} \\ \mathbf{0} & K\left(\delta\mathrm{h}_{\text{shift}}\right)\end{Bmatrix},\ \hat{\mathbf{K}}_{\text{boost}}\left(\delta\mathrm{e}_{\text{boost}}\right)=\begin{Bmatrix} K\left(-\delta\mathrm{e}_{\text{boost}}\right) & \mathbf{0} \\ \mathbf{0} & K\left(\delta\mathrm{e}_{\text{boost}}\right)\end{Bmatrix}, \quad (164)$$

and the associated rotational matrices are those of the angular momentum of the entire system, which will now be denoted simply as $\hat{\mathbf{J}}$, together with the contra-angular momentum $\hat{\mathbf{Q}}$:

$$\hat{\mathbf{J}}(\delta\boldsymbol{\omega})=\begin{Bmatrix} \hat{J}(\delta\boldsymbol{\omega}) & \mathbf{0} \\ \mathbf{0} & \hat{J}(\delta\boldsymbol{\omega})\end{Bmatrix},\quad \hat{\mathbf{Q}}(\delta\boldsymbol{\upsilon})=\begin{Bmatrix} \hat{J}(-\delta\boldsymbol{\upsilon}) & \mathbf{0} \\ \mathbf{0} & \hat{J}(\delta\boldsymbol{\upsilon})\end{Bmatrix}. \quad (165)$$

The full Lorentz operators of the shift and boost subgroups share the same angular momentum operator $\hat{\mathbf{J}}$, and are independent of $\hat{\mathbf{Q}}$:

$$\hat{\mathbf{L}}_{\text{shift}}\left(\delta\boldsymbol{\eta}_{\text{shift}},\delta\boldsymbol{\omega}\right)=\hat{\mathbf{K}}_{\text{shift}}\left(\delta\boldsymbol{\eta}_{\text{shift}}\right)\hat{\mathbf{J}}(\delta\boldsymbol{\omega}),\quad \hat{\mathbf{L}}_{\text{boost}}\left(\delta\boldsymbol{\varepsilon}_{\text{boost}},\delta\boldsymbol{\omega}\right)=\hat{\mathbf{K}}_{\text{boost}}\left(\delta\boldsymbol{\varepsilon}_{\text{boost}}\right)\hat{\mathbf{J}}(\delta\boldsymbol{\omega}). \quad (166)$$

The manifold of the operator $\hat{\mathbf{Q}}$ is populated only by cross products between the shift and boost operators like $\hat{\mathbf{K}}_{\text{shift}}\left(\boldsymbol{\eta}_{\text{shift}}\right)\hat{\mathbf{K}}_{\text{boost}}\left(\boldsymbol{\varepsilon}_{\text{boost}}\right)$ and lies entirely outside the shift and boost Lorentz subgroups themselves.

In the interactive representation the infinitesimal transformations of the double Lorentz group become

$$\hat{\mathbf{L}}_{\text{int}}\left(\delta\boldsymbol{\eta}_{\text{shift}},\delta\boldsymbol{\varepsilon}_{\text{boost}},\delta\boldsymbol{\omega},\delta\boldsymbol{\upsilon}\right)=\mathbf{1}-i\delta\eta_{\text{shift}}{}^{j}\hat{\mathbf{K}}_{\text{shift},j}-i\delta\varepsilon_{\text{boost}}{}^{j}\hat{\mathbf{K}}_{\text{boost},j}-i\delta\omega^{j}\hat{\mathbf{J}}_{j}-i\delta\upsilon^{j}\hat{\mathbf{Q}}_{j}. \quad (167)$$

It is easy to see that the generators are twelve matrices of rank eight, occurring as four three-vectors (with $i = 1,2,3$):

$$\hat{\mathbf{K}}_{\text{shift},j} = \begin{Bmatrix} \hat{K}_j & \mathbf{0} \\ \mathbf{0} & \hat{K}_j \end{Bmatrix}, \quad \hat{\mathbf{K}}_{\text{boost},j} = \begin{Bmatrix} -\hat{K}_j & \mathbf{0} \\ \mathbf{0} & \hat{K}_j \end{Bmatrix}, \quad \hat{\mathbf{J}}_j = \begin{Bmatrix} \hat{J}_j & \mathbf{0} \\ \mathbf{0} & \hat{J}_j \end{Bmatrix}, \quad \hat{\mathbf{Q}}_j = \begin{Bmatrix} -\hat{J}_j & \mathbf{0} \\ \mathbf{0} & \hat{J}_j \end{Bmatrix}. \tag{168}$$

The commutators belonging to the shift and boost subgroups are in each case isomorphic with those of the ordinary Lorentz group. They include the usual angular momentum commutator

$$\left[\hat{\mathbf{J}}_m, \hat{\mathbf{J}}_n\right] = i\varepsilon^{mnl}\hat{\mathbf{J}}_l, \tag{169}$$

the shift commutators

$$\text{(a)} \ \left[\hat{\mathbf{K}}_{\text{shift},m}, \hat{\mathbf{K}}_{\text{shift},n}\right] = -i\varepsilon^{mnl}\hat{\mathbf{J}}_l, \quad \text{(b)} \ \left[\hat{\mathbf{K}}_{\text{shift},m}, \hat{\mathbf{J}}_n\right] = i\varepsilon^{mnl}\hat{\mathbf{K}}_{\text{shift},l}, \tag{170}$$

and the boost commutators

$$\text{(a)} \ \left[\hat{\mathbf{K}}_{\text{boost},m}, \hat{\mathbf{K}}_{\text{boost},n}\right] = -i\varepsilon^{mnl}\hat{\mathbf{J}}_l, \quad \text{(b)} \ \left[\hat{\mathbf{K}}_{\text{boost},m}, \hat{\mathbf{J}}_n\right] = i\varepsilon^{mnl}\hat{\mathbf{K}}_{\text{boost},l}. \tag{171}$$

The remaining commutators involve the vector operator $\hat{\mathbf{Q}}$ and are outside the Lorentz subgroups. They are

$$\begin{aligned} &\text{(a)} \ \left[\hat{\mathbf{K}}_{\text{shift},m}, \hat{\mathbf{K}}_{\text{boost},n}\right] = -i\varepsilon^{mnl}\hat{\mathbf{Q}}_l, \\ &\text{(b)} \ \left[\hat{\mathbf{K}}_{\text{shift},m}, \hat{\mathbf{Q}}_n\right] = i\varepsilon^{mnl}\hat{\mathbf{K}}_{\text{boost},l}, \quad \text{(c)} \ \left[\hat{\mathbf{K}}_{\text{boost},m}, \hat{\mathbf{Q}}_n\right] = i\varepsilon^{mnl}\hat{\mathbf{K}}_{\text{shift},l}, \\ &\text{(d)} \ \left[\hat{\mathbf{J}}_m, \hat{\mathbf{Q}}_n\right] = i\varepsilon^{mnl}\hat{\mathbf{Q}}_l, \qquad \text{(e)} \ \left[\hat{\mathbf{Q}}_m, \hat{\mathbf{Q}}_n\right] = i\varepsilon^{mnl}\hat{\mathbf{J}}_l. \end{aligned} \tag{172}$$

For the angular momentum algebra of the system we can also define the operators
$\hat{\mathbf{J}}^2$ and $\hat{\mathbf{Q}}^2$. These are Casimir operators for the entire double rotation group $\mathcal{R}^2$, the rotational subgroup of the double Lorentz group $\mathcal{L}^2$, with the vanishing commutators

$$\left[\hat{\mathbf{Q}}^2, \hat{\mathbf{J}}^2\right] = \left[\hat{\mathbf{J}}_j, \hat{\mathbf{J}}^2\right] = \left[\hat{\mathbf{J}}_j, \hat{\mathbf{Q}}^2\right] = \left[\hat{\mathbf{Q}}_j, \hat{\mathbf{J}}^2\right] = \left[\hat{\mathbf{Q}}_j, \hat{\mathbf{Q}}^2\right] = 0. \tag{173}$$

In addition, by Eq. (6)(d) we also have

$$\left[\hat{\mathbf{J}}_j, \hat{\mathbf{Q}}_j\right] = 0. \tag{174}$$

The four operators $\left(\hat{\mathbf{J}}^2, \hat{\mathbf{J}}_j, \hat{\mathbf{Q}}^2, \hat{\mathbf{Q}}_j\right)$ thus form a mutually commuting set.

We can now define the usual raising and lowering operators for the ordinary angular momentum $\hat{\mathbf{J}}$ together with a set of analogous operators constructed from the contra-angular momentum $\hat{\mathbf{Q}}$ as well:

$$\hat{\mathbf{J}}_\pm := \left(\hat{\mathbf{J}}_1 \pm i\hat{\mathbf{J}}_2\right), \quad \hat{\mathbf{Q}}_\pm := \left(\hat{\mathbf{Q}}_1 \pm i\hat{\mathbf{Q}}_2\right), \tag{175}$$

They have various commutation relations:

$$\text{(a)}\ \left[\hat{\mathbf{J}}_3, \hat{\mathbf{J}}_\pm\right] = \pm\hat{\mathbf{J}}_\pm \quad \text{(b)}\ \left[\hat{\mathbf{J}}_+, \hat{\mathbf{J}}_-\right] = 2\hat{\mathbf{J}}_3; \tag{176}$$

$$\text{(a)}\ \left[\hat{\mathbf{Q}}_3, \hat{\mathbf{Q}}_\pm\right] = \pm\hat{\mathbf{J}}_\pm \quad \text{(b)}\ \left[\hat{\mathbf{Q}}_+, \hat{\mathbf{Q}}_-\right] = 2\hat{\mathbf{J}}_3; \tag{177}$$

$$\text{(a)}\ \left[\hat{\mathbf{Q}}_3, \hat{\mathbf{J}}_\pm\right] = \pm\hat{\mathbf{J}}_\pm, \quad \text{(b)}\ \left[\hat{\mathbf{Q}}_-, \hat{\mathbf{J}}_+\right] = 2\hat{\mathbf{J}}_3, \quad \text{(c)}\ \left[\hat{\mathbf{J}}_-, \hat{\mathbf{Q}}_+\right] = 2\hat{\mathbf{J}}_3, \quad \text{(d)}\ \left[\hat{\mathbf{J}}_3, \hat{\mathbf{Q}}_\pm\right] = \pm\hat{\mathbf{J}}_\pm. \tag{178}$$

We can now evaluate also

$$\hat{\mathbf{J}}_1{}^2 + \hat{\mathbf{J}}_2{}^2 = \left(\hat{\mathbf{J}}_+\hat{\mathbf{J}}_- + \hat{\mathbf{J}}_-\hat{\mathbf{J}}_+\right)/2 = \hat{\mathbf{J}}_+\hat{\mathbf{J}}_- + \hat{\mathbf{J}}_3 = \hat{\mathbf{J}}_-\hat{\mathbf{J}}_+ - \hat{\mathbf{J}}_3, \tag{179}$$

$$\hat{\mathbf{Q}}_1{}^2 + \hat{\mathbf{Q}}_2{}^2 = \left(\hat{\mathbf{Q}}_+\hat{\mathbf{Q}}_- + \hat{\mathbf{Q}}_-\hat{\mathbf{Q}}_+\right)/2 = \hat{\mathbf{Q}}_+\hat{\mathbf{Q}}_- + \hat{\mathbf{J}}_3 = \hat{\mathbf{Q}}_-\hat{\mathbf{Q}}_+ - \hat{\mathbf{J}}_3, \tag{180}$$

The operators of $\hat{\mathbf{J}}_\pm$ and $\hat{\mathbf{Q}}_\pm$ are seen to behave for the most part like other angular momentum operators. However, because of Eq. (4)(e) and its consequences Eq. (40)(a), the raising and lowering operators for $\hat{\mathbf{Q}}_3$ are not components of $\hat{\mathbf{Q}}$ itself, but they are identically $\hat{\mathbf{J}}_\pm$, those of the ordinary angular momentum.

As usual the operators $\hat{\mathbf{J}}^2$ and $\hat{\mathbf{J}}_3$ have the quantum numbers j, m_j and the eigenvalues $j(j+1)\hbar^2, m_j\hbar$. By using the raising and lowering operators $\hat{\mathbf{J}}_\pm$ we get for the operator $\hat{\mathbf{Q}}_3$ the integrally spaced quantum numbers m_q and eigenvalues $m_q\hbar$. It can then be shown that the eigenvalues of $\hat{\mathbf{Q}}^2$ are $q(q+1)\hbar^2$, and its quantum numbers are q, integer or half-integer, with the usual rules.

In a doubly hyperbolic geometry and kinematics, each particle may be labeled by all four of these spin and contraspin quantum numbers j, m_j, q, m_q. In the classical limit, correspondingly, dynamics in a doubly hyperbolic universe has available for each particle additional three degrees of freedom and three additional integrals of the motion, beyond those usually known, those associated with the contra-angular momentum vector $\hat{\mathbf{Q}}$.

We can now refer to the commutators to review the routes by which the various subsets in $\mathcal{P}_{\mathrm{H}}$ are populated. Rotation of a member of any of the subsets $\mathcal{R}, \breve{\mathcal{K}}_{\mathrm{shift}}, \breve{\mathcal{K}}_{\mathrm{boost}}, \breve{\mathcal{Q}}$ produces effects inside the subset exclusively, and can be neglected. The rotational subgroup $\mathcal{R}$ is populated directly by rotations and quadratically by the products $\hat{\mathbf{K}}_{\mathrm{shift}} \times \hat{\mathbf{K}}'_{\mathrm{shift}}, \hat{\mathbf{K}}_{\mathrm{boost}} \times \hat{\mathbf{K}}'_{\mathrm{boost}}, \hat{\mathbf{Q}} \times \hat{\mathbf{Q}}'$. The translational subset $\breve{\mathcal{K}}_{\mathrm{shift}}$ is populated directly by translations and quadratically only by the product $\hat{\mathbf{K}}_{\mathrm{boost}} \times \hat{\mathbf{Q}}$. The boost subset $\breve{\mathcal{K}}_{\mathrm{boost}}$ is populated directly by velocity boosts and quadratically only by the product $\hat{\mathbf{K}}_{\mathrm{shift}} \times \hat{\mathbf{Q}}$. For the quasirotational subset $\breve{\mathcal{Q}}$ a direct process of population does not occur within $\mathcal{P}_{\mathrm{H}}$, but the quadratic process $\hat{\mathbf{K}}_{\mathrm{boost}} \times \hat{\mathbf{K}}_{\mathrm{shift}}$ will populate it. The existence of this process is a fundamental consequence of three-space curvature, and makes it necessary to go beyond the apparatus of the ordinary Poincaré group. The hyperbolic Poincaré group provides a starting point for the study of other geometries in this regard.

3. Time Displacement and the Mass Operator

In the hyperbolic Poincaré group the twelve generators of the double Lorentz group are supplemented by the generator of time displacement. This displacement is in cosmological proper time, and the generator is a mass operator. It can be expressed through the elementary displacement

$$\Lambda(\delta t) = \mathbf{1}_8 + \left(\frac{\delta t}{2t_0}\right)\mathbf{M}; \qquad \mathbf{M} = \begin{pmatrix} \mathbf{1}_4 & 0 \\ 0 & -\mathbf{1}_4 \end{pmatrix}. \tag{181}$$

Unlike P^0 in the ordinary Poincaré group, $\mathbf{M}$ commutes with all the operators of the double Lorentz group. The complete infinitesimal displacement can now be written

$$\Lambda\left(\delta t;\delta\boldsymbol{\eta}_{\text{shift}},\delta\boldsymbol{\varepsilon}_{\text{boost}},\delta\boldsymbol{\omega},\delta\boldsymbol{\upsilon}\right) = \mathbf{1}+\left(\delta t/2t_0\right)\hat{\mathbf{M}}-i\,\delta\eta_{\text{shift}}{}^{j}\hat{\mathbf{K}}_{\text{shift},j}-i\,\delta\varepsilon_{\text{boost}}{}^{j}\hat{\mathbf{K}}_{\text{boost},j}-i\,\delta\omega^{j}\hat{\mathbf{J}}_{j}-i\delta\upsilon^{j}\hat{\mathbf{Q}}_{j}\,. \tag{182}$$

In the flat space limit the equivalent expression is

$$\Lambda(\delta t;\delta\mathbf{r},\delta\boldsymbol{\varepsilon},\delta\boldsymbol{\omega}) = \mathbf{1}+c\,\delta t\hat{P}_0 - i\delta r^{j}\hat{P}_j - i\delta\varepsilon^{j}\hat{K}_j - i\,\delta\omega^{j}\hat{J}_j\,. \tag{183}$$

Comparing Eqs. (47) (13) and (6), we can immediately connect the generators of the hyperbolic Poincaré group one to one with those of the ordinary Poincaré group, with the exception that the three contrarotation generators $\hat{\mathbf{Q}}_j$ effectively disappear in the transition to flat space.

4. The Casimir Operators of the Hyperbolic Poincaré Group

In the ordinary Poincaré group the Casimir operators are the total energy invariant and the Pauli-Lubanski invariant. Their generalization to the hyperbolic Poincaré group is facilitated by the identification of the generators illustrated by the parallels between Eqs. (47) and (6) above. However, since the Casimir operators must commute with all the operators of the group, it is important to employ the elements of the group in their simplest and most symmetrical form. For this purpose, we must look to the geometric representation of the double Lorentz group and not the interaction representation in which Eq. (4) is expressed.

The operators P_j are generators of translation in position space. In the geometric representation we must generalize from this and associate them with a subset of the operators $\hat{\mathbf{L}}_{\text{pos},\mu\nu}$ of the entire Lorentz subgroup $\mathcal{L}_{\text{pos}}$. This leaves the operators $\hat{\mathbf{L}}_{\text{vel},\mu\nu}$ of the subgroup $\mathcal{L}_{\text{vel}}$ to be identified with the operators $\hat{L}_{\mu\nu}$ of the homogeneous Lorentz subgroup of the ordinary Poincaré group, an obviously satisfactory arrangement.

a. The Invariant of Action and Mass

The first Casimir invariant is

$$W = \boldsymbol{\Lambda}_{\mu\nu}\boldsymbol{\Lambda}^{\mu\nu} = \mathbf{L}_{\text{pos}}{}^{\mu\nu}\mathbf{L}_{\text{pos},\mu\nu} + \mathbf{L}_{\text{vel}}{}^{\mu\nu}\mathbf{L}_{\text{vel},\mu\nu} = w_{\text{pos}} + w_{\text{vel}}\,, \tag{184}$$

where w_{pos} and w_{vel} are the invariants of the subgroups they belong to. Because all the elements of each of these geometric subgroups commute with those of the other, w_{pos} and w_{vel} are each Casimir invariants of the larger group.

If we multiply W by its associated mass and by the Hubble-Lorentz constant σ we get the mass or action invariant.

b. The Pauli-Lubanski Vector and its Invariant

In flat space the Pauli-Lubanski four-vector is the antisymmetric product

$$\Omega^{\lambda} = \varepsilon^{\lambda\mu\nu\rho} L_{\mu\nu} P_{\rho} / 2. \tag{185}$$

Replacing L by $\mathbf{L}_{\text{vel}}$ and P by $\mathbf{L}_{\text{pos}}$, and recognizing that four-vectors must be replaced by second-rank tensors, we can propose the structure

$$\mathbf{\Omega}^{\kappa\lambda} = \varepsilon^{\kappa\lambda\mu\nu\rho\sigma} \mathbf{L}_{\text{vel},\mu\nu} \mathbf{L}_{\text{pos},\rho\sigma} \tag{186}$$

for the Pauli-Lubanski tensor. Its invariant magnitude is

$$w_{\text{PL}}{}^{2} = \mathbf{\Omega}^{\kappa\lambda} \mathbf{\Omega}_{\kappa\lambda} . \tag{187}$$

The commutation properties of this invariant, as well as the possible role of a third Casimir invariant arising from w_{pos} and w_{vel}, deserve further investigation.

5. The Operator Q: Possibilities of Measurement.

The development of the hyperbolic Poincaré group has brought to light a new type of angular momentum associated with the operator $\mathbf{Q}$ and the angular variables $\boldsymbol{\upsilon}$ or $\boldsymbol{\beta}$ in the kinematics of a universe with curvature in the geometry of its position and velocity spaces. This angular momentum and its angular coordinates is hidden in a perfectly flat geometry, and has not been noticed in our human view of the universe from a location in which the curvature is very small. Its role can be elucidated from the information revealed by the Lie algebra of the group.

To examine the possibility of finding an observable associated with it we can start with the analogy between the occurrence of the ordinary angular momentum in Eq. (45)(a), a) $\left[\hat{\mathbf{K}}_{\text{boost},m}, \hat{\mathbf{K}}_{\text{boost},n}\right] = -i\varepsilon^{mnl}\hat{\mathbf{J}}_{l}$, and the parallel appearance of the contra-angular momentum $\hat{\mathbf{Q}}$ in Eq. (17)(a),

$\left[\hat{\mathbf{K}}_{\text{shift},m},\hat{\mathbf{K}}_{\text{boost},n}\right]=-i\varepsilon^{mnl}\hat{\mathbf{Q}}_l$. In the former case we are familiar with the associated operation $\left(\delta\mathbf{v}_i\times\delta\mathbf{v}_j/c^2\right)$ and its consequence in the Thomas precession. In the latter case we can construct the operator $\boldsymbol{\beta}=\boldsymbol{\eta}\hat{\times}\boldsymbol{\varepsilon}\cong\left(\delta\mathbf{r}_i\times\delta\mathbf{v}_j\gamma_j/c\rho\right)=\left(\delta\mathbf{r}_i\times\delta\mathbf{u}_j/c\rho\right)$ on a similar plan. Observable consequences might be sought either in astrophysical systems at high redshifts and moderate values of v/c or under terrestrial conditions with a system involving velocities approaching c. In view of the very small factor of $\delta r/\rho=\delta r/c_0H_0^{-1}$ in $\boldsymbol{\beta}$ any such effect will be very hard to detect in a region of the universe where gravitational curvature is very small. As an effect of such curvature, however, its properties and the possibility of their measurement should be explored.

X. Compatibility of Kinematic Symmetry with General Relativity

A satisfactory symmetric special relativity should be compatible with a relativistic theory of gravitation that retains the principal results of the description of gravitational fields in General Relativity. Born showed how the Principle of Reciprocity could be applied to gravitation when he developed his proposal to reconcile General Relativity and quantum mechanics with its help [1]. However, his implementation of this reconciliation was incomplete, because it neglected the need to bring special relativity also into agreement with the reciprocity principle. Born's modification of General Relativity therefore failed to lead to physically useful consequences. This failure may in turn have reinforced the common opinion that Hamiltonian symmetries and methods and the quantum mechanical structures that are associated with them are incompatible with relativity, both special and general.

In this work I have shown that the Principle of Reciprocity can be applied successfully to Special Relativity. To do this requires applying the reciprocity principle not between the variables of position and ordinary momentum but rather to a generalized Hamiltonian coordinate-momentum pair with the mass-weighting $\overline{\mathbf{q}}=m^{1/2}\mathbf{x},\overline{\mathbf{p}}=m^{-1/2}\mathbf{p}=m^{1/2}\mathbf{v}$. This form of the Hamiltonian reciprocity exhibits simultaneously a long-neglected symmetry between position and velocity in kinematics and dynamics, Kinematic Symmetry. The resulting combination of Born's reciprocity with kinematic position-velocity symmetry and Hamiltonian coordinate-momentum symmetry leads to an improved and symmetric form of special relativity provided we make several additional modifications which conform with observed features of the universe but sometimes differ from previously accepted assumptions:

(1) The position space of special relativity is not the flat space of a Minkowski metric but rather the expanding hyperbolic space of an open, homogeneous FRW universe.

(2) The curvature length of the hyperbolic position space is the time-dependent Hubble length $\rho(t) = cH^{-1}(t)$.

(3) The velocity c establishing the curvature of the hyperbolic velocity space of special relativity is cosmologically time dependent, $c(t)$, in symmetry with $\rho(t)$ in such a way that their product is constant, $c(t)\rho(t) = c_0{}^2 H_0{}^{-1} = \sigma$, the Hubble-Lorentz constant, while their ratio $c(t)/\rho(t) = H(t)$ is the measurable Hubble function.

(4) The Hubble effect—the effect of distance on an observed velocity—is to be recognized as the quantitative converse of the Lorentz transformation—the effect of a velocity difference on an observed length or time.

The result of these adjustments is a symmetric form of special relativity with many valuable properties and consequences. Among them are a symmetric and covariant solution of the problem of the centers of mass and momentum in special relativity; the construction of a relativistic Hamiltonian that is fully covariant; and the consequent establishment of fully covariant expressions for the relativistic Schrödinger equation and the n-body Dirac equation. Numerous other consequences follow.

This whole development strongly suggests that these symmetry principles should also be satisfied by a new version of general relativistic gravitation. A promising line of approach will be to modify Born's application of the reciprocity principle to general relativity, by first incorporating the new features of symmetric special relativity as the gravitation-free limit of the theory.

The new point of view requires that we recognize the following conclusions of symmetric special relativity before attempting to follow Born's pattern for generalizing the gravitational metric to comply with the new symmetry requirements:

(a) Application of the symmetry principle to special relativity confirms Fock's suggestion that the field-free limit of the general-relativistic metric should be spatially Lobachewskian (hyperbolic) rather than Minkowskian.

(b) The symmetry of the reciprocity principle is to be taken as applying not between the ordinary position and momentum spaces, but between position and velocity, and especially to their hyperbolic representatives, the separation η and the rapidity ε. The associated Hamiltonian symmetry can be expressed through the four-by-four tensors $\overline{\mathbf{X}}_i = m_i^{1/2}\mathbf{X}_i, \overline{\mathbf{P}}_i = m_i^{1/2}\mathbf{U}_i$. Both the position-velocity and the Hamiltonian symmetries should survive in general relativity.

(c) The fundamental symmetry between the Hubble effect of a change in position on an observed velocity vector, and the Lorentz effect of a change in velocity on an observed position vector, shows that both of these are intimately related and express a deep feature in the structure of the physical universe. It follows that the Hubble effect is not to be interpreted primarily as an effect of gravitation. The symmetry that these phenomena exhibit must be continued unimpaired in the domain of general relativity.

(d) The Lorentz effect and the Hubble effect differ in a vital change in sign, the sign alternation that also appears in the symplectic symmetry of Hamiltonian dynamics and quantum mechanics. This symplectic symmetry is a characteristic feature of the hyperbolic Poincaré group, and must carry over into general relativity.

(e) The velocity of light and the Hubble length are both time-dependent in the universal expansion, obeying the equations $c(t)\rho(t) = \sigma$, a constant, and $H(t) = c(t) / \rho(t)$, the observable Hubble expansion function.

(f) The nonrelativistic concept of “time” turns out to cover an asymptotic merger of two different concepts that need to be distinguished and treated differently under relativistic conditions. The time t of the universal expansion is a universal proper time variable, an invariant under velocity boosts and position shifts. The local time τ appears as one component of a four-vector or tensor of location or interval, and is dependent on the velocity and position frame of the object with respect to the observer.

(g) The properties of a doubly hyperbolic universe together with its history of a universal expansion from an origin at a finite proper time past show that the position-space metric in the absence of gravitation does not have the Minkowski metric’s symmetry, but rather the symmetries associated with the FRW metric of an expanding hyperbolic space.

(h) Kinematics in a doubly curved universe requires recognition of a previously unrecognized three additional degrees of rotational freedom per particle, the degrees of freedom of a contrarotation between position and velocity space. These degrees of freedom must be recognized whenever position space is curved.

In applying these conclusions to the description of gravitating systems we must take into account the major features that have been observed in the real universe, but we do not have to accept the theoretical models that have been advanced in the past to account for some of them. In particular, we take note of the following circumstances:

(i) While general relativity is traditionally described by a Riemannian metric in four-dimensional space-time, all the effects of gravitational curvature accessible to human measurement can be described by a metric with local inhomogeneities only in the three dimensions of position space. The principle of parsimony would seem recommend such a procedure.

(j) Cosmological information appears to indicate that the universe accessible to our observation is on the average extraordinarily close to flat. At the same time, we know that gravitational inhomogeneities on a local scale ensure that it essentially nowhere locally flat. If this is the case, the local positive curvature prevalent in our own galaxy and near mass concentrations in general must be compensated by negative curvature in the vast voids revealed by cosmological mapping. It is largely through these voids that the electromagnetic signals pass that enable us to probe the most distant galaxies.

In his proposal for a new general relativistic quantum mechanics, Born [1] proposed introducing two limiting metrics in a quantum-adapted general relativity, one in usual position-time four-space and the other in a momentum-energy four-space. Each of these would be applicable in its own limiting case, and between the two limits quantum uncertainty conditions would connect the two cases. Drawing on the above considerations, it can be suggested that Born's program may be applied (a) in the geometries of position and velocity rather than position and momentum, and (b) with the gravitational inhomogeneities confined to the metric of a position three-space and a countermetric in velocity three-space, rather than being extended to space-time (or velocity-energy) four-spaces. The limiting metrics can be supposed to apply in the respective three-spaces of the separation vector $\boldsymbol{\eta}$ and its conjugate rapidity vector $\boldsymbol{\varepsilon}$.

A new approach to the problem of applying Born's Principle to general relativity is now warranted, based in a thorough development of symmetric special relativity as a pregravitational approximation.

XI. Concluding Remarks

This work was initiated under the impulse of the unsatisfactory lack of continuity in the boundary zone from low to relativistic energies in the dynamical theory of two or more spatially separated bodies. These issues of continuity have been especially recognized in quantized systems, but their origin turns out to lie in a defect of special relativity itself. Its cure turns out to have profound consequences propagating into many fields of physics.

This paper has been devoting to establishing the basic principles of the new kinematically symmetric form of special relativity and surveying some of their consequences in the case of the field-free kinematics of a single particle. Its conclusions can be extended immediately to the treatment to two-body and many-body systems, relying on the use of a single cosmological proper time variable applicable to all particles of a system. This allows the development of a symmetric and thoroughly relativistic treatment of the center of mass and momentum, establishment of the relativistic Hamiltonian for n-body systems, and construction of the relativistic Schrödinger and n-body Dirac equations. These developments will be reported in a separate paper.

It is a task for the future to develop the extension of general relativity that will embody the kinematic symmetry between position and velocity spaces developed here.

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