Uncertainty Relations and the Operator Problem in Quantum Mechanics

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ABSTRACT This article pinpoints the interpretation of the uncertainty relation in quantum mechanics. Interestingly, uncertainty relations are shown to be mathematical relations that follow without using the quantum mechanics. They are exclusively based on the definitions of the associated probability densities using the theory of Fourier analysis. To increase the conceptual understanding of the quantum theory, we analyze various relationships between the Schrödinger equation and the Liouville equation. A solution to the operator problem in quantum mechanics is provided such that quantization of the kinetic energy in a curved space can be achieved.

KEYWORDS Liouville process, Schrödinger equation, uncertainty relations, Heisenberg, operator problem.

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1 Introduction

The Liouville equation [10] is present in most standard textbooks of statistical physics. The partial differential Liouville equation for the joint probability density for position and velocity follows exclusively from the given deterministic ordinary differential equations. It is well known that the Liouville equation follows when one uses the deterministic classical equation and introduces the stochastic nature by only assuming that the initial values of position and velocity are of stochastic nature. By “counting up” the independent and different tracks (realizations), i.e. position and velocity versus time, the joint probability densities for the position and velocity can be constructed and is a solution of the Liouville equation. By inserting an initial Dirac delta distribution into the Liouville equation, the distribution remains a Dirac delta distribution for all times. The Liouville equation supported the Laplace world view; that the future can be foreseen and the past can be recovered to any desired accuracy by finding sufficiently precise initial data and finding sufficiently powerful laws of nature[1].

In the early 1900s Poincaré supplemented this view by pointing out the possibility that very small differences in the initial conditions may produce large differences in the final phenomena[2]. Poincaré further argued that the initial conditions are always uncertain [8]. He reasoned that “a very small cause which escapes our notice determines a considerable effect that we cannot fail to see, and then we say that the effect is due to chance”. He thus early on grasped the foundation for modern deterministic chaos theory [17].

In the early 1900s Markov introduced a new kind of stochastic theory [7]. In contrast to the Liouville approach, where drawing from a probability

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[1] Laplace [28] on determinism: “We ought to regard the present state of the universe as the effect of its anterior state and as the cause of the one which is to follow. Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it- an intelligence vast to submit this data to analysis- it would embrace in the same formulae the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes.”

[2] Poincaré [8] on chaos: “A very small cause which escapes our notice determines a considerable effect that we cannot fail to see, and then we say that the effect is due to chance. If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at the following moment. But, even if it were the case that the natural laws had no longer any secrets for us, we could still only know the initial conditions approximately. If that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon has been predicted, that is governed by laws. But it is not always so; it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce very great ones in the final phenomena. Predictions become impossible, and we have the fortuitous phenomenon”.

distribution happens only once initially, in the Markov approach there is drawing from a probability distribution at each new time interval. It may be argued that for a so called Markov process, the chaos aspect is so dominant that the predictably of state n+1 given state n is impossible for all arbitrarily small positive time differences between the states. However, the probability of remaining in state n increases to one when the time step approaches zero. By “counting up” the independent and different tracks (realizations) the probability density can again be constructed.

Influenced by Einstein’s [12] explanation of the photoelectric effect in terms of particle-like properties associated with light waves, Louis de Broglie [11] proposed in 1924 that wavelike properties are associated with moving particles. Davisson and Germer [27] presented the first experimental verification of the existence of such wavelike properties. In 1926 Schrödinger [13] proposed a differential equation for the associated matter wave. Born [14] thereafter suggested that the square of the length of the wave function could be thought of as a probability density in time and space.

It is easily shown that the Liouville approach and the Markov approach give linear equations for the joint probability density. Hence a linear combination of two solutions of either one of these equations is also a solution of the equation. It is easily found that the same applies for the marginal densities, i.e. linear combinations of marginal density solutions are also solutions.

The linearity is caused by the possibility of generating the density by “counting up” the independent and different tracks (realizations), i.e. position and velocity versus time. Such summation, with equal or unequal weight to each realization, by its very nature generates linearity. The name stochastic process is the familiar name as a subcategory for such stochastic theories.

The Schrödinger equation gives non linear solutions for the probability densities. Hence the equation does not allow for constructing a method for generating independent realizations such that the density can be found by counting up independent and different tracks. Since independent realizations can not be identified, we are not dealing with a stochastic process in particular, though we are of course dealing with a stochastic theory. The non linearity leads to interference results, of which the double slit experiments are the most famous example. The interference phenomenon is complicated to handle since by measuring state variables, e.g. position at one or several points in time, the received quantum theory states that the interference pattern disappears [9].

Although the quantum theory has proven to be very successful, the conceptual foundation of quantum mechanics has been widely discussed [1]. Feynman [15] developed the concept of interfering alternative tracks, which may prove useful, particularly in developing conceptual understanding. Nelson [18] gave a description based on classical concepts applying Brownian motion. De Broglie [20] also addressed various deficiencies and paradoxes. Based on de Broglie's conception that certain wavelike properties are associated with moving particles, Fitzgerald [26] proposed that wave mechanics should be supplemented by a differential equation for momentum transfer where the interaction term is due to relative velocity and not relative position.

The article intends to increase our conceptual understanding of quantum theory. One objective is to pinpoint relations that are valid for most stochastic theories, where we define what is meant by "most", and relations that are specific for quantum theory. Of special interest is the relationship between the quantum theory and the Liouville process. In developing these relations we provide a solution to an operator problem, i.e. how to quantize terms of the type H(q) p^2 where q is position, p is momentum, and H() is a function. Different quantization rules give different answers since the common procedure is to let $p \to \left(-i\hbar\partial/\partial q\right)$ through quantization, but the answer depends on the algebraic order of the term that is to be quantified [1],[19]. Thus for instance the algebraic orders q p p q and p q q p give different results when quantization is applied. The operator ordering has to be specified if quantization of the kinetic energy in a curved space is to be achieved.

Section 2 presents interpretations of the uncertainty relation and analyzes these. Section 3 presents various operator relations. Section 4 gives a solution to the operator problem. Section 5 concludes.

2 The uncertainty relation

The double slit experiment and the interpretation of the uncertainty relation in quantum mechanics are widely discussed [1]. The literature offers many different interpretations of the inequality $\Delta q\, \Delta p \geq 1/2$, where q means position, p means momentum, and Δ means an increment. The two most important interpretations are in our view the following, where we choose the mass, time and length unit such that m=1, $\hbar = 1$.

1. The standard deviation of the momentum times the standard deviation of the position is larger than one half. The problem with this interpretation is that according to the interpretation of the double slit experiments there are

no paths in quantum mechanics, and certainly no derivative path, which means that the momentum concept is not defined.

2. The attempt to measure the position with some given accuracy Δq implies an uncontrolled disturbance Δp of the momentum, where $\Delta p \geq (1/2)/\Delta q$. The problem with this interpretation is that it disconnects the theory and the experimental measuring process. That is, an algebraic manipulation of an inequality is not sufficient to explain the connection between a theoretical statement and an empirical observation.

Assume as an example that the classical equation is given by

$$\ddot{N}_{1t} \overset{mod}{=} -f_2(N_{1t}) \Rightarrow \left\{ \dot{N}_{1t} = N_{2t}, \quad \dot{N}_{2t} \overset{mod}{=} -f_2(N_{1t}) \right\}, \tag{2.1}$$

where “mod” means model assumption, $f_2()$ is a function, N_{1t} and N_{2t} are stochastic variables which do not need to be position or velocity, and t is time. Drawing only the initial values at time t_0 of N_{1t_0}, N_{2t_0}, $\dot{N}_{1t_0}$, $\dot{N}_{2t_0}$ randomly[3], the Liouville equations follow for the joint probability density $\rho^L(t,n_1,n_2)$, i.e.

$$\dot{\rho}^L(t,n_1,n_2) = -D_1\left(n_2\rho^L(t,n_1,n_2)\right) + D_2\left(f_2(n_1)\rho^L(t,n_1,n_2)\right), \tag{2.2}$$

[3] Taylor and Karlin [21] state that “a stochastic process is a family of random variables X_t, where t is a parameter running over a suitable index set T,” where “an old-fashioned but very useful and highly intuitive definition describes a random variable as a variable that takes on its values by chance.” For ease of presentation we take the liberty in this article of using N_t also for classical processes traditionally conceived as “deterministic”, where the “chance” element is introduced by allowing the initial value N_{t_0} at time t_0 to be drawn from a probability distribution. Readers are free to interpret N_t as the conventional $N(t)$ for the classical equations in this article. Thus the Liouville equation follows from a special stochastic process where the change element is only introduced initially.

where D_i means derivation w.r.t. n_i. The marginal densities are defined by

$$\rho_1{}^L(t,n_1) \overset{def}{=} \int_{-\infty}^{\infty} \rho^L(t,n_1,n_2)dn_2, \quad \rho_2{}^L(t,n_2) \overset{def}{=} \int_{-\infty}^{\infty} \rho^L(t,n_1,n_2)dn_1, \tag{2.3}$$

where "def" means definition. Applying (2.2) and integrating each of the variables separately give

$$\dot\rho_1{}^L(t,n_1) + D_1\left(\rho_1{}^L(t,n_1)v_1^L(t,n_1)\right) = 0, \quad \dot\rho_2{}^L(t,n_2) + D_2\left(\rho_2{}^L(t,n_2)v_2^L(t,n_2)\right) = 0, (a)$$

$$v_1^L(t,n_1) \overset{def}{=} \left(1/\rho_1{}^L(t,n_1)\right)\int_{-\infty}^{\infty} n_2\rho^L(t,n_1,n_2)dn_2, (b)$$

$$v_2^L(t,n_2) \overset{def}{=} \left(1/\rho_2{}^L(t,n_2)\right)\int_{-\infty}^{\infty} f_2(n_1)\rho^L(t,n_1,n_2)dn_1, (c) \tag{2.4}$$

where $v_i^L(t,n_1)$ are the so-called stream velocities, i=1,2. Equation (2.2) implies

$$\begin{aligned}
&\dot E^L\left(N_{1t}\right) = E^L\left(N_{2t}\right), \quad \dot E^L\left(N_{1t}\right) = -E^L\left(f_2(N_{1t})\right), (a)\\
&E^L\left(N_{1t}N_{2t}\right) = \dot E^L\left(N_{1t}{}^2\right), \dot E^L\left(N_{2t}^2\right) = -2E^L\left(N_{2t}f_2(N_{1t})\right), (b)\\
&\dot E^L\left(N_{2t}{}^2\right) = (1/2)\ddot E^L\left(N_{1t}{}^2\right) + E^L\left(f_2(N_{1t})N_{1t}\right), (c)\\
&\dot{Var}^L\left(N_{1t}\right) = 2Cov^L\left(N_{1t},N_{2t}\right), \quad \dot{Var}^L\left(N_{2t}\right) = -2Cov^L\left(N_{2t}, f_1(N_{1t})\right), (d)\\
&\dot{Cov}^L\left(N_{1t},N_{2t}\right) = Var^L\left(N_{2t}\right) - Cov^L\left(N_{1t}, f_2(N_{1t})\right), (e)\\
&\dot E^L\left(h\left(N_{1t}\right)\right) = E^L\left(H\left(N_{1t}\right)N_{2t}\right), D_1h\left(N_{1t}\right) = H\left(N_{1t}\right), (f)
\end{aligned} \tag{2.5}$$

which is a closed set of equations for the expectations, variances and covariance when $f_2(N_{1t})$ is linear in N_{1t}. $H()$ is an arbitrary function.

Generalizing beyond the Liouville process, we consider the arbitrary density function $\rho_1(t,n_1)$, which could be the Liouville density or some other density, where n_1 is a variable. We now define the complex function

$$\theta_1(t,n_1) \overset{def}{=} \sqrt{\rho_1(t,n_1)}e^{i\,s_1(t,n_1)} \Rightarrow \rho_1(t,n_1) = \theta_1(t,n_1)^*\theta_1(t,n_1) \qquad (2.6)$$

where $s_1(t,n_1)$ so far is any arbitrary real function. This means that $\theta_1(t,n_1)$ and $\theta_1(t,n_1)^*$ are arbitrary functions. The Fourier transform is defined by

$$\theta_1^f(t,n_1) \overset{def}{=} \left(1/(2\pi)\right)\int_{-\infty}^{\infty} \theta_1(t,u)\, e^{-in_1u}\, du, \qquad (2.7)$$

The inverse Fourier transform follows when using that $\delta(x) = (1/(2\pi))\int_{-\infty}^{\infty} e^{ixy} dy$, to read

$$\theta_1(t,n_1) = \int_{-\infty}^{\infty} \theta_1^f(t,u)\, e^{iun_1}\, du. \qquad (2.8)$$

where u is a dummy variable.

Theorem 1. The function

$$\rho_1^f(t,n_1) \overset{def}{=} \theta_1^f(t,n_1)^*\theta_1^f(t,n_1) \qquad (2.9)$$

is a probability density.

Proof. Note first that $\rho_1^f(t,n_1) \geq 0$. Inserting (2.6) and (2.7) into (2.9) gives

$$\int_{-\infty}^{\infty} \rho_1^f(t,u)du = \int_{-\infty}^{\infty} \theta_1^f(t,u)\theta_1^f(t,u)^* du = (1/(2\pi))\int_{-\infty}^{\infty} \theta_1(t,y)^*\theta_1(t,y')e^{ui(y-y')}dy'dydu$$

$$= \int_{-\infty}^{\infty} \theta_1(t,y)^*\theta_1(t,y')\delta(y-y')dy'dy = \int_{-\infty}^{\infty} \theta_1(t,y)^*\theta_1(t,y)^*dy = \int_{-\infty}^{\infty} \rho_1(t,v)dv = 1. \quad Qed.$$

(2.10)

That is, defining an arbitrary density function $\rho_1(t,n_1)$ in a specific manner, as in (2.6), it follows in Theorem 1 that the Fourier transform $\rho_1^f(t,m_1)$ is also a density function. We could define M_{1t} as the stochastic variable drawn from this density.

Theorem 2.

$$E\left(M_{1t}^n\right)=\int_{-\infty}^{\infty}\theta_1(t,u)^*\left(-iD_1\right)^n\theta_1(t,u)du, \tag{2.11}$$

Proof. Applying the definition for expectation and (2.6)-(2.8) give

$$E\left(M_{1t}^n\right) \overset{def}{=} \int_{-\infty}^{\infty}u^n\rho_1^f(t,u)du=(1/(2\pi))\int_{-\infty}^{\infty}\theta_1(t,y)^*\theta_1(t,y')i^{-n}(\partial/\partial y)^n e^{ui(y-y')}dy'dydu$$

$$=\int_{-\infty}^{\infty}\theta_1(t,y)^*\theta_1(t,y')i^{-n}(\partial/\partial y)^n\delta(y-y')dy'dy=\int_{-\infty}^{\infty}\theta_1(t,y)^*(-iD_1)^n\theta_1(t,y)dy \tag{2.12}$$

Theorem 2 states that the expected value $E\left(M_{1t}^n\right)$ of the Fourier transformed raised to the power n can be expressed as an integral which involves the arbitrary functions $\theta_1(t,n_1)$ and $\theta_1(t,n_1)^*$ defined in (2.6), and $\left[-iD_1\right]^n$. Equation (2.11) does not assume or require any special stochastic theory.

Theorem 3. The uncertainty relation.

$$\sqrt{Var(N_{1t})}\sqrt{Var(M_{1t})}\geq 1/2. \tag{2.13}$$

Proof. Define

$$I(\alpha) \overset{def}{=} \int_{-\infty}^{\infty} \left\{ i\alpha\left[u\theta_1(t,u) - E(N_{1t})\theta_1(t,u)\right] + (1/i)D_1\theta_1(t,u) - E(M_{1t})\theta_1(t,u)\right\}$$

$$\times\left\{ i\alpha\left[u\theta_1(t,n_1) - E(N_{1t})\theta_1(t,n_1)\right] + (1/i)D_1\theta_1(t,u) - E(M_{1t})\theta_1(t,u)\right\} * du \geq 0. \quad (2.14)$$

Expansion of (2.14) and integration by parts give

$$I(\alpha) = \alpha^2 Var(N_{1t}) + Var(M_{1t}) + \alpha \geq 0 \quad \Leftrightarrow \quad Var(M_{1t}) \geq -\alpha\left(1 + \alpha\, Var(N_{1t})\right), \quad (2.15)$$

where the RHS has a maximum when $\alpha = -1/\left(2Var(N_{1t})\right)$, implying (2.13). *Qed.*

Theorem 3 is purely of mathematical nature and can be interpreted as an uncertainty relation. It states that the uncertainty relation between the standard deviation of the arbitrary probability density $\rho_1(t,n_1)$ and the standard deviation of the corresponding Fourier transformed probability density $\rho_1^f(t,m_1)$, is exclusively a mathematical relation that does not need quantum mechanics to be developed. Heisenberg's formulation is that "The more precisely the POSITION is determined, the less precisely the MOMENTUM is known" [16]. In Heisenberg's formulation, position and momentum have specific physical interpretations. In Theorem 3, in contrast, $\rho_1(t,n_1)$ does not have to be the position density and $\rho_1^f(t,m_1)$ does not have to be a momentum density.

3 The operator relations and joint probabilities

Interesting relations appear when $s_1(t,n_1)$ is connected with the stream velocity $v_1(t,n_1)$, defined by the general conservation equation for a general stochastic theory in space, where n_1 is the position for the remainder of this article. The conservation of probability gives

$$\dot{\rho}_1(t,n_1) + D_1\left(\rho_1(t,n_1)v_1(t,n_1)\right) \overset{mod}{=} 0 \quad (3.1)$$

Assuming the specific model

$$s_1(t,n_1) \overset{mod}{=} \int_{-\infty}^{n_1} v_1(t,u)du, \theta_1(t,n_1) = \rho_1(t,n_1)^{1/2} e^{i\, s_1(t,n_1)},$$

$$\rho_1(t,n_1) = \theta_1(t,n_1)^* \theta_1(t,n_1), \tag{3.2a}$$

it follows from (3.1) and (3.2a)

$$D_1 s_1(t,n_1) = v_1(t,n_1) = \frac{1}{2i} D_1 \,\mathrm{Ln}\left(\frac{\theta_1(t,n_1)}{\theta_1(t,n_1)^*}\right) = \frac{1}{2i}\left[\frac{D_1\theta_1(t,n_1)}{\theta_1(t,n_1)} - \frac{D_1\theta_1(t,n_1)^*}{\theta_1(t,n_1)^*}\right] \tag{3.2b}$$

Theorem 4. When h is arbitrary,

$$\dot{E}\left(h(N_{1t})\right) = (1/2)\int_{-\infty}^{\infty} \theta_1(t,n_1)^* \left\{H(n_1)(-iD_1)\theta_1(t,n_1) + (-iD_1)H(n_1)\theta_1(t,n_1)\right\} dn_1$$

$$= (1/i)\int_{-\infty}^{\infty} \theta_1(t,n_1)^* H(n_1) D_1\left(\theta_1(t,n_1)\right) dn_1 + E\left(D_1 H(n_1)\right)/(2i), D_1 h(n_1) \overset{def}{=} H(n_1) \tag{3.3}$$

Proof.

Using the definitions and (3.1) and (3.2) give

$$\dot{E}\left(h(N_{1t})\right) = \frac{d}{dt}\int_{-\infty}^{\infty} h(n_1)\rho_1(t,n_1)dn_1 = \int_{-\infty}^{\infty} h(n_1)\dot{\rho}_1(t,n_1)dn_1 = \int_{-\infty}^{\infty} v_1(t,n_1)\rho_1(t,n_1)D_1 h(n_1)\, dn_1$$

$$= \int_{-\infty}^{\infty} \theta_1(t,n_1)^*\theta_1(t,n_1)\left(1/(2i)\right)\left[\left(D_1\theta_1(t,n_1)\right)/\theta_1(t,n_1) - \left(D_1\theta_1(t,n_1)\right)^*/\theta_1(t,n_1)^*\right] H(n_1)dn_1$$

$$= (1/i)\int_{-\infty}^{\infty} \theta_1(t,n_1)^* H(n_1) D_1\left(\theta_1(t,n_1)\right) dn_1 + E\left(D_1 H(N_{1t})\right)/(2i)$$

$$= (1/2)\int_{-\infty}^{\infty} \theta_1(t,n_1)^* \left\{H(n_1)(-iD_1)\theta_1(t,n_1) + (-iD_1)\left[H(n_1)\theta_1(t,n_1)\right]\right\} dn_1. \quad Qed. \tag{3.4}$$

Theorem 5. First connection between position distribution and Fourier transformed distribution.

$$\dot{E}(N_{1t}) = E(M_{1t}), \tag{3.5}$$

Proof. Follows directly from Theorems 2 and 4 by choosing $h(n_1) = n_1$.

Theorem 5 states that the time derivative of the expectation is equal to the expectation of the Fourier inverse constructed density. Observe from (2.5a) and (3.5) that for the Liouville process we additionally get $E(M_{1t}) = E\left(N_{2t}\right)$. Thus the Fourier transformed density gives the same expectation as the expectation of the velocity, for the Liouville process. But be aware that (3.5) as such does not demand any Liouville process or the Schrödinger equation.

Lemma 1.

$$E\left(f_2(N_{1t})\right) = -E(M_{2t}) \tag{3.6}$$

Proof. Applying (2.1) gives $\dot{E}(N_{1t}) = E(N_{2t})$ and $\dot{E}(N_{2t}) = -E\left(f_2(N_{1t})\right)$. Further it is easy to show that $\dot{E}(N_{2t}) = E\left(M_{2t}\right)$. *Qed.*

Theorem 6. Second connection between velocity distribution and Fourier transformed distribution.

$$\textit{Schrödinger equation} \Rightarrow E(M_{1t}{}^2) = (1/2)\ddot{E}^{sc}\left(N_{1t}{}^2\right) + E^{Sc}\left(f_2(N_{1t})N_{1t}\right). \tag{3.7}$$

Proof. The Schrödinger equation is given by

$$-(1/2)D_1{}^2\theta_1^{Sc}(t,n_1) + V(n_1)\theta_1^{Sc}(t,n_1) \overset{mod}{=} i\dot{\theta}_1^{Sc}(t,n_1), (a) \tag{3.8}$$

$$\theta_1^{Sc}(t,n_1) = \rho_1^{Sc}(t,n_1)^{1/2} e^{i\, s_1(t,n_1)}, \quad D_1 s(t,n_1) = v_1^{Sc}(t,n_1), \quad D_1 V(n_1) \overset{def}{=} f_2(n_1), (b)$$

which implies

$$
\begin{aligned}
&(1/2)\ddot{E}^{Sc}\left(N_{1t}{}^{2}\right) = (1/2)\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty} n_1{}^2\theta_1^{Sc}(t,n_1)*\theta_1^{Sc}(t,n_1)dn_1 \\
&= \int_{-\infty}^{\infty}\theta_1^{Sc}(t,n_1)*\left(D_1(1/i)\right)^2\theta_1^{Sc}(t,n_1)dn_1 - \int_{-\infty}^{\infty}\theta_1^{Sc}(t,n_1)*\theta_1^{Sc}(t,n_1)f_2(n_1)n_1dn_1 \\
&= E^{Sc}\left(M_{1t}{}^{2}\right) - E^{Sc}\left(f_2(N_{1t})\right)
\end{aligned}
\tag{3.9}
$$

where (3.8a) is used twice. *Qed.*

Theorem 6 is established by explicitly using the Schrödinger equation. For the special case of the Liouville process in (2.1), the RHS in (3.7) is equal to $E(N_{2t}{}^{2})$. This does not mean that $E(M_{1t}{}^{2}) = E(N_{2t}{}^{2})$ for the Liouville process since we used (3.8a) to establish Theorem 6. It is well known that for some special solutions of the Liouville equation the solution is equal with a corresponding Schrödinger solution. For those cases we accordingly get $E(M_{1t}{}^{2}) = E(N_{2t}{}^{2})$.

4 The operator problem in quantum mechanics

The Schrödinger equation in (3.8a) does not provide the complete solutions to problems in quantum mechanics. Additionally, the operator of physical quantities is often needed. Introducing an operator of physical quantities is one way of providing complete solutions. This section applies a rule to find the operator corresponding to the term $E(H(N_{1t})M_{1t}{}^{2})$ which covers the terms necessary to quantize the kinetic energy in a curved space.

The most familiar operator rules are the Weyl rule in (4.1a), the symmetry rule in (4.1b), and the Born-Jordan rule in (4.1c). According to the familiar notation and our notation these are given by

$$E(q^n p^m) \xrightarrow{Weyl} \left(\frac{1}{2}\right)^n \sum_{l=0}^{l=n} \binom{n}{l} q^{n-l} \left(-i\frac{\partial}{\partial q}\right)^m q^l,$$
$$E_W^{Sc}\left(N_{1t}^n M_{1t}^m\right) = \int_{-\infty}^{\infty} \theta_1(t,n_1)^* \left[\left(\frac{1}{2}\right)^n \sum_{l=0}^{l=n} \binom{n}{l} n_1^{\,n-l} \left(-iD_1\right)^m n_1^{\,l}\right] \theta_1(t,n_1) dn_1$$

(4.1a)

$$q^n p^m \xrightarrow{Symmetry} \left(\frac{1}{2}\right)\left(q^n \left(-i\frac{\partial}{\partial q}\right)^m + \left(-i\frac{\partial}{\partial q}\right)^m q^n\right),$$
$$E_S^{Sc}\left(N_{1t}^n M_{1t}^m\right) = \int_{-\infty}^{\infty} \theta_1(t,n_1)^* \left[\left(\frac{1}{2}\right)\left(n_1^{\,n} \left(-iD_1\right)^m + \left(-iD_1\right)^m n_1^{\,n}\right)\right] \theta_1(t,n_1) dn_1$$

(4.1b)

$$q^n p^m \xrightarrow{Born-Jordan} \left(\frac{1}{m+1}\right) \sum_{l=0}^{l=m} \left(-i\frac{\partial}{\partial q}\right)^{m-1} q^n \left(-i\frac{\partial}{\partial q}\right)^l,$$
$$E_{BJ}^{Sc}\left(N_{1t}^n M_{1t}^m\right) = \int_{-\infty}^{\infty} \theta_1(t,n_1)^* \left[\left(\frac{1}{m+1}\right) \sum_{l=0}^{l=m} \left(-iD_1\right)^{m-1} q^n \left(-iD_1\right)^l\right] \theta_1(t,n_1) dn_1$$

(4.1c)

The substitution is $q = N_{1t}, p = M_{1t}$. The Weyl and symmetry rules keep a constant power of the momentum operator. The symmetry and Born-Jordan rules keep a constant power of the position operator. Thus only the symmetry rule keeps a constant power of both the momentum and position operators.

Theorem 2 defines operator relations, which are valid for any stochastic theory in general. Quantum theory states in addition to the Schrödinger equation the following relation

$$E^{Sc}\left(H(N_{1t}) M_{1t}\right) = \dot{E}^{Sc}\left(h(N_{1t})\right), D_1 h = H \tag{4.2}$$

Equation (2.5f) implies that stating the same for the Liouville process gives $E^L(H(N_{1t})M_{1t})=E^L(H(N_{1t})N_{2t})$, thus relating M_{1t} and N_{2t}. Equation (4.2) and Theorem 4 give

$$E^{Sc}\left(H(N_{1t})M_{1t}\right)=\int_{-\infty}^{\infty} v_1(t,n_1)\rho_1(t,n_1)H(n_1)\,dn_1$$
$$=(1/i)\int_{-\infty}^{\infty}\theta_1(t,n_1)^* H(n_1)D_1\left(\theta_1(t,n_1)\right)dn_1 - i\,E\left(D_1H(N_{1t})\right)/2 \qquad (4.3)$$
$$=\int_{-\infty}^{\infty}\theta_1(t,n_1)^*\left\{\frac{(-iD_1)H(n_1)+H(n_1)(-iD_1)}{2}\right\}\theta_1(t,n_1)dn_1.$$

It is easily observed from (4.1) and (4.3) that

$$E^{Sc}\left(H(N_{1t})M_{1t}\right)=E^{Sc}_S\left(H(N_{1t})M_{1t}\right)=E^{Sc}_{BJ}\left(H(N_{1t})M_{1t}\right) \qquad (4.4)$$

The Weyl rule is more difficult to relate but we readily find from (4.1a) that

$$E^{Sc}_W\left(N_{1t}{}^2M_{1t}\right)=\int_{-\infty}^{\infty}\theta_1(t,n_1)^*\frac{(-i)}{4}\left[n_1{}^2D_1+2n_1D_1n_1+D_1n_1{}^2\right]\theta_1(t,n_1)dn_1$$
$$=\int_{-\infty}^{\infty}\theta_1(t,n_1)^*\frac{(-i)}{4}\left[4n_1{}^2D_1+4n_1\right]\theta_1(t,n_1)dn_1=E^{Sc}\left(N_{1t}{}^2M_{1t}\right) \qquad (4.5)$$

In general the Weyl rule, the rule of symmetry and the rule of Born-Jordan give the same answer when only the first power (m=1) of M_{1t} applies in the expectation.

One question is whether the relations in (4.3) are valid for the Liouville process. Is $E^L\left(H(N_{1t})M_{1t}\right)=\dot{E}^L\left(h(N_{1t})\right)$? The answer is yes, if we want! We can construct almost whatever relation we want for $E^L\left(H(N_{1t})M_{1t}\right)$. The reason is that the Liouville theory does not specify any relations for the joint distribution. Only the marginal densities are defined according to (2.7) and (2.8).

It easily follows from traditional stochastic theory that there is a one to one relation between a joint distribution and the corresponding operator ordering. Thus the Weyl rule corresponds to the Wigner function, while the symmetry rule corresponds with the Margenau–Hill function. The Born and Jordan rule also give a specific joint distribution. The Dirac and the rule of von Neumann rule are shown to be internally inconsistent [29][4]. See reference reference [1]-[6], [19], [21]- [25] for further discussions concerning this matter.

From the Liouville equation in (2.2) it follows by integration that

$$\dot{\rho}_1^{\,L}(t,n_1)+D_1\left(\rho_1^{\,L}(t,n_1)v_1^L(t,n_1)\right)=0,(a)$$

$$\dot{v}_1^L(t,n_1)+v_1^L(t,n_1)D_1v_1^L(t,n_1)=-f_2\left(n_1\right)+\frac{D_1\left[\rho_1^{\,L}(t,n_1)\left(v_1^L(t,n_1)^2-v_1^{L2}(t,n_1)\right)\right]}{\rho_1^{\,L}(t,n_1)},(b)$$

$$v_1^L(t,n_1)\overset{def}{=}\left(1/\rho_1^{\,L}(t,n_1)\right)\int_{-\infty}^{\infty}n_2\rho^L(t,n_1,n_2)dn_2,\,v_1^{L2}(t,n_1)\overset{def}{=}\left(1/\rho_1^{\,L}(t,n_1)\right)\int_{-\infty}^{\infty}n_2^{\,2}\rho^L(t,n_1,n_2)dn_2,(c)$$

$$E^L\left(H(N_{1t})N_{2t}^{\,2}\right)=\int_{-\infty}^{\infty}H(n_1)v_1^{L2}(t,n_1)\rho_1^L(t,n_1)dn_1,(d)$$

(4.6)

By multiplying (4.6b) with $\rho_1^{\,L}(t,n_1)$ and using (4.6a) it also follows that

$$\dot{\rho}_1^{\,L}(t,n_1)+D_1\left(\rho_1^{\,L}(t,n_1)v_1^L(t,n_1)\right)=0,(a)$$

$$\dot{v}_1^L(t,n_1)+v_1^L(t,n_1)D_1v_1^L(t,n_1)=-f_2\left(n_1\right)+\frac{D_1\left[\rho_1^{\,L}(t,n_1)\left(v_1^L(t,n_1)^2-v_1^{L2}(t,n_1)\right)\right]}{\rho_1^{\,L}(t,n_1)},(b) \qquad (4.7)$$

$$v_1^L(t,n_1)\overset{def}{=}\left(1/\rho_1^{\,L}(t,n_1)\right)\int_{-\infty}^{\infty}n_2\rho^L(t,n_1,n_2)dn_2,\,v_1^{L2}(t,n_1)\overset{def}{=}\left(1/\rho_1^{\,L}(t,n_1)\right)\int_{-\infty}^{\infty}n_2^{\,2}\rho^L(t,n_1,n_2)dn_2,(c)$$

$$E^L\left(H(N_{1t})N_{2t}^{\,2}\right)=\int_{-\infty}^{\infty}H(n_1)v_1^{L2}(t,n_1)\rho_1^L(t,n_1)dn_1,(d),$$

$$\frac{\partial}{\partial t}\left(\rho_1^L(t,n_1)v_1^L(t,n_1)\right)=-D_1\left(\rho_1^L(t,n_1)v_1^{L2}(t,n_1)\right)-f_2(n_1)\rho_1^L(t,n_1),(e)$$

When the initial position is a Dirac function it follows from the definitions in (4.7c) that $v_1^L(t,n_1)^2=v_1^{L2}(t,n_1)$, which gives agreement with the Hamilton-Jacobi formalism in classical mechanics in (4.6). We cannot

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[4] The Dirac rule is $\{A,B\}\to(-i/\hbar)[A_{op},B_{op}]$. The rule of von Neumann is $A\to A_{op},g(A)\to g(A_{op})$.

freely choose the initial values of $v_1^L(t_0,n_1)$ in (4.7b) since $v_1^L(t_0,n_1)$ follows exclusively from the initial values of the joint probability $\rho_1^L(t_0,n_1,n_2)$. In addition, when the joint probability is given, $\rho_1^L(t_0,n_1)$ follows. The opposite implication is not is valid. That is, for given functions $\rho_1^L(t_0,n_1)$ and $v_1^L(t_0,n_1)$, it is easy to show that the joint probability does not follow. Thus if we choose (4.7a) and (4.7b) as our only constitutive equations, this enlarges our equation set and in general disconnects the theory from the Liouville theory since the relation to the joint probability is disconnected. The disconnection requires providing a relation for $v_1^{L2}(t,n_1)$ to close the equation set. Such closure is exactly what the quantum theory provides.

The linearity is not easily observed in (4.7), but it is shown when using $\rho_1^L(t,n_1)$ and $\rho_1^L(t,n_1)$ multiplied with $v_1^L(t,n_1)$ as the unknown functions.

Moving onto the quantum mechanics, the de Broglie expression of quantum mechanics follows when inserting (3.8b) into the Schrödinger equation (3.8a), which gives after some algebraic manipulations

$$\begin{aligned}
&\dot{\rho}_1^{Sc}(t,n_1) + D_1\left(\rho_1^{Sc}(t,n_1)v_1^{Sc}(t,n_1)\right) = 0,(a)\\
&\dot{v}_1^{Sc}(t,n_1) + v_1^{Sc}(t,n_1)D_1v_1^{L}(t,n_1) = -f_2\left(n_1\right) + (1/2)D_1\left[\left(D_1{}^2\rho_1^{Sc}(t,n_1)^{1/2}\right)\Big/\rho_1^{Sc}(t,n_1)^{1/2}\right]\\
&= -f_2\left(n_1\right) + (1/4)\frac{D_1\left[D_1{}^2\rho_1^{Sc}(t,n_1) - \left(D_1\rho_1^{Sc}(t,n_1)\right)^2/\rho_1^{Sc}(t,n_1)\right]}{\rho_1^{Sc}(t,n_1)}\\
&= -f_2\left(n_1\right) + \frac{D_1\left[\rho_1^{Sc}(t,n_1)\left(v_1^{Sc}(t,n_1)^2 - v_1^{Sc2}(t,n_1)\right)\right]}{\rho_1^{Sc}(t,n_1)},(b
\end{aligned} \tag{4.8}$$

where

$$v_1^{Sc2}(t,n_1) \overset{def}{=} v_1^{Sc}(t,n_1)^2 + (1/4)\left[\left(D_1\rho_1^{Sc}(t,n_1)\right)^2\Big/\rho_1^{Sc}(t,n_1)^2 - D_1{}^2\rho_1^{Sc}(t,n_1)\Big/\rho_1^{Sc}(t,n_1)\right], \tag{4.9}$$

Observe the analogy between (4.8a)-(4.8b) and (4.7a)-(4.7b). But while the joint probability defines $v_1^{L2}(t,n_1)$ in the Liouville process, the Schrödinger equation gives (4.9) for this analogous quantity. Although the Liouville equation in general does not give (4.9), but instead gives $v_1^{L2}(t,n_1)$ from a joint probability, some special cases allow $v_1^{L2}(t,n_1)$ to be written as (4.9).

Multiplying (4.8b) with $\rho_1^{Sc}(t,n_1)$ and using (4.8a) gives the analogous expression to (4.7e), that is

$$\frac{\partial}{\partial t}\left(\rho_1^{Sc}(t,n_1)v_1^{Sc}(t,n_1)\right) = -D_1\left(\rho_1^{Sc}(t,n_1)v_1^{Sc2}(t,n_1)\right) - f_2(n_1)\rho_1^{Sc}(t,n_1) \quad (4.10)$$

Analogously to the quantum postulate in equation (4.2), we propose as a supplement to the quantum theory the following operator relation

$$E^{Sc}\left(H(N_{1t})M_{1t}^{\ 2}\right) \overset{mod}{=} \int_{-\infty}^{\infty} H(n_1)v_1^{Sc2}(t,n_1)\rho_1^{Sc}(t,n_1)dn_1 \quad (4.11)$$

which supplements the more familiar postulate in (4.2). The rationale behind (4.11) is (4.6d) for the Liouville approach.

(4.9) gives that

$$\begin{aligned}
&\rho_1^{Sc}(t,n_1)v_1^{Sc2}(t,n_1) \\
&= -\left(\theta_1(t,n_1)*\theta_1(t,n_1)\right)\frac{1}{4}\left[\left(\frac{D_1\theta_1(t,n_1)}{\theta_1(t,n_1)}\right)^2 - 2\left(\frac{D_1\theta_1(t,n_1)}{\theta_1(t,n_1)}\right)\left(\frac{D_1\theta_1(t,n_1)*}{\theta_1(t,n_1)*}\right) + \left(\frac{D_1\theta_1(t,n_1)*}{\theta_1(t,n_1)*}\right)^2\right] \\
&+ \frac{1}{4}\left[\left(\theta_1(t,n_1)D_1\theta_1(t,n_1)* + \theta_1(t,n_1)*D_1\theta_1(t,n_1)\right)^2 \Big/ \left(\theta_1(t,n_1)*\theta_1(t,n_1)\right)\right] - (1/4)D_1^{\ 2}\rho_1^{Sc}(t,n_1) \\
&= \left(D_1\theta_1(t,n_1)*\right)\left(D_1\theta_1(t,n_1)\right) - (1/4)D_1^{\ 2}\rho_1^{Sc}(t,n_1)
\end{aligned} \quad (4.12)$$

Then (4.11) and (4.12) imply

$$E^{Sc}\left(H(N_{1t})M_{1t}{}^2\right)=\int_{-\infty}^{\infty}\left(D_1\theta_1(t,n_1)^*\right)H(n_1)D_1\theta_1(t,n_1)dn_1-(1/4)E^{Sc}\left(D_1{}^2H(n_1)\right)$$
$$=-\int_{-\infty}^{\infty}\theta_1^{Sc}(t,n_1)^*D_1H(n_1)D_1\theta_1^{Sc}(t,n_1)dn_1-(1/4)\int_{-\infty}^{\infty}\theta_1^{Sc}(t,n_1)^*\theta_1^{Sc}(t,n_1)D_1{}^2H(n_1)dn_1$$
$$=\left(1/4\right)\int_{-\infty}^{\infty}\theta_1^{Sc}(t,n_1)^*\left((-iD_1)^2H(n_1)+2(-iD_1)H(n_1)(-iD_1)+H(n_1)(-iD_1)^2\right)\theta_1^{Sc}(t,n_1)dn_1 \tag{4.13}$$

When H() =1 (4.13) is in agreement with Theorem 2 as a special case. The three rules in (4.1a)-(4.1c) is in general different from (4.13).

Inserting $H(n_1)=n_1{}^2$ into (4.13) gives

$$E^{Sc}\left(N_{1t}{}^2M_{1t}{}^2\right)=\frac{(-i)^2}{4}\int_{-\infty}^{\infty}\theta_1^{Sc}(t,n_1)^*\left(D_1{}^2n_1^2+2D_1n_1D_1+n_1^2D_1^2\right)\theta_1^{Sc}(t,n_1)dn_1$$
$$=\frac{(-i)^2}{4}\int_{-\infty}^{\infty}\theta_1^{Sc}(t,n_1)^*\left(4n_1^2D_1^2+8n_1D_1+2\right)\theta_1^{Sc}(t,n_1)dn_1 \tag{4.14}$$

The Weyl rule gives from (4.1a)

$$E_W^{Sc}\left(N_{1t}^2M_{1t}^2\right)=\int_{-\infty}^{\infty}\theta_1(t,n_1)^*\frac{(-i)^2}{4}\left[n_1{}^2D_1{}^2+2n_1D_1^2n_1+D_1^2n_1{}^2\right]\theta_1(t,n_1)dn_1 \tag{4.15}$$
$$=\int_{-\infty}^{\infty}\theta_1(t,n_1)^*(-i)^2\left[n_1{}^2D_1^2+2n_1D_1+1/2\right]\theta_1(t,n_1)dn_1=E^{Sc}\left(N_{1t}^2M_{1t}^2\right)$$

The symmetry rule gives from (4.1b)

$$E_S^{Sc}\left(N_{1t}^2M_{1t}^2\right)=\int_{-\infty}^{\infty}\theta_1(t,n_1)^*\left(\frac{(-i)^2}{2}\right)\left[D_1^2n_1^2+n_1{}^2D_1{}^2\right]\theta_1(t,n_1)dn_1 \tag{4.16}$$
$$=\int_{-\infty}^{\infty}\theta_1(t,n_1)^*(-i)^2\left[n_1{}^2D_1^2+2n_1D_1+1\right]\theta_1(t,n_1)dn_1\neq E^{Sc}\left(N_{1t}^2M_{1t}^2\right)$$

The Born-Jordan rule gives from (4.1b)

$$E_{BJ}^{Sc}\left(N_{1t}^2M_{1t}^2\right)=\int_{-\infty}^{\infty}\theta_1(t,n_1)^*\left(\frac{(-i)^2}{3}\right)\left[D_1^2n_1^2+D_1n_1^2D_1+n_1{}^2D_1{}^2\right]\theta_1(t,n_1)dn_1$$
$$=\int_{-\infty}^{\infty}\theta_1(t,n_1)^*(-i)^2\left[n_1{}^2D_1^2+2n_1D_1+2/3\right]\theta_1(t,n_1)dn_1\neq E^{Sc}\left(N_{1t}^2M_{1t}^2\right) \tag{4.17}$$

The rule in (4.13) is in correspondence with the Weyl rule in this special case where $H(n_1) = n_1^2$ but not in correspondence with the symmetry rule and the Born-Jordan rule. These three rules are treated in more detail in [1]-[6].

5 Conclusion

This article pinpoints the interpretation of the uncertainty relation in quantum mechanics. Uncertainty relations are shown to be mathematical relations that follow without using the quantum mechanics. They are exclusively based on the definitions of the associated probability densities using the theory of Fourier analysis. To increase our conceptual understanding of quantum theory, we analyze various relationships between the Schrödinger equation and the Liouville approach. Theorem 2 shows that the rule where $p^n \to p_{op}{}^n$ where $p_{op} \to \left(-i\hbar\partial / \partial q\right)$ follows from most general stochastic theories. The rule that $pH\left(q\right) \to (1/2)\left(p_{op}H\left(q\right) + H\left(q\right)p_{op}\right)$ follows from the familiar additional quantum postulate. This rule is equivalent to what follows from the Weyl, symmetry, and Born-Jordan rules. We additionally find that $p^2H\left(q\right) \to \left(p_{op}{}^2H\left(q\right) + 2p_{op}H\left(q\right)p_{op} + H\left(q\right)p_{op}\right)/4$. This rule is generally different from what follows from the Weyl, symmetry, and Born-Jordan rules. For one specific example we demonstrate correspondence between this rule and the Weyl rule, but not with the symmetry and Born-Jordan rules. The closed solution given to this operator problem in quantum mechanics can be used to quantize the kinetic energy in a curved space.

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