Higher Derivative Scalar Field Theory in the First Order Formalism

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ABSTRACT. The scalar field theory with higher derivatives is considered in the first order formalism. The field equation of the forth order describes scalar particles possessing two mass states. The first order relativistic wave equation in the 10-dimensional matrix form is derived. We find the relativistically invariant bilinear form and corresponding Lagrangian. The canonical energy-momentum tensor and density of the electromagnetic current are obtained. Dynamical and non-dynamical components of the wave function are separated and the quantum-mechanical Hamiltonian is found. Projection operators extracting solutions of field equations for definite energy and different mass states of particles are obtained. The canonical quantization of scalar fields with two mass states is performed, and propagators are found in the formalism considered.

1 Introduction

It is known that the gravity theory based on the Einstein-Hilbert action is non-renormalizable in four dimensions [1]. By including curvature squared terms in the action [2], the theory becomes renormalizable but the higher derivative (HD) theory. There are also HD fields in generalized electrodynamics [3]. HD theories allow us to improve renormalization of the theory and to avoid ultraviolet divergences [4]. However, it was discovered soon that there are some difficulties with negative norm (ghosts), and unitarity in HD theories [5], [6]. Much attention, therefore, was made to study the simplest version of the HD theory of scalar fields (see [7], [8], [9], [10], [11] and references therein). It should be mentioned that scalar fields play very important role in the theory of inflation of the universe. So, inflation scalar fields can be candidates of dark matter of the universe. HD scalar fields are also present in SUSY field theories in extra dimensions (see, for example, [12]). Although HD field theories lead to ghosts, it was shown that problems with negative probabilities and S-matrix unitarity can be solved [9]. The purpose of this paper is to formulate the forth order equation for scalar fields in the form of the first order relativistic wave equation, to obtain the Lagrangian, conserved currents, the quantum-mechanical Hamiltonian, and to perform the quantization.

The paper is organized as follows. In Sec. 2, we derive the first order relativistic wave equation for scalar fields in the 10-dimensional matrix form, the relativistically invariant bilinear form and the Lagrangian formulation. In Sec. 3, the canonical energy-momentum tensor and the electromagnetic current density are obtained. Dynamical and non-dynamical components of the wave function are separated and the quantum-mechanical Hamiltonian is found in Sec. 4. In Sec. 5 projection operators extracting solutions of field equations for definite energy and different mass states of particles are obtained. The canonical quantization of scalar fields with two mass states is performed, and the propagators of scalar fields were found in the formalism of the first order in Sec. 6. We discuss results obtained in Sec. 7. In Appendix, some useful products of matrices are given. The system of units $\hbar = c = 1$ is chosen, Greek and Latin letters run $1,2,3,4$ and $1,2,3$, correspondingly.

2 Scalar Field Equation of the Forth Order

Consider the forth order field equation describing scalar particles possessing two mass states [7], [9]:

$$\left(\partial^2 - m_1^2\right)\left(\partial^2 - m_2^2\right)\varphi(x) = 0, \tag{1}$$

where $\partial^2 \equiv \partial_\nu^2$, $\partial_\nu = \partial/\partial x_\nu = (\partial/\partial x_m, \partial/\partial(it))$. It is obvious that Eq. (1) has two solutions corresponding to mass m_1 and m_2.

Let us introduce the 10-dimensional wave function

$$\phi(x) = \{\phi_A(x)\} = \begin{pmatrix} \varphi(x) \\ \widetilde{\varphi}(x) \\ \varphi_\mu(x) \\ \widetilde{\varphi}_\mu(x) \end{pmatrix} \qquad (A = 0, \widetilde{0}, \mu, \widetilde{\mu}), \tag{2}$$

where $\phi_0(x) = \varphi(x)$, $\phi_{\widetilde{0}}(x) = \widetilde{\varphi}(x)$, $\phi_\mu(x) = \varphi_\mu(x)$, $\phi_{\widetilde{\mu}}(x) = \widetilde{\varphi}_\mu(x)$,

$$\widetilde{\varphi}(x) = \frac{1}{m_1 m_2}\partial^2\varphi(x), \qquad \varphi_\mu(x) = \frac{1}{m_1 + m_2}\partial_\mu\varphi(x),$$

$$\widetilde{\varphi}_\mu(x) = \frac{1}{m_1 + m_2}\partial_\mu \widetilde{\varphi}(x). \tag{3}$$

The function $\phi(x)$ represents the direct sum of two scalars $\varphi(x)$, $\widetilde{\varphi}(x)$, and two four-vectors $\varphi_\mu(x)$, $\widetilde{\varphi}_\mu(x)$.

Introducing the elements of the entire matrix algebra $\varepsilon^{A,B}$ (see, for example [15]) with matrix elements and products

$$\left(\varepsilon^{M,N}\right)_{AB} = \delta_{MA}\delta_{NB}, \qquad \varepsilon^{M,A}\varepsilon^{B,N} = \delta_{AB}\varepsilon^{M,N}, \tag{4}$$

where $A, B, M, N = 0, \widetilde{0}, \mu, \widetilde{\mu}$, Eq. (1) can be cast in the form of the first order equation

$$\begin{gathered}\partial_\mu \left[\varepsilon^{0,\widetilde{\mu}} - \varepsilon^{\widetilde{0},\mu} - \sigma\varepsilon^{0,\mu} - \frac{m}{m_1+m_2}\left(\varepsilon^{\mu,0} + \varepsilon^{\widetilde{\mu},\widetilde{0}}\right)\right]_{AB} \phi_B(x) \\ +m\left[\varepsilon^{0,0} + \varepsilon^{\widetilde{0},\widetilde{0}} + \varepsilon^{\mu,\mu} + \varepsilon^{\widetilde{\mu},\widetilde{\mu}}\right]_{AB} \phi_B(x) = 0,\end{gathered} \tag{5}$$

where

$$m = \frac{m_1 m_2}{m_1 + m_2}, \qquad \sigma = \frac{m_1^2 + m_2^2}{m_1 m_2},$$

and there is a summation over all repeated indices. We define 10-dimensional matrices as follows:

$$\rho_\mu = \varepsilon^{0,\widetilde{\mu}} - \varepsilon^{\widetilde{0},\mu} - \sigma\varepsilon^{0,\mu} - \frac{m}{m_1+m_2}\left(\varepsilon^{\mu,0} + \varepsilon^{\widetilde{\mu},\widetilde{0}}\right), \tag{6}$$

$$I_{10} = \varepsilon^{0,0} + \varepsilon^{\widetilde{0},\widetilde{0}} + \varepsilon^{\mu,\mu} + \varepsilon^{\widetilde{\mu},\widetilde{\mu}}, \tag{7}$$

where I_{10} is a unit 10-dimensional matrix. Then Eq. (5) becomes the relativistic wave equation of the first order:

$$(\rho_\mu \partial_\mu + m)\,\phi(x) = 0. \tag{8}$$

So, we reformulated the higher derivative equation for scalar fields (1) in the form of the first order Eq. (8). Now, one can apply general methods to investigate the first order matrix equation [13]. The spectrum of the particle mass of Eq. (8) is given by m/λ_i, where λ_i are the eigenvalues of the matrix ρ_4. It is not difficult to verify that the matrix

$$\rho_4 = \varepsilon^{0,\widetilde{4}} - \varepsilon^{\widetilde{0},4} - \sigma\varepsilon^{0,4} - \frac{m}{m_1+m_2}\left(\varepsilon^{4,0} + \varepsilon^{\widetilde{4},\widetilde{0}}\right)$$

satisfies the matrix equation

$$\rho_4^4 - \frac{m_1^2 + m_2^2}{(m_1 + m_2)^2}\rho_4^2 + \frac{m_1^2 m_2^2}{(m_1 + m_2)^4}\Lambda = 0, \tag{9}$$

where

$$\Lambda = \varepsilon^{0,0} + \varepsilon^{\widetilde{0},\widetilde{0}} + \varepsilon^{4,4} + \varepsilon^{\widetilde{4},\widetilde{4}} \tag{10}$$

is the projection operator extracting four-dimensional subspace, $\Lambda^2 = \Lambda$. As the eigenvalue of the matrix Λ is unit, we obtain from Eq. (9) eigenvalues of the matrix ρ_4 as follows:

$$\lambda_1 = \pm\frac{m_2}{m_1 + m_2}, \quad \lambda_2 = \pm\frac{m_1}{m_1 + m_2}. \tag{11}$$

So, positive masses of scalar particles described by the first order Eq. (8) are $m/\lambda_1 = m_1$, $m/\lambda_2 = m_2$.

The Lorentz group generators in the representation space are [15]

$$J_{\mu\nu} = \varepsilon^{\mu,\nu} - \varepsilon^{\nu,\mu} + \varepsilon^{\widetilde{\mu},\widetilde{\nu}} - \varepsilon^{\widetilde{\nu},\widetilde{\mu}}, \tag{12}$$

and obey the commutation relations

$$[J_{\mu\nu}, J_{\alpha\beta}] = \delta_{\nu\alpha}J_{\mu\beta} + \delta_{\mu\beta}J_{\nu\alpha} - \delta_{\nu\beta}J_{\mu\alpha} - \delta_{\mu\alpha}J_{\nu\beta}. \tag{13}$$

The relativistic form-invariance of Eq. (8) follows from the relationship

$$[\rho_\lambda, J_{\mu\nu}] = \delta_{\lambda\mu}\rho_\nu - \delta_{\lambda\nu}\rho_\mu. \tag{14}$$

One may verify with the help of Eq. (4), (6), (12) that Eq. (14) is valid.

The Hermitianizing matrix η has to obey the relations [13]

$$\eta\rho_m = -\rho_m^+\eta^+, \qquad \eta\rho_4 = \rho_4^+\eta^+ \qquad (m = 1, 2, 3). \tag{15}$$

We obtain

$$\begin{aligned}\eta = \varepsilon^{0,0} - \varepsilon^{\widetilde{0},\widetilde{0}} - \frac{\sigma(m_1 + m_2)}{m}\left(\varepsilon^{m,m} - \varepsilon^{4,4}\right)\\ + \frac{(m_1 + m_2)}{m}\left(\varepsilon^{m,\widetilde{m}} + \varepsilon^{\widetilde{m},m} - \varepsilon^{4,\widetilde{4}} - \varepsilon^{\widetilde{4},4}\right).\end{aligned} \tag{16}$$

The matrix η is the Hermitian matrix, $\eta^+ = \eta$. Introducing the matrix $\overline{\phi}(x) = \phi^+(x)\eta$ ($\phi^+(x)$ is the Hermitian-conjugate wave function), one obtains from Eq. (8) the "conjugate" equation

$$\overline{\phi}(x)\left(\rho_\mu \overleftarrow{\partial}_\mu - m\right) = 0. \tag{17}$$

The Lagrangian is given by the standard equation

$$\mathcal{L} = -\frac{1}{2}\left[\overline{\phi}(x)\left(\rho_\mu\partial_\mu + m\right)\phi(x) - \overline{\phi}(x)\left(\rho_\mu\overleftarrow{\partial}_\mu - m\right)\phi(x)\right], \qquad (18)$$

so that the relativistically invariant bilinear form is $\overline{\phi}(x)\phi(x) = \phi^+(x)\eta\phi(x)$.

With the aid of Eq. (2),(3),(16), the Lagrangian (18) becomes

$$\mathcal{L} = -\frac{1}{m_1+m_2}\left\{\frac{1}{2m_1m_2}\left[\varphi^*\partial^4\varphi + \left(\partial^4\varphi^*\right)\varphi\right] - \frac{\sigma}{2}\left[\varphi^*\partial^2\varphi + \left(\partial^2\varphi^*\right)\varphi\right] + m_1m_2\varphi^*\varphi\right\}. \qquad (19)$$

Up to total derivatives, which do not influence on an equation of motion, the Lagrangian (19) takes the compact form

$$\mathcal{L} = -\frac{1}{m_1m_2(m_1+m_2)}\left[\left(\partial_\mu\partial_\nu\varphi^*\right)\left(\partial_\mu\partial_\nu\varphi\right) + \left(m_1^2+m_2^2\right)\left(\partial_\mu\varphi^*\right)\left(\partial_\mu\varphi\right) + m_1^2m_2^2\varphi^*\varphi\right]. \qquad (20)$$

One may verify that the Euler-Lagrange equation [14]

$$\frac{\partial\mathcal{L}}{\partial\varphi^*} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)} + \partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\varphi^*)} = 0, \qquad (21)$$

for the higher derivative Lagrangian (20), leads to the field equation (1).

3 The Energy-Momentum Tensor and Electromagnetic Current

With the help of the general expression [16]

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi(x)\right)}\partial_\nu\phi(x) + \partial_\nu\overline{\phi}(x)\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\overline{\phi}(x)\right)} - \delta_{\mu\nu}\mathcal{L}, \qquad (22)$$

we obtain from the Lagrangian (18) the canonical energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{2}\left(\partial_\nu\overline{\phi}(x)\right)\rho_\mu\phi(x) - \frac{1}{2}\overline{\phi}(x)\rho_\mu\partial_\nu\phi(x). \qquad (23)$$

It was taken into account that $\mathcal{L} = 0$ for functions $\phi(x)$, $\overline{\phi}(x)$ satisfying Eq. (8), (17). Using Eq. (2)-(4), (6), one finds from Eq. (23) the expression as follows:

$$T_{\mu\nu} = \frac{1}{2(m_1+m_2)}\left\{\left(m_1^2+m_2^2\right)\left[\varphi^*\partial_\mu\partial_\nu\varphi - \left(\partial_\mu\varphi^*\right)\partial_\nu\varphi\right] - \varphi^*\partial_\mu\partial_\nu\partial^2\varphi\right.$$

$$- \left(\partial^2\varphi^*\right)\partial_\mu\partial_\nu\varphi + \left(\partial_\mu\varphi^*\right)\partial_\nu\partial^2\varphi + \left(\partial_\nu\varphi^*\right)\partial_\mu\partial^2\varphi\Big\} + c.c., \tag{24}$$

where c.c. means the complex conjugate expression. The energy density and the momentum density are given by $\mathcal{E} = T_{44}$, $P_m = iT_{m4}$.

The electric current density is [16]

$$j_\mu(x) = i\left(\overline{\phi}(x)\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\overline{\phi}(x)\right)} - \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi(x)\right)}\phi(x)\right). \tag{25}$$

Replacing Eq. (18) into Eq. (25), one obtains the electric current density

$$j_\mu(x) = i\overline{\phi}(x)\rho_\mu\phi(x), \tag{26}$$

so that $\partial_\mu j_\mu(x) = 0$. With the help of Eq. (2)-(4), (6), we find

$$\begin{aligned} j_\mu = &\frac{i}{m_1m_2(m_1+m_2)}\Big\{\left(m_1^2+m_2^2\right)\left[\left(\partial_\mu\varphi^*\right)\varphi - \varphi^*\partial_\mu\varphi\right] \\ &+\varphi^*\partial_\mu\partial^2\varphi - \left(\partial_\mu\partial^2\varphi^*\right)\varphi + \left(\partial^2\varphi^*\right)\partial_\mu\varphi - \left(\partial_\mu\varphi^*\right)\partial^2\varphi\Big\}. \end{aligned} \tag{27}$$

For the real scalar fields, $\varphi = \varphi*$, the electric current vanishes.

4 Quantum-Mechanical Hamiltonian

Introducing an interaction of scalar particles with electromagnetic fields by the substitution $\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu$, where A_μ is the four-vector potential of electromagnetic fields (the minimal electromagnetic interaction), Eq. (8) may be represented as follows:

$$i\rho_4\partial_t\phi(x) = \left[\rho_a D_a + m + eA_0\rho_4\right]\phi(x). \tag{28}$$

Let us introduce two auxiliary functions

$$\Xi(x) = \Lambda\phi(x), \quad \Omega(x) = \Pi\phi(x), \tag{29}$$

where the projection operator Λ is given by Eq. (10), and the projection operator Π is

$$\Pi = 1 - \Lambda = \varepsilon^{m,m} + \varepsilon^{\widetilde{m},\widetilde{m}}, \tag{30}$$

so that $\Xi(x) + \Omega(x) = \phi(x)$. Acting on Eq. (28) by the operator

$$\frac{(m_1+m_2)^2}{m_1^2m_2^2}\rho_4\left[m_1^2+m_2^2-\left(m_1+m_2\right)^2\rho_4^2\right],$$

and taking into consideration Eq. (9), we obtain

$$
\begin{aligned}
i\partial_t\Xi(x) = eA_0\Xi(x) \\
+\frac{(m_1+m_2)^2}{m_1^2m_2^2}\rho_4\left[m_1^2+m_2^2-(m_1+m_2)^2\rho_4^2\right](\rho_a D_a+m)\,\phi(x).
\end{aligned}
\tag{31}
$$

Multiplying Eq. (28) by the operator Π, one finds

$$
\Pi\rho_a D_a\Xi(x)+m\Omega(x)=0. \tag{32}
$$

We took into account here that $\Pi\rho_4=0$, $\Pi\rho_m\Pi=0$. It follows from Eq. (31),(32) that the dynamical components of wave function are given by the $\Xi(x)$, and non-dynamical components by the $\Omega(x)$. The $\Xi(x)$ possesses four components, and the auxiliary function $\Omega(x)$ has six components. To separate the dynamical and non-dynamical components, we express the auxiliary function $\Omega(x)$ from Eq. (32), and replace it into Eq. (31). As a result, one obtains the equation in the Hamiltonian form:

$$
i\partial_t\Xi(x)=\widehat{\mathcal{H}}\Xi(x), \tag{33}
$$

$$
\begin{aligned}
\widehat{\mathcal{H}} = eA_0 + \frac{(m_1+m_2)^2}{m_1^2m_2^2}\left[m_1^2+m_2^2-(m_1+m_2)^2\rho_4^2\right]\rho_4 \\
\times(\rho_a D_a+m)\left(1-\frac{1}{m}\Pi\rho_m D_m\right).
\end{aligned}
\tag{34}
$$

Two components of the function $\Xi(x)$ correspond to the state with the mass m_1 (with positive and negative energy) and the other two - to the state with the mass m_2. With the help of products of matrices, given in Appendix, the Hamiltonian (34) becomes:

$$
\begin{aligned}
\widehat{\mathcal{H}} = eA_0 - \sigma m\varepsilon^{\widetilde{4},\widetilde{0}} - (m_1+m_2)\left(\varepsilon^{0,4}+\varepsilon^{\widetilde{0},\widetilde{4}}\right) + m\left(\varepsilon^{\widetilde{4},0}-\varepsilon^{4,\widetilde{0}}\right) \\
+\frac{1}{m_1+m_2}\left(\varepsilon^{\widetilde{4},\widetilde{0}}+\varepsilon^{4,0}\right)D_m^2.
\end{aligned}
\tag{35}
$$

We have omitted the linear term in derivatives $L\equiv\left(\varepsilon^{4,m}+\varepsilon^{\widetilde{4},\widetilde{m}}\right)D_m$ in the Hamiltonian, because $L\Xi(x)=0$. Taking into consideration Eq. (2),(4),(10),(35), we can rewrite Eq. (33) in the component form

$$
i\partial_t\varphi = eA_0\varphi-(m_1+m_2)\,\varphi_4, \qquad i\partial_t\widetilde{\varphi} = eA_0\widetilde{\varphi}-(m_1+m_2)\,\widetilde{\varphi}_4,
$$

$$
i\partial_t\varphi_4 = eA_0\varphi_4 - m\widetilde{\varphi}+\frac{1}{m_1+m_2}D_m^2\varphi, \tag{36}
$$

$$i\partial_t\widetilde{\varphi}_4 = eA_0\widetilde{\varphi}_4 - \sigma m\widetilde{\varphi} + m\varphi + \frac{1}{m_1+m_2}D_m^2\widetilde{\varphi}.$$

The system of equations (36) may be obtained from Eq. (1), (3), after the exclusion of components

$$\varphi_m = \frac{1}{m_1+m_2}\partial_m\varphi, \qquad \widetilde{\varphi}_m = \frac{1}{m_1+m_2}\partial_m\widetilde{\varphi}.$$

According to Eq. (33) only components with time derivatives enter Eq. (36), and describe the evolution of fields in time.

5 Mass Projection Operators

Consider solutions of Eq. (8) with definite energy and momentum for two mass states, $\tau = 1, 2$, in the form of plane waves:

$$\phi_\tau^{(\pm)}(x) = \sqrt{\frac{m_\tau^2}{p_0 V m}}\, v_\tau(\pm p)\exp(\pm ipx), \tag{37}$$

where V is the normalization volume, $p^2 = -m_\tau^2$ (no summation in index τ). We imply that four momentum $p = (\mathbf{p}, ip_0)$ possesses the additional quantum number $\tau = 1, 2$ corresponding two masses. The 10-dimensional function $v_\tau(\pm p)$ obeys the equation

$$(i\hat{p} \pm m)\, v_\tau(\pm p) = 0, \tag{38}$$

where $\hat{p} = \rho_\mu p_\mu$. It is convenient to use the normalization conditions

$$\int_V \overline{\phi}_\tau^{(\pm)}(x)\rho_4\phi_\tau^{(\pm)}(x)d^3x = \pm 1, \qquad \int_V \overline{\phi}_\tau^{(\pm)}(x)\rho_4\phi_\tau^{(\mp)}(x)d^3x = 0, \tag{39}$$

where $\overline{\phi}_\tau^{(\pm)}(x) = \left(\phi_\tau^{(\pm)}(x)\right)^+\eta$. Normalization conditions (39) lead to relations for functions $v_\tau(\pm p)$:

$$\overline{v}_\tau(\pm p)\rho_\mu v_\tau(\pm p) = \mp\frac{imp_\mu}{m_\tau^2}, \qquad \overline{v}_\tau(\pm p)v_\tau(\pm p) = 1. \tag{40}$$

It is not difficult to verify, with the help of Eq. (4), (6) (see Appendix), that the minimal equation for the matrix $\hat{p}$ is

$$\hat{p}^5 - \frac{\left(m_1^2+m_2^2\right)p^2}{\left(m_1+m_2\right)^2}\hat{p}^3 + \frac{m^2p^4}{\left(m_1+m_2\right)^2}\hat{p} = 0. \tag{41}$$

Now we obtain the projection matrices corresponding to definite energy, momentum, and quantum number τ:

$$\Pi_\tau(\pm p) = \frac{(m_1+m_2)^4\, i\hat{p}\,(i\hat{p}\mp m)}{2m_1^2 m_2^2\,(m_\tau^4 - m_1^2 m_2^2)}\left[\widehat{p}^2 + \frac{m_\tau^4}{(m_1+m_2)^2}\right]. \tag{42}$$

The projection operators (42) obey equations as follows:

$$(ip \pm m)\,\Pi_\tau(\pm) = 0, \tag{43}$$

$$\Pi_\tau(\pm p)^2 = \Pi_\tau(\pm p), \qquad \Pi_\tau(+p)\Pi_\tau(-p) = 0, \qquad \mathrm{tr}\Pi_\tau(\pm p) = 1. \tag{44}$$

Projection matrices (42) can be represented (see [17]) as matrix-dyads

$$\Pi_\tau(\pm p) = v_\tau(\pm p)\cdot \overline{v}_\tau(\pm p), \tag{45}$$

so that matrix elements of matrix-dyads are

$$\left(v_\tau(\pm p)\cdot \overline{v}_\tau(\pm p)\right)_{MN} = \left(v_\tau(\pm p)\right)_M \left(\overline{v}_\tau(\pm p)\right)_N .$$

Projection operators (42) extract solutions of Eq. (38) for definite energy and different mass states of particles. Eq. (42), (45) allow us to calculate matrix elements of different processes of interactions of scalar particles in the covariant form.

6 Field Quantization

One can obtain the momenta from Eq. (18):

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi(x)} = \frac{i}{2}\overline{\phi}(x)\rho_4, \tag{46}$$

$$\overline{\pi}(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0\overline{\phi}(x))} = -\frac{i}{2}\rho_4\phi(x), \tag{47}$$

where the fields $\phi(x)$, $\overline{\phi}(x)$ are independent "coordinates". The Hamiltonian density is given by the equation

$$\begin{aligned}\mathcal{H} &= \pi(x)\partial_0\phi(x) + \left(\partial_0\overline{\phi}(x)\right)\overline{\pi}(x) - \mathcal{L}\\ &= \frac{i}{2}\overline{\phi}(x)\rho_4\partial_0\phi(x) - \frac{i}{2}\left(\partial_0\overline{\phi}(x)\right)\rho_4\phi(x),\end{aligned} \tag{48}$$

It follows from Eq. (23) that the value $\mathcal{H} = T_{44}$ is the energy density.

In the quantized theory the field operators can be written as follows

$$\phi_\tau(x) = \sum_p \left[a_{\tau,p}\phi_\tau^{(+)}(x) + b^+_{\tau,p}\phi_\tau^{(-)}(x)\right],$$
$$\overline{\phi}_\tau(x) = \sum_p \left[a^+_{\tau,p}\overline{\phi_\tau^{(+)}}(x) + b_{\tau,p}\overline{\phi_\tau^{(-)}}(x)\right] \tag{49}$$

where positive and negative parts of the wave function are given by Eq. (37). The creation and annihilation operators of particles $a^+_{\tau,p}$, $a_{\tau,p}$, and creation and annihilation operators of antiparticles $b^+_{\tau,p}$, $b_{\tau,p}$ obey the commutation relations:

$$[a_{\tau,p}, a^+_{\tau',p'}] = \delta_{\tau\tau'}\delta_{pp'}, \quad [a_{\tau,p}, a_{\tau',p'}] = [a^+_{\tau,p}, a^+_{\tau',p'}] = 0,$$
$$[b_{\tau,p}, b^+_{\tau',p'}] = \delta_{\tau\tau'}\delta_{pp'}, \quad [b_{\tau,p}, b_{\tau',p'}] = [b^+_{\tau,p}, b^+_{\tau',p'}] = 0, \tag{50}$$
$$[a_{\tau,p}, b_{\tau',p'}] = [a_{\tau,p}, b^+_{\tau',p'}] = [a^+_{\tau,p}, b_{\tau',p'}] = [a^+_{\tau,p}, b^+_{\tau',p'}] = 0.$$

With the aid of Eq. (48)-(50), and normalization condition (39), we obtain the Hamiltonian

$$H = \int \mathcal{H} d^3x = \sum_{\tau,p} p_0 \left(a^+_{\tau,p}a_{\tau,p} + b_{\tau,p}b^+_{\tau,p}\right). \tag{51}$$

It is not difficult to find from Eq. (49)-(50) commutation relations as follows:

$$[\phi_{\tau M(x)}, \phi_{\tau N}(x')] = [\overline{\phi}_{\tau M}(x), \overline{\phi}_{\tau N}(x')] = 0, \tag{52}$$

$$[\phi_{\tau M}(x), \overline{\phi}_{\tau N}(x')] = N_{\tau MN}(x,x'), \tag{53}$$

$$N_{\tau MN}(x,x') = N^+_{\tau MN}(x,x') - N^-_{\tau MN}(x,x'),$$
$$N^+_{\tau MN}(x,x') = \sum_p \left(\phi_\tau^{(+)}(x)\right)_M \left(\overline{\phi_\tau^{(+)}}(x')\right)_N, \tag{54}$$
$$N^-_{\tau MN}(x,x') = \sum_p \left(\phi_\tau^{(-)}(x)\right)_M \left(\overline{\phi_{\tau,}^{(-)}}(x')\right)_N.$$

One obtains from Eq. (49):

$$N^\pm_{\tau MN}(x,x') = \sum_p \frac{m_\tau^2}{p_0 V m} \left(v_\tau(\pm p)\right)_M \left(\overline{v}_\tau(\pm p)\right)_N \exp[\pm ip(x-x')], \tag{55}$$

Taking into consideration Eq. (42), (45), and the relation $p^2 = -m_\tau^2$, we find from Eq. (55):

$$N^{\pm}_{\tau MN}(x,x') = \sum_p \left\{ \frac{i(m_1+m_2)^4 m_\tau^2 \widehat{p}\,(i\widehat{p} \mp m)}{2p_0 V m m_1^2 m_2^2 \,(m_\tau^4 - m_1^2 m_2^2)} \left[\widehat{p}^2 + \frac{m_\tau^4}{(m_1+m_2)^2}\right]\right\}_{MN}$$

$$\times \exp[\pm ip(x-x')] = \left\{ \frac{(m_1+m_2)^4 m_\tau^2\,(\pm\rho_\mu\partial_\mu)\,(\pm\rho_\mu\partial_\mu \mp m)}{m m_1^2 m_2^2\,(m_\tau^4 - m_1^2 m_2^2)} \right. \tag{56}$$

$$\left. \times \left[\frac{m_\tau^4}{(m_1+m_2)^2} - (\rho_\mu\partial_\mu)^2\right]\right\}_{MN} \sum_p \frac{1}{2p_0 V} \exp[\pm ip(x-x')],$$

With the help of the singular functions [16]

$$\Delta_+(x) = \sum_p \frac{1}{2p_0 V}\exp(ipx), \qquad \Delta_-(x) = \sum_p \frac{1}{2p_0 V}\exp(-ipx),$$

$$\Delta_0(x) = i\left(\Delta_+(x) - \Delta_-(x)\right),$$

we obtain from Eq. (54), (56)

$$N_{\tau MN}(x,x') = -i\left\{ \frac{(m_1+m_2)^4 m_\tau^2\,(\rho_\mu\partial_\mu)\,(\rho_\mu\partial_\mu - m)}{m m_1^2 m_2^2\,(m_\tau^4 - m_1^2 m_2^2)} \right.$$
$$\left. \times \left[\frac{m_\tau^4}{(m_1+m_2)^2} - (\rho_\mu\partial_\mu)^2\right]\right\}_{MN} \Delta_0(x-x'). \tag{57}$$

For the points x and x', which are separated by the space-like interval $(x-x') > 0$, the commutator $[\phi_M(x), \overline{\phi}_N(x')]$ equals zero due to the properties of the function $\Delta_0(x)$ [16]. If $t = t'$, $[\phi_M(\mathbf{x},0), \overline{\phi}_N(\mathbf{x}',0)] = N_{\tau MN}(\mathbf{x}-\mathbf{x}',0)$, and the function $N_{\tau MN}(\mathbf{x}-\mathbf{x}',0)$ can be obtained from Eq. (57) with the help of equations

$$\partial_0^{2n}\Delta_0(x)|_{t=0} = 0, \qquad \partial_m^n \Delta_0(x)|_{t=0} = 0, \qquad \partial_0\Delta_0(x)|_{t=0} = \delta(\mathbf{x}), \tag{58}$$

where $n = 1,2,3,....$ One may verify, using Eq. (41), the validity of the equation

$$(\rho_\mu\partial_\mu + m)\,N^{\pm}_{\tau}(x,x') = 0. \tag{59}$$

The propagator of scalar fields (the vacuum expectation of chronological pairing of operators) can be defined in our formalism as

$$\langle T\phi_{\tau M}(x)\overline{\phi}_{\tau N}(y)\rangle_0 = N^c_{\tau MN}(x-y)$$

$$= \theta\left(x_0 - y_0\right) N^{+}_{\tau MN}(x-y) + \theta\left(y_0 - x_0\right) N^{-}_{\tau MN}(x-y), \tag{60}$$

where $\theta(x)$ is the well known theta-function. We obtain from Eq. (56):

$$\langle T\phi_{\tau M}(x)\overline{\phi}_{\tau N}(y)\rangle_0 = \left\{ \frac{(m_1+m_2)^4 m_\tau^2 \left(\rho_\mu\partial_\mu\right)\left(\rho_\mu\partial_\mu - m\right)}{m m_1^2 m_2^2 \left(m_\tau^4 - m_1^2 m_2^2\right)} \right.$$
$$\left. \times \left[\frac{m_\tau^4}{(m_1+m_2)^2} - \left(\rho_\mu\partial_\mu\right)^2\right]\right\}_{MN} \Delta_c(x-y), \tag{61}$$

and the function $\Delta_c(x-y)$ is given by

$$\Delta_c(x-y) = \theta\left(x_0 - y_0\right)\Delta_+(x-y) + \theta\left(y_0 - x_0\right)\Delta_-(x-y). \tag{62}$$

Propagators (61) are finite only for $m_1 \neq m_2$. It is seen from Eq. (61) that propagators have different signs for $\tau = 1$ and $\tau = 2$. This means that one of states of the scalar field is the ghost [9].

7 Conclusion

We have formulated the scalar field equation with higher derivatives in the form of the 10-component first order relativistic wave equation. This equation describes scalar particles possessing two mass states. It should be noted that the second order equation for scalar fields with one mass state is formulated in the 5-component matrix form [18], [19]. The relativistically invariant bilinear form, and the Lagrangian were obtained, and this allowed us to find the canonical energy-momentum tensor and density of the electromagnetic current by the standard procedure. We found the quantum-mechanical Hamiltonian by the separation of dynamical and non-dynamical components of the wave function. The wave function entering the Hamiltonian equation possesses four components to describe two mass states of scalar fields with positive and negative energies. The first order relativistic wave equation as well as the Hamiltonian equation are convenient for different applications. The density matrix (matrix-dyad) found can be used for calculations of electromagnetic possesses. The first order formalism allowed us to quantize HD scalar fields in a simple manner. The HD scalar field theory considered can by applied for a model of inflation of the universe, where one of states of the scalar field is identified with Quintessence and another, the ghost, with Phantom [20].

8 Appendix

With the help of Eq. (4), we find useful products of matrices entering the Hamiltonian (34):

$$\rho_4\rho_m = \frac{m}{m_1+m_2}\left(\sigma\varepsilon^{4,m} + \varepsilon^{\widetilde{4},m} - \varepsilon^{4,\widetilde{m}}\right), \tag{63}$$

$$\rho_4^3\rho_m = \left(\frac{m}{m_1+m_2}\right)^2\left[\left(\sigma^2-1\right)\varepsilon^{4,m} + \sigma\varepsilon^{\widetilde{4},m} - \sigma\varepsilon^{4,\widetilde{m}} - \varepsilon^{\widetilde{4},\widetilde{m}}\right], \tag{64}$$

$$\rho_4\rho_m\Pi\rho_n = \delta_{mn}\left(\frac{m}{m_1+m_2}\right)^2\left(\varepsilon^{4,\widetilde{0}} - \varepsilon^{\widetilde{4},0} - \sigma\varepsilon^{4,0}\right), \tag{65}$$

$$\rho_4^3\rho_m\Pi\rho_n = \delta_{mn}\left(\frac{m}{m_1+m_2}\right)^3\left[\varepsilon^{\widetilde{4},\widetilde{0}} + \sigma\varepsilon^{4,\widetilde{0}} - \sigma\varepsilon^{\widetilde{4},0} + \left(1-\sigma^2\right)\varepsilon^{4,0}\right], \tag{66}$$

$$\rho_4\Pi = 0. \tag{67}$$

Using the definition $\hat{p} = p_\mu\rho_\mu$, we obtain:

$$\hat{p}^2 = \frac{mp^2}{m_1+m_2}\left(\sigma\varepsilon^{0,0} + \varepsilon^{\widetilde{0},0} - \varepsilon^{0,\widetilde{0}}\right) - \frac{m}{m_1+m_2}p_\mu p_\nu\left(\varepsilon^{\mu,\widetilde{\nu}} - \varepsilon^{\widetilde{\mu},\nu} - \sigma\varepsilon^{\mu,\nu}\right), \tag{68}$$

$$\hat{p}^3 = \frac{mp^2}{m_1+m_2}p_\mu\left[\sigma\left(\varepsilon^{0,\widetilde{\mu}} - \varepsilon^{\widetilde{0},\mu}\right) + \left(1-\sigma^2\right)\varepsilon^{0,\mu} + \varepsilon^{\widetilde{0},\widetilde{\mu}}\right] + \frac{m^2p^2}{(m_1+m_2)^2}p_\mu\left(\varepsilon^{\mu,\widetilde{0}} - \varepsilon^{\widetilde{\mu},0} - \sigma\varepsilon^{\mu,0}\right), \tag{69}$$

$$\hat{p}^4 = \frac{mp^2\sigma}{m_1+m_2}\hat{p}^2 - \frac{m^2p^2}{(m_1+m_2)^2}\left[p^2\left(\varepsilon^{0,0} + \varepsilon^{\widetilde{0},\widetilde{0}}\right) + p_\mu p_\nu\left(\varepsilon^{\mu,\nu} + \varepsilon^{\widetilde{\mu},\widetilde{\nu}}\right)\right], \tag{70}$$

$$\hat{p}^5 = \frac{\left(m_1^2+m_2^2\right)p^2}{\left(m_1+m_2\right)^2}\hat{p}^3 - \frac{m^2p^4}{\left(m_1+m_2\right)^2}\hat{p}, \tag{71}$$

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