

## Quantum Mechanics Without The Quantum

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ABSTRACT. The two Heisenberg Uncertainties entail an *incompatibility* between the two pairs of conjugated variables  $E, t$  and  $p, q$ . But incompatibility comes in *two* kinds, exclusive of one another. There is incompatibility defineable as:  $(p \rightarrow \neg q) \wedge (q \rightarrow \neg p)$  **or** defineable as  $[(p \rightarrow \neg q) \wedge (q \rightarrow \neg p)] \Leftrightarrow r$ . The former kind is *unconditional*, the latter conditional. The former, in accordance, is fact *independent*, and thus ascertainable by virtue of *logic*, the latter fact dependent, and thus ascertainable by virtue of fact. The two types are therefore diametrically opposed.

In spite of this, however, the existing derivations of the Uncertainties are shown here to entail both types of incompatibility simultaneously.  $\Delta E \Delta t \geq h$ , for example, is known to derive from the quantum relation  $E = h\nu$  plus the Fourier relation  $\Delta\nu \Delta t \simeq 1$ . And the Fourier relation assigns a *logical* incompatibility between a  $\Delta\nu = 0$ ,  $\Delta t = 0$ . (No frequency defineable at an instant.) Which is therefore fact independent and unconditional. How can one reconcile this with the fact that  $\Delta E \Delta t$  *if and only if*  $h > 0$ , which latter supposition is a *factual* truth, entailing that a  $\Delta E = 0$ ,  $\Delta t = 0$  incompatibility should itself be fact dependent?

To then say that the incompatibility at hand is only logical, i.e. that resulting from  $\Delta\nu \Delta t \geq 1$ , is to treat  $\Delta E = 0$ ,  $\Delta t = 0$  as *unconditionally* incompatible, since this is what their equivalents,  $\Delta\nu = 0$ ,  $\Delta t = 0$  are, and therefore as incompatible independently of the *quantum*. And to say that it is only factual, amounts to disputing  $E = h\nu$  itself, whose presence alone is what necessitates application of the -logical-relation  $\Delta\nu \Delta t \geq 1$ . Since either option sacrifices an equally essential requirement, it can only follow that this Uncertainty expresses both a conditional *and* an unconditional form of incompatibility.

We continue by tracing the exact same phenomenon right within the heart of the noncommutative formalism of QM. The *fact* dependent  $p, q$  noncommutativity, expressed in  $pq \neq qp$  as derived from the relation

$pq - qp = i\hbar I$ , has given its place to the abstract Hilbertian, fact independent noncommutativity, expressed in  $AB \neq BA$ , without explicit or implicit reference to  $\hbar$ . Hence, to identify the two would lead to a contradiction comparable to the previous.

## 1 Distinguishing Within Incompatibility

In a series of previous works one of us [1–3] has argued that the incompatibility of the two pairs of conjugated variables, comprising the action products  $Et$  and  $pq$ , as manifested in the two corresponding quantum uncertainties (UR here after), comes in two, *antithetic* types, once because this is how theorists as a rule tend to argue (unfortunately), twice because this is an option contained in the nature of incompatibility itself. Indeed, and contrary to appearances or common opinion, Incompatibility as such is a *twofold* concept. It is not too difficult to establish this in formal logic. The dichotomy can be immediately seen (and felt) in the following way:

- a.  $(p \rightarrow \neg q) \wedge (q \rightarrow \neg p)$  however, also
- b.  $[(p \rightarrow \neg q) \wedge (q \rightarrow \neg p)] \Leftrightarrow r$ .

These two expressions of Incompatibility are contradictory to one another. For the possible value ascription  $\neg r$  in [b] we will obtain “*p and q*”, a possibility which will never come up within the contents of formula [a]. For ascription  $\neg r$  to [b], formulae [a] and [b] immediately develop incompatible truth tables. This is because formula [a] expresses unconditional incompatibility between  $p$  and  $q$ , while formula [b] only a conditional incompatibility between them, conditional, to be exact, on  $r$ . [b] reads: “*p* excludes *q* and *q* excludes *p* if and only if *r*.” But not otherwise. Since the possibility that  $\neg r$  stands for “otherwise”, for the ascription  $\neg r$  the two propositional variables,  $p$  and  $q$ , will cease to be incompatible. But in formula [a] they never cease to be, come what may.

In other words, the incompatibility between the two propositional variables expressed in formula [b], as being conditional on the presence of an additional factor ( $r$ ), is one which obtains only in *some* cases. Namely, iff  $r$ . Since, however, the incompatibility expressed in formula [a] is not conditional on anything, this latter obtains independently of all other factors what so ever and hence obtains for *all possible* cases instead. It therefore (trivially) follows that no two pairs of concepts can

be ever both, conditionally as well as unconditionally incompatible for no two pairs of concepts can ever be both, incompatible in all possible cases and also incompatible only in some.

The problem in QM is that, once this distinction is explicitly drawn as above (which it never is), we frequently find ourselves obliged to conclude that the classical concepts,  $E$  with  $t$  and  $p$  with  $q$ , are indeed both. But of this later.

At present our task is to determine why and how -viz. under what specific conditions Incompatibility presents itself in two antithetic ways. To put the point differently, it may be clear to us how the concepts "one" and "many" may be incompatible to one another. They are as a matter of *definition*. Hence, the incompatibility between "one" and "many" is a straightforward matter of logic. And is therefore unexceptional. (Unconditional). But we have already seen in our definition that not all kinds of Incompatibility are unexceptional; that is to say, [b]. How then does the incompatibility expressed by [b] come about?

Here is an example: I have a daughter and, besides, 10.000 dollars in the bank. No problem there. But then my daughter is kidnapped and I receive a ransom note for 10.000 dollars. I can no longer have both, my daughter *and* 10.000 dollars in the bank. In view of the specific circumstances confronting me, a pair of hitherto compatible situations have been rendered "mutually exclusive", Bohr's known term for inaugurating his introduction to Complementarity (CTY hereafter). "Having one's daughter" and "having 10.000 dollars in the bank" are not incompatible states *per se* in the least. But they can be made incompatible, provided that the right sort of suitable *conditions* are introduced. In their face any two, hitherto compatible states (or concepts) can be rendered incompatible, *on condition* that a suitable set of physical conditions are obtaining or provisionally introduced, thereby forbidding their hitherto recorded mutual compatibility for the entire duration of their presence. It goes without saying that, once the presence of such conditions is removed, the two (temporarily) disjunctive states will become mutually compatible once again in their usual, peaceful coexistence.

We have seen, therefore, that in the case of conditional incompatibility between a pair of states (i.e. of incompatibility type [b]), it takes the intervention of an *additional fact*, if it is to ever result. This we may entitle "*the prohibitive fact*". ([2], p. 188). It should be stressed that the "prohibitive" element in question is invariably and uniquely an *additional fact* and nothing over and above a (mere) fact. And as such,

unexpected from a formal point of view. By contrast, unconditional incompatibility should never be unexpected from a formal point of view, because it is a matter of *logic*. Not a matter of fact which could, formally at least, have gone the other way. The two pairs of antithetic clusters, “factual/conditional” - “logical/unconditional”, are therefore individually coextensive, respectively.

Conditional incompatibility cannot result by virtue of the *definitions* of the (currently) incompatible pair of states (or pair of concepts). If it could, their incompatibility would be logical and, as such, unexceptional; in other words, unconditional. It therefore follows that all instances of conditional incompatibility will invariably turn out to be factual, whereas, by evident contrast, all instances of unconditional incompatibility invariably logical. From this realization follows a further consequence, the importance of which to the overall argument we can hardly overemphasize. The preceding considerations have unequivocally established that unconditional incompatibility is *self sufficient*. By contrast, once again, conditional incompatibility is never self sufficient. It invariably requires an *additional*, prohibitive factor, capable of driving the two thus related states to incompatibility, an incompatibility which otherwise -and in absence of the said factor - would itself be impossible to result.

The two formal (and exhaustive<sup>1</sup>) definitions of incompatibility previously specified, i. e. those of incompatibility type [a] and incompatibility type [b], succeed in reflecting the property of Self-Sufficiency -or its absence- quite explicitly. In formula [a] the incompatibility is *confined* to the two related variables,  $p$  and  $q$ , at the exclusion of all other conditions, and we are forbidden to go looking beyond the two variables *per se* for its establishment. In fact, to go looking *beyond* the two variables of relation  $(p \rightarrow \neg q) \wedge (q \rightarrow \neg p)$  for tracing or grounding their incompatibility is, quite simply, contradictory to the assumption. If only to repeat the point, unconditional incompatibility is (*intolerantly*) self sufficient.

In formula [b], however, the situation is altogether different. The biconditional connective,  $\Leftrightarrow$ , speaks for itself. The variables  $p$  and  $q$  will never in the context of formula [b] enter a relation of mutual incompatibility without help from outside.  $p$  and  $q$  will simply be compatible without such help, as can be seen immediately from assuming  $\neg r$ . The outside help is withdrawn and the variables become compatible. Conse-

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<sup>1</sup>Exhaustive, that is to say, in Two-Valued Propositional Calculus.

quently, on the whole, unconditional incompatibility is synonymous with self-sufficient incompatibility and, accordingly, conditional incompatibility synonymous with self-*insufficient* incompatibility. Emphasis on this provision, though it may seem pedantic to most readers at this stage, is nonetheless well warranted and many quantum surprises will depend on it.

Now that the distinction between conditional and unconditional Incompatibility has been defined and understood as above it is time to turn and ask the next, natural question: Do the two quantum uncertainties,  $\Delta E \Delta t \geq h$ ,  $\Delta p \Delta q \geq h$  express conditional or do they express unconditional incompatibility between their two related sets of variables,  $E$  with  $t$  and  $p$  with  $q$ ? In view of the preceding reasoning the answer to this question is similarly natural. The reciprocal uncertainties in the values of the two pairs of conjugated variables,  $E$  with  $t$  and  $p$  with  $q$ , as presently joined, obviously express *conditional* incompatibility between these variables. Conditional (obviously) on  $h$  itself. Clearly, for  $h = 0$  both clusters of related uncertainties would vanish. On the other hand, they do emerge for  $h > 0$ . Consequently,  $\Delta E \Delta t$ ,  $\Delta p \Delta q$  are uncertainties which are there because and *only* because of  $h$ . And would be removed in its absence. This reads, respectively,

$$\Delta E \Delta t \Leftrightarrow h > 0 \text{ and, accordingly, } \Delta p \Delta q \Leftrightarrow h > 0$$

which both precisely correspond to logical formula [b]. Evidently, then, the two UR express conditional incompatibility between their related variables, conditional, that is, on nothing other than  $h$ .

The interpretation thus suggested can then be integrated just as naturally as all the other elements so far were in the following (natural) fashion: The two pairs of conjugate classical variables,  $E$  and  $t$ ,  $p$  and  $q$ , yielding the two action products  $Et$  and  $pq$  of the corresponding uncertainties, are rendered incompatible in QM because, simply, the latter theory incorporates an *additional fact*, hitherto unacknowledged and unanticipated by the classical theory, namely, action quantization, and it is the intervention of this precise fact, absent in classical assumptions, which is responsible for the incompatibility in the joint determinations of  $E$  and/or  $t$  and  $p$  and/or  $q$  below its limit,  $h$ . The two sets of incompatibilities are therefore *fact dependent*, that is to say, conditional on a *fact*;  $h$ . And therefore, trivially, express conditional incompatibility only. One of the most reliable commentators in the field, C.A.Hooker, certainly seems to think so and not at all without reason:

Bohr believes that while it has seemed to us at the macro-level of classical physics that the conditions were in general satisfied for the joint applicability of all classical concepts, we have *discovered* this century that this is not accurate and that the conditions required for the applicability of some classical concepts are actually incompatible with those required for the applicability of other classical concepts. This is the burden of the doctrine (B4) [=CTY.]

This conclusion is necessitated by the discovery of the quantum of action *and only because of its existence*. It is not therefore a purely *conceptual* discovery that could have been made *a priori* through a more critical *analysis* of classical concepts. It is a discovery of the **factual absence** of the conditions required for the joint applicability of certain classical concepts.[[9] Dark letters for the author's italics.]

This, therefore, is exactly as foretold. The incompatibility above referred to is factual, because it is not the product of concept analysis, disclosing a logical discrepancy between the disjunctive concepts (and as such available a priori) and, therefore, as being fact *dependent*, it is eo ipso conditional. Conditional, that is, on the fact itself upon which it is dependent, and which we have previously labelled “the prohibitive fact”. In other words, the quantum. Perhaps a fleeting allusion to the spontaneity of the author's account of the matter, its ‘naturalness’ so to speak, would not be entirely out of place. Spontaneity is important in this instance because it serves in crosschecking the two accounts, ours, which is in conscious awareness of a contrast between these two types of incompatibility, formally defined, and Hooker's, which is rather intuitive and reflexive at this stage<sup>2</sup>.

On the whole, therefore, at first it seems a safe bet that the two pairs of classical variables of QM, when featuring pairwise in the two

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<sup>2</sup>And at a subsequent stage he has repudiated it altogether! Indeed, at a later time Hooker has actually expressed his *scepticism* as concerns the viability of the distinction between logical and factual incompatibility as formulated by [2] or, even, its usefulness as such. In his letter to Antonopoulos, dated 18 December 1989, he says that when it comes to “formal” as opposed to “factual” aspects of the problem at hand “naturalists like me [him] cannot make a sharp distinction between the two kinds of truth”. Well, up *there* he has! Not too consciously, it would appear, but nonetheless most definitely. Which is all to the better, really, for, when not too much undue sophistication has come by just yet, to hold one captive to wavering amphiboly, first thoughts are best thoughts.

corresponding quantum uncertainties, they should express conditional incompatibility between the thus related concepts and nothing but that. On closer inspection, however, the situation appears a great deal more complex than initially assumed. Closer inspection in fact reveals that, when analyzed and examined all across the logico-mathematical board, the quantum uncertainties manifest and force upon us an incompatibility which is *both*; conditional *and* unconditional for one and the same pair of classical concepts.

Amazingly, the same phenomenon is noticeable, as we shall demonstrate later on, right within the noncommutative formalism itself. To the fundamental, noncommutative formalism inaugurated by the formula  $pq - qp = i\hbar$  there is now erected the noncommutative formalism of  $AB \neq BA$ . In other words, a noncommutativity without the quantum! There is hardly ever a commentator who would not treat the two commutativities as interchangeable, with the sole exception, in our knowledge, of Hilgevoord and Uffink [8]. But they are not really interchangeable at all. One is the *mathematical* consequence of non-diagonal matrix multiplication, yielding noncommutativity by definition, the other a noncommutativity *due to*  $\hbar$ . There's a difference.

## 2 Applying the Distinction

### 2.1 Application to Wave-Particle Duality

The results of applying the Conditional vs Unconditional Incompatibility distinction to quantum problems appear quite startling when viewed in this light. As a rule observed by nearly all physicists, the quantum uncertainties and Wave-Particle Duality (WPD hereafter) are treated as if intimately associated. And, indeed, there is a strong temptation to associate them. According to this association a certain group of classical variables by nature relate to the particle, their complementary variables by nature to the wave. But particles are local entities, so particles are small. By contrast, waves are nonlocal entities, so waves are large. And the opposition between large and small is logical, that is to say, fact *independent*. Hence, waves and particles are self-sufficiently incompatible. This is why, besides, waves and particles are incompatible ***also in classical mechanics***. And classical mechanics does not contain the quantum.

Well, then. If the two UR are a consequence of WPD, one set of variables belonging to the wave, the other set to the particle, then, since

waves (large) and particles (small) exclude one another *self-sufficiently*, and hence without any help from the quantum, the variables appearing in the two UR, as derived from WPD, would *also* exclude one another self-sufficiently, and hence exclude one another without any help from the quantum. In fact, they do not need any help from anything at all, except of course the *self contained* opposition between “large” and “small” itself. Which opposition, as remarked, obtains independently of the quantum. Consequently, either the two UR have nothing to do with WPD or else they have to do with WPD<sup>3</sup>, but then they have nothing to do with the ...*quantum*, on which, however, they are supposed to depend!

In other words, how can the incompatibility contained in WPD, which is self sufficient enough to obtain full force even in *classical mechanics*, ever be responsible for the incompatibility between the classical conjugate variables, which latter results only on the basis of *quantum* assumptions? Or, to put the point differently, how can a fact independent incompatibility, as that belonging to WPD, ever be responsible for a fact dependent incompatibility, as that demanded by the two quantum uncertainties?

Some people still believe that WPD is the epitome of the quantum uncertainties, if not indeed the epitome of QM as such. But once the Conditional/Unconditional Incompatibility contrast is applied to it, it simply proves to be an incoherence. The uncertainties, exactly as Hooker stressed, must absolutely depend on the quantum or be nothing at all. But if the uncertainties are constructed upon the logical model afforded by WPD, they will thereby express a self-sufficient type of incompatibility and, as we have seen, such incompatibility -trivially- has no need of the quantum. To be precise, cannot even make room for the quantum, except contradictorily. People think that WPD furnishes the right sort of quantum incompatibility required by the UR. We have just shown that it furnishes the wrong sort, if there ever was one.

## 2.2 Application to $\Delta E \Delta t \geq h$

But the real trouble does not lie in the comparison between an invalid derivation and a -let us say- valid one. The invalid one can be discarded at no cost. The real trouble lies within the frame of the valid derivation itself. For that too is equally open to both accounts, the conditional *and* the unconditional. Consider the Fourier reasoning applied to the

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<sup>3</sup>Which is what one of us has been insisting for two decades now; See [1] and, especially, [2].



quantum relation  $E = h\nu$ . (See [8] and Bohr's own work referred to there; for more detail see Marmet, 1994, p.343; see, finally, [1])

Fourier's known relation,  $\Delta\nu\Delta t \geq \frac{1}{2\pi}$ , was based on the observation that it is a logical impossibility to determine the frequency at an instant  $dt = 0$ . Frequency is by definition a repetitive phenomenon and hence by definition such as requires a time latitude to be exemplified, if at all. Obviously, I cannot define the regular reoccurrence of a certain event over even time intervals within a time  $dt = 0$ , i.e. a time so narrow that won't allow the event to occur even once. As D.M. Mackay has remarked almost fifty years ago, the idea of defining a frequency at an instant  $dt \rightarrow 0$  is self contradictory. "This is not physics but logic", he says [10].

Once the quantum relation,  $E = h\nu$  is (factually!) established, by simply substituting for  $\nu = \frac{E}{h}$  in Fourier's above mentioned relation, we immediately obtain  $\frac{\Delta E}{h}\Delta t \geq \frac{1}{2\pi}$  and, finally,  $\Delta E\Delta t \geq h$ . Now, what *sort* of incompatibility does this relation express, if derived in this way? Well, it should express precisely the sort of incompatibility which  $\nu$  itself, the frequency, does in Fourier's relation. Are we not constantly reminded that "energy" is the frequency in QM? Mackay, for instance, speaks of

the *identification* of energy with frequency [10].

and, in much more recent times, we are told in no uncertain terms that

This simple Planck relationship between energy and light frequency in effect says that energy and frequency are *the same thing*, measured in different units [6].

A shorthand expression of the whole idea is the relation  $E \approx \nu$ , which says it all. So to the task of specifying the syllogistic mechanism involved:

- Premise 1: Energy is logically equivalent with the frequency.
- Premise 2: Frequency is logically incompatible with an exact time.
- Conclusion: Hence, *energy* is logically incompatible with an exact time.

Here *Conclusion* follows from Premises 1 and 2 as trivially as the proverbial mortality of Socrates follows from "All Men Are Mortal" and

“Socrates Is A Man”. Attention should be paid to the subordinate predicate, “logically”, modally conditioning the primary predicate, “incompatible”. If Energy is coextensional with the Frequency, this predicate must necessarily be included in the *Conclusion*, otherwise, and in its absence, the latter will not be validly drawn, contradicting their coextensionality. Hence, in the most straightforward and valid of manners, energy is above shown to be *logically* incompatible with an exact time, just as frequency previously was. But, as we have seen, concepts incompatible in this sense are self - sufficiently so. And concepts which are self - sufficiently incompatible are concepts whose incompatibility is in fact independent. And therefore such that cannot even *relate* to a fact, e.g. *h*. Hence, in accordance with the Fourier treatment of the relation  $E = h\nu$ , we obtain an uncertainty  $\Delta E\Delta t$  due to a fact, *h*, with which it cannot even relate.

The reactions to this conclusion are not too difficult to foresee. Fortunately, we have at our disposal something a good deal more substantial than mere foresight to get our hands on, namely, an actual objection recently raised. It is the following:

I cannot share the author’s diagnosis. The energy - time uncertainty relation can be derived from two premises: (1) $E = h\nu$  (2) $\Delta\nu\Delta t \geq 1$ . Here it is clear that the second relation is the result of Fourier analysis, and therefore independent of any physical assumption. The first however is clearly a non-trivial *physical* assumption, that need not hold in physical theories other than QM. (1) and (2) together imply  $\Delta E\Delta t \geq h$  (3). The diagnosis is simply this: since conclusion (3) depends on two premises, one of which is dependent upon a physical assumption, the conclusion<sup>4</sup> is dependent on this [physical] assumption too.<sup>5</sup>

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<sup>4</sup>The conclusion should be dependent on this physical assumption, *h* or  $E = h\nu$ , says the author of the passage, correcting us. But we have never denied that it *depends* on this assumption. Anything but. We have only raised the question, whether the physical assumption referred to *retains its identity*. We have never denied *whether*  $E = h\nu$  is a premise to the argument. This is precisely what we have stressed. We have only questioned the *nature* of this premise and whether the *context* of the argument, imposing the logical relation  $\Delta\nu\Delta t \geq 1$ , allows it to retain its original logical properties or whether it retrodictively cancels them, given the overall pressures of the said context.

<sup>5</sup>Extract from a report on a previous version of the argument, dated 4th November 2003. Italics, brackets and initial ours. The report was written for *Studies in the*

And hence must be dependent on  $h$ . This is all so nice and cozy and so consonant with quantum tradition that hardly anyone would resist the temptation of replying just thus, a referee all the more so. However, it takes but one word to spoil it all, its hopes and plans included, though not necessarily the fun as well: Substitution. Once this word is properly attended to, this objection is exposed in all its circular and dogmatic incorrigibility.

What is the true essence of the entire Fourier derivation? It is, in a word, the *substitution* of  $\nu$  in  $\Delta\nu\Delta t \geq 1$  by  $E/h$  in order to derive  $\Delta E\Delta t \geq h$ . And in order that one can be at all entitled to substitute  $E/h$  for  $\nu$  one needs to presuppose that the two of them, the substitute and the substituted, will just have to be identical, or equal, or equivalent or what have you, provided they are so intimately linked as to license and, indeed, entail the substitution. You name it, they have to be it. In consequence,  $E/h$ , which replaces  $\nu$ , the frequency, *is* the frequency, or else the substitution is illegitimate and has no business being there in the first place. And then, since  $E/h$  is the frequency, what is true of the frequency must be true of its substitute,  $E/h$ . And then, since what is true of the frequency is that it is unconditionally incompatible with time, which this referee openly concedes,  $E/h$  is also unconditionally incompatible with time, which he inconsistently does not. It is either that or else the substitution is sheer bogus and no  $\Delta E\Delta t \geq h$  of any kind will result, coherent or otherwise.

By right of mathematical law, the law of substitutions,  $E/h(=\nu)$  is unconditionally incompatible with time, *even* if it deceitfully contains  $h$  in the denominator of the fraction just to mislead (some of) us. The conclusion can now be denied at the pain of contradiction. By inserting  $E/h$  in the place of  $\nu$  in  $\Delta\nu\Delta t \geq 1$ , we commit ourselves to making  $E/h$  whatever  $\nu$  is, thus deriving a logical uncertainty and, therefore, a fact independent one that cannot even relate to this very  $h$ , which it has itself put there! In the face of our distinction the Fourier treatment of  $E = h\nu$  leads to incoherence and absurdity comparable to that of WPD previously encountered (essentially it is the same exact problem), if not indeed to a worse kind. Valid reasoning is reasoning which transmits the logical properties of the premises down to the last conclusion. And the logical properties of premise  $\Delta\nu\Delta t \geq 1$  is that it incorporates a self-sufficient type of incompatibility, which renders  $h$  redundant.

The essence of the problem here encountered stems from the fact that, in view of the distinction here introduced (and hitherto absent in all quantum theorizing),  $E = h\nu$  proves a full scale logical *hybrid*. Initially,  $E = h\nu$  states a factual truth -a startling one at that- so whatever  $E$ ,  $t$  incompatibility is subsequently destined to result on its basis, it should only be fact-dependent in this particular context. However, what this (unique) factual truth reveals right after is, that the concept which is (factually) equivalent with  $E$ , i.e. the frequency,  $\nu$ , is itself *logically* incompatible with an exact  $t$ , thereby rendering the thus resulting  $E$ ,  $t$  incompatibility a fact independent one, in this other context. Given the surrounding, 'outer' context, i.e. the factualness of  $E = h\nu$ ,  $E$  and  $t$  must be conditionally incompatible. But given the *surrounded*, 'inner' context, i.e. the logical incompatibility between a  $\Delta\nu = 0$  with a  $\Delta t = 0$ ,  $E$  and  $t$  must now be unconditionally incompatible.

When, in other words,  $E = h\nu$  is considered in its (outward) relation to reality, it must in this capacity be a factual truth. But when considered in itself (inwardly), in this other capacity it incorporates a logical truth. What should we say then? That what  $E = h\nu$  really asserts is that, on its basis,  $E$  and  $t$  are unconditionally incompatible concepts *on condition* that  $E = h\nu$  is true? On the basis of the distinction here introduced this is exactly what we have to say. Though, of course, in its absence, we wouldn't have to.

### 3 The hybrid nature of Quantum Mechanical Formalism

In Heisenberg's paper of 1925[7] there is mentioned a type of multiplication between the quantities characteristic of a quantum system directly leading to relations of noncommutativity between them. Such multiplication was subsequently identified by Born and Jordan[5] as a multiplication of matrices corresponding to the physical quantities attributable to a quantum system. This was the inauguration of transformation theory which in turn developed into the widely disseminated axiomatic foundation of von Neumann's[11].

In the following pages of the present essay we shall attempt to classify Heisenberg's quantum multiplication -this is how it will be referred to from now on so that it will be distinguished from matrix multiplication as such-, the multiplication of matrices and their concomitant noncommutativities, and finally the resulting uncertainties, on the basis of the distinction established in the first part of the paper. In particular, the relations mentioned in Heisenberg's paper are satisfied by physical sys-

tems *on condition that action is quantized and on that condition alone*. By contrast, the mathematical treatment, which was initiated by Born and Jordan, constitutes a “hybrid” for the second time running, because the premises of this latter hypothesis may lead to the noncommutative relation  $pq - qp \neq 0$  for the variables  $p, q$ , but the specific relation  $pq - qp = i\hbar I^6$  is not intrinsically derivable from within it. It is extrinsically introduced on the basis of further assumptions.

In other words, the latter noncommutativity is inherent in advance within the chosen formalism, as a self-subsisting mathematical property, contrary to Heisenberg’s multiplication, which is factually *dependent* on the quantum and cannot result in its absence. Whereupon, the noncommutativity in question must become system-specific in order to be applied to the relevant phenomena.

Commencing, Heisenberg denounces the classical picture of an electron’s kinematics and proceeds to the adoption of a different interpretation of the function  $x(t)$  whose classical interpretation would be the particle’s position in the space of intuition. In a parallel course, however, considering the correspondence principle, he retains the differential equation which governs the said function (Newton’s second law) in the classical treatment. Thus he accepts that the equation  $\ddot{x}(t) + f(x) = 0$  regulates the connection of  $x(t)$  with the outwardly exerted force  $f(x)$ .

In what follows he analyzes  $x(t)$  in “Fourier” fashion, so that the resulting expression will harmonize itself with the quantum conditions. In the classical case, if  $\nu(n, a)$  is the frequency observed during the transition from state  $n$  to state  $a$ , then  $\nu(n, a) = \frac{a}{\hbar} \frac{dW(n)}{dn}$  (1) where  $W(n)$  is the energy of the said state. By contrast, due to the presence of discontinuous states in the quantum case, the frequency during the transition from state  $W(n)$  to state  $W(n - a)$  is characterized by emission of radiation  $\nu(n, n - a) = \frac{W(n) - W(n - a)}{\hbar}$  (2). Suppose then that  $x(n, t)$  is  $x(t)$  in the specific case that the electron is in the state  $W(n)$ . Then, classically,  $x(n, t)$  would be expanded as  $\int_{-\infty}^{+\infty} U_a(n) e^{i\omega(n)at} da$  (3), where  $U_a(n)$  is now a complex quantity, i.e. the transition amplitude, whose squared measure furnishes the probability that an electron will pass from state  $W(n)$  to state  $W(a)$ . Quantum mechanically,  $x(n, t)$  is expanded into a series  $\sum_{a=-\infty}^{+\infty} U(n, n - a) e^{i\omega(n, n - a)t}$  (4)<sup>7</sup>, where  $U(n, n - a)$  plays a

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<sup>6</sup>This formula is referred to as canonical commutation relation. In [7] there is only a specific form of this relation.

<sup>7</sup>With the assumption that the system is periodic or multiperiodic. Else the series

part analogous with  $U_a(n)$  and  $\omega(n, n - a) = 2\pi\nu(n, n - a)$ . Then in accordance with the form assumed by  $f(x)$  there are obtained retrodictive formulae for the quantities  $A(n, n - a)$  and  $\omega(n, n - a)$ , where  $A(n, n - a) = \text{Re}\{U(n, n - a)\}$ , introducing the expansion into the differential equation.

Consider then two quantities

$$\alpha(t) = \sum_{a=-\infty}^{+\infty} U(n, n-a)e^{i\omega(n, n-a)t}, \beta(t) = \sum_{a=-\infty}^{+\infty} V(n, n-a)e^{i\omega(n, n-a)t}.$$

Then  $\alpha(t)\beta(t) = \sum_{b=-\infty}^{+\infty} Z_1(n, n-b)e^{i\omega(n, n-b)t}$  and

$$\beta(t)\alpha(t) = \sum_{b=-\infty}^{+\infty} Z_2(n, n-b)e^{i\omega(n, n-b)t},$$

where  $Z_1(n, n-b) = \sum_{a=-\infty}^{+\infty} U(n, n-a)V(n-a, n-b)e^{i\omega(n, n-b)t}$  and

$$Z_2(n, n-b) = \sum_{a=-\infty}^{+\infty} V(n, n-a)U(n-a, n-b)e^{i\omega(n, n-b)t}.$$

Whereupon, in general, we obtain  $\alpha(t)\beta(t) - \beta(t)\alpha(t) \neq 0$ . That is to say, the multiplication of the two quantities ceases being commutative. And this noncommutativity exists by virtue of the quantum of action. Were it not for the quantum, we would not have observed a discontinuous and countable sequence of states, starting from the ground state. In consequence, it would not be formula of minuses nr.(2), which would then obtain, but nr.(1). But then, as can be verified by a single calculation,  $\alpha(t)\beta(t) - \beta(t)\alpha(t) = 0$ . This result is directly specified for the magnitudes  $x(t)$ , of position, and  $p = m\dot{x}(t)$ , of momentum.

We conclude by contending that the noncommutative quantum multiplication here obtained is satisfied only on the basis that  $h > 0$  and would reduce to ordinary, commutative multiplication, were the limitation  $h > 0$  to be withdrawn. As will soon become evident this is no longer true for transformation theory and, by extension, for von Neumann's axiomatization. Noncommutativity is still there but it is now of a different type, resulting as a self contained property of the mathematical scheme employed. And which is therefore unconditionally true, i.e. such that obtains independently of  $h$ . To compensate for this, it then has to be remodified in order to accord itself with the phenomenon investigated, and thus it is inconsistently reshaped into a noncommutativity by virtue of facts on top of the previous. This may not suffice to condemn all parts of the formalism, but it can be a severe problem.

Commencing their treatment Born and Jordan assume that the conjugate dynamical variables of the system under investigation,  $p, q$ , constitute hermitian matrices of the form  $p = (p(nm)e^{i\omega(nm)t})$  and  $q =$

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has to be replaced by the integral  $\int_{-\infty}^{+\infty} U_a(n)e^{i\omega(n, n-a)t} da$  without any essential change in the foregoing argument.

$(q(nm)e^{i\omega(nm)t})$ , where  $m, n \in \mathbb{N}$ ,  $\omega(nm) = \omega(n, m) = 2\pi \frac{W(n) - W(m)}{\hbar}$  in Heisenberg's symbolism and  $q(nm), p(nm) \in \mathbb{C}$  generally. Due to this the products  $pq$  and  $qp$  are considered to be the products of a pair of matrices. It is commonplace knowledge that matrix multiplication is the archetype of noncommutative multiplication in Mathematics. Therefore, generally  $pq - qp \neq 0$  (5) without the quantum playing a part in this effect. What is more, it is shown next in the work referred to that the relation  $pq - qp = i\hbar I$ , where  $I$  is the identity matrix, by means of Bohr's quantum formula  $J = \int_0^{\frac{1}{\nu}} pdq = nh$ . It is more than evident that this last formula constitutes the premise, so as to make matrix mechanics agree with the phenomenon, which in Heisenberg's argument has perfectly sufficed for deriving the quantal multiplication without any external help. The authors of the paper themselves remark that relation (5) constitutes a direct statement of the correspondence principle. That is to say, when  $\hbar \rightarrow 0$ ,  $pq - qp \rightarrow 0$ , and the quantities become commutative. However, this statement is misleading.

It is true that  $pq - qp \rightarrow 0$ , when  $\hbar \rightarrow 0$ , but this "0" is the zero matrix and not the zero of real numbers. If that relation constituted a genuine statement of the correspondence principle,  $p$  and  $q$  would in the end turn out real numbers, and this is not what happens. Quite simply, from noncommutative matrices they turn into commutative ones. This will not of course degenerate all the eigenvalues of each and every *matrix* into one, and so the quantum fluctuations will remain invariant. The difference is that the conjugate variables coevolve without excluding one another. Although the two multiplications, the quantal and the matrix, display the same formal attitude, this will not coerce their referents to become identical.

By contrast, in Heisenberg's reasoning, when  $\hbar \rightarrow 0$ , the "Fourier" series assumes a classical expansion, since we can now differentiate by using formula (1) and the two quantities turn out to be real numbers. In this way, while the two multiplications, the quantal and the matrix, manifest the same outward effects, the mathematical objects which each one refers are quite distinct.

And it is at this point that the inconsistent deviation from Heisenberg's multiplication is inaugurated. Conflating the two cases, as we have seen it done before (1st Part) Born, Jordan and von Neumann proceed on the supposition that the two noncommutativities were analogous or identical and, hence, that the two formalisms were. But they are not. For while in Heisenberg's multiplication, on the assumption that " $\hbar = 0$ "

noncommutativity is eliminated, in the case of [5] where a zero matrix is obtained instead, noncommutativity is not. It just assumes a different look by putting on a mask, yet without ever departing from its authentic identity.

On our understanding of the matter, the problem is rather simple to state though not necessarily simple to solve. The noncommutativity in Heisenberg's multiplication is eliminable in principle because it is conditional on the quantum, hence removable in its absence, but the noncommutativity of Born, Jordan and von Neumann is ineliminable in principle, because it is self subsisting. It is a fact-independent noncommutativity and therefore ineliminable come what may. It is a noncommutativity ascertainable in advance, a clear cut mathematical phenomenon the residues of which stay on in one form or another, once this mathematics is employed. Not being conditional on anything except its own self it therefore continues to tacitly obtain even when the quantum is removed, yielding for classical expectations a zero matrix only, instead of zero just, bearing witness to its own fact-independent origin.

Consequently, it will either be consistently regarded in its pure mathematical essence, whereupon however it cannot even relate to the quantum, or else incorporate Heisenberg's conditional noncommutativity, if to be at all able to apply to specific quantum problems, but then do so at the price of an antinomy. The very antinomy detected in our first part of the argument, where the logical  $\Delta\nu\Delta t \geq 1$  is turned into the factual  $\Delta E\Delta t \geq h$ , when the empirical relation  $E = h\nu$  is introduced. Yielding in both instances the same logical hybrid.

Before we conclude we wish to make explicit the situation in von Neumann's abstract axiomatization<sup>8</sup>. Let us list the axioms in this approach:

1. To each quantum system there corresponds a complex separable Hilbert space  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  where  $\langle \cdot, \cdot \rangle$  is the inner product. Every  $\psi \in \mathcal{H}$  with  $\|\psi\| = \sqrt{\langle \psi, \psi \rangle} = 1$  corresponds to a state of the system. By equivalence, the projective space  $P\mathcal{H} = \{[\psi] : (\psi \in \mathcal{H}) \wedge (\|\psi\| = 1)\}$  constitutes the set of the states of the system.
2. The observables of the system in question are symmetrical operators over the space  $(\mathcal{H}, \langle \cdot, \cdot \rangle)$  that is to say, observable  $A$

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<sup>8</sup>We shall study autonomous (isolated) systems. So the Hamiltonian is time-independent and corresponds to the total energy of the system



is represented by a linear operator  $A$  such that  $\langle A\phi, \psi \rangle = \langle \phi, A\psi \rangle, \forall \phi, \psi \in \mathcal{H}$ . The values of the observable  $A$  are the spectrum of the corresponding symmetrical operator,  $\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda Id_{\mathcal{H}} \text{ is singular}\}^9$ , where  $Id_{\mathcal{H}}$  is the identity mapping. The mean value of the observable,  $E(A)$ , is given by the corresponding spectral measure,  $E(A) = \langle \psi, A\psi \rangle$ , where  $\psi$  the state of the system.

3. The Hamiltonian of the system,  $H$ , constitutes an observable and its corresponding operator is self-adjoint. In Heisenberg's representation, if  $A$  is an observable, then its temporal evolution is determined by the equation  $\dot{A} = \frac{i}{\hbar}[H, A]$ , where  $[H, A] = HA - AH$  is the commutator of  $H$  and  $A$ . In Schroedinger's picture, when the system is in state  $\psi(t_0)$  at the time  $t_0$ , it is in the state  $\psi(t) = \exp(-\frac{i}{\hbar}H(t - t_0))\psi(t_0)^{10}$  (formally the Schroedinger equation is then written as  $i\hbar \frac{\partial \psi}{\partial t} = H\psi$ ). In every one of the cases the Hamiltonian constitutes an infinitesimal generator of the evolution of the dynamical quantities.

The first two axioms introduce a self-contained noncommutativity, since the observables are represented as symmetric operators in a Hilbert space. The sole case, when the symmetric operators always commute, is that of a one-dimensional Hilbert space, that is to say  $\mathcal{H} = \mathbb{C}$ . But then the system should have only one state which could occupy. It is evident why we can't accept this strongly artificial situation. If we demand, however, of the quantities of momentum and position to fulfill the normal rules of commutativity, the resulting space must now be one of infinite dimensionality, as Born and Jordan themselves remark. (To be precise, the operators of momentum and position cannot be bounded [14, 15], and this is why in the foregoing axioms we speak of symmetric rather than of self-adjoint operators.)

Consequently, noncommutativity is in this case inherent within the axiomatic system a priori, as is inherent in the statement, "If  $X > 10$ , then  $X > 5$ ", the statement that "then  $X > 6$ ". Which latter is a trivial consequence of the previous, hypothetical statement. Similarly the noncommutativity in question is a tautological consequence of the axioms 1 and 2 and has nothing to do with the quantum of action. Further ahead, in the third axiom the quantum emerges in the evolution

<sup>9</sup>Cause of the operator's symmetry the spectrum is a subset of the real numbers

<sup>10</sup>We've used here the Stone theorem, see [13]

equation of each picture. This, however, constitutes a further introduction for settling the matter, exactly as it has in the papers of Born and Jordan, and that of Born, Jordan and Heisenberg which followed.

Operating within the frame of von Neumann's axiomatization we can demonstrate Robertson's general uncertainty relation: Let  $A, B$  be two observables. Then the inequality  $(Var(A)Var(B))^{\frac{1}{2}} \geq \frac{1}{2}|E([A, B])|$  obtains, where  $Var(A)$  the variance of  $A$ ,  $Var(A) = E((A - E(A))^2)$  (the same for  $Var(B)$ ).

This inequality is also self-sufficient and obtains without mediation of the quantum. If we consider the relation  $[p, q] = i\hbar Id_{\mathcal{H}}$  for position and its conjugated momentum then  $(Var(p)Var(q))^{\frac{1}{2}} \geq \frac{\hbar}{2}$  or equivalent  $\Delta p \Delta q \geq \frac{\hbar}{2}$ , if  $\Delta A = \sqrt{Var(A)}$  is the uncertainty dispersion of quantity  $A$ , that is to say, Heisenberg's uncertainty relation for momentum and position. But this uncertainty should be dependent on the quantum and still its derivation, down to its terminal conclusion, has been quite possible without it.

We therefore notice that although Robertson's inequality reflects a self-sufficient noncommutativity and hence independent of  $\hbar$ , by introducing further premises, we end up with a hybrid noncommutativity, due to and not due to  $\hbar$  in the end. In other words an uncertainty relation that is fact dependent and fact independent at the same time. Higevoord and Uffing argue[8] that Robertson's inequality presents problems additional to the one we have detected.

In concluding we contend that the mathematical formalism has since 1925 been tracking a most mysterious and confusing path. On the one hand it imposes upon the physical magnitudes involved noncommutativities warranted *a priori*, and hence such as were true in advance of the factual discovery that  $\hbar > 0$ , and on the other hand introduces a different kind of noncommutativity altogether, in order to force the former to come to agreement with the physical content it purports to reflect. We are not pursuing the deeper causes or motives behind this contradictory tendency but we do believe that steps towards its amendment should be taken, so that the formalism may maintain its applicability and at the same time restore the authenticity of the physical ideas which have given rise to it.

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