Objective quantum theory based on the CP(N-1) affine gauge fields

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The ordinary linear quantum theory predicts the quantum correlations at any distance (the universal superposition principle). It creates the decoherence problem since quantum interactions entangle states into non-separable combination. On the other hand the linear quantum theory prevents the existence of the localizable solutions, and after all, leads to the divergences problem in the quantum field theory. In order to overcome these difficulties the non-perturbative nonlinearity originated by the curvature of the compact quantum phase space has been used.

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1 Introduction

Non-linearity in quantum theory has been invoked in order to build the objective quantum theory and to prevent the unlimited spread out of the observable fields by the gravitational self-potential [1, 2]. But Newtonian quantum gravity in the present form is not effective for the shaping wave-packets of elementary particle size since the characteristic scale of the ground-state wave-packet obtained from the gravitational Schrödinger equation for nucleon masses is around $10^{23}m$ [2].

There is a different group of works that emphasize the formulation of *the standard quantum mechanics* in quantum phase space (QPS) represented by the complex projective Hilbert space $CP(N-1)$ [3, 4, 5, 6, 7]. I think, however, that a consistent and prolific theory based on such QPS should be connected with serious deviations from the standard quantum scheme. Such a modification must, of course, preserve all achievements of de Broglie-Heisenberg-Schrödinger-Dirac linear theory in a natural way. One may think about attempts to establish a deductive approach to the quantum theory.

Standard quantum mechanics (QM) treats the electron as a pointwise particle but it is ‘wrapped’ in so-called de Broglie-Schrödinger fields of probability. Quantum field theory (QFT) uses the same classical space-time coordinates of the pointwise particle as ‘indices’ whereas the fields are operators acting in some Hilbert state space (frequently in Fock space). QM and QFT take account of the non-commutative nature of the dynamical variables but the interaction between pointwise particles and the relativistic invariance are borrowed from the classical theory. These are the sources of the singular functions involved in QFT. It is useful to understand the true reason of these difficulties.

The special and general relativity is based on the possibility to detect locally the coincidence of two pointwise events of different nature. As such the “state” of the local clock gives us local coordinates - the “state” of the incoming train [8]. In the classical case the notions of the “clock” and the “train” are intuitively clear. Furthermore, Einstein especially notes that he did not discuss the inaccuracy of the simultaneity of two *approximately coinciding events* which should be overcome by some abstraction [8]. This abstraction is of course the neglect of finite sizes (and all internal degrees of freedom) of both the real clock and the train. It gives the representation of these “states” by mathematical points in space-time. Thereby the local identification of two events is the formal source of the classical relativistic theory. But in the quantum case this is impossible since the localization of quantum particles is state-dependent [9, 10, 11]. Hence the identification of quantum events (transitions) requires a physically motivated operational procedure with corresponding mathematical description.

Therefore it is inconsistent to start the development of the quantum theory from the space-time symmetries because just the space-time properties should be established in some approximation to internal quantum dynamics, i.e. literally *a posteriori*. Namely, the quantum measurement with help of the “quantum question” leads locally to the Lorentz transformations of its spinor components, and, on the other hand, to dynamical (state-dependent) introduction of space-time coordinates. Therefore, instead of the representation of the Poincare group in some extended Hilbert space, I use an “inverse representation” of the $SU(N)$ by solutions of relativistic quasi-linear partial derivative equations (PDE) in the dynamical space-time. It is in fact one of the possible realizations of L.

de Broglie's idea about the "wave with a hump" [12].

In the present article I propose a non-linear relativistic 4D field model originated by the internal dynamics in QPS $CP(N-1)$ [13, 14]. This is the development of the ideas used in [15]. There is no initial distinction between 'particle' and 'field', and the space-time manifold is derivable. Quantum measurements will be described in terms of parallel transport of the local dynamical variables.

2 The Action Quantization

Schrödinger sharply denied the existence of so-called "quantum jumps" during the process of emission/absorption of the quanta of energy (particles) [16, 17]. Leaving the question about the nature of quantum particles outside of consideration, he thought about these processes as a resonance of the de Broglie waves that phenomenologically may look like "jumps" between two "energy levels". The second quantization method formally avoids these questions but there are at least two reasons for its modification:

First. In the second quantization method one formally assumes particles the properties of which are defined by some commutation relations between creation-annihilation operators. Note, that the commutation relations are only the simplest consequence of the curvature of the dynamical group manifold in the vicinity of the group's unit (in algebra). Dynamical processes require, however, finite group transformations and, hence, the global group structure. In this paper the main technical idea is to use vector fields over a group manifold instead of Dirac's abstract q-numbers. This scheme therefore tries to elucidate the dynamical nature of the creation and annihilation processes of quantum particles.

Second. The quantum particles (energy bundles) should gravitate. Hence, strictly speaking, their behavior cannot be described as a linear superposition. Therefore the ordinary second quantization method (creation-annihilation of free particles) is merely a good approximate scheme due to the weakness of gravity. Thereby the creation and annihilation of particles are time consuming dynamical non-linear processes. So, linear operators of creation and annihilation (in Dirac sense) do exist as approximate quantities.

POSTULATE 1.

There are elementary quantum states $|\hbar a>, a = 0, 1, ...$ *belonging to the Fock space of an abstract Planck oscillator whose states correspond to* *the quantum motions with given number of Planck action quanta.*

One may imagine some *"elementary quantum states" (EAS)* $|\hbar a>$ *as a quantum motion with entire number* a *of the action quanta.* These $a, b, c, ...$ replace of the "principal quantum number" serving as discrete indices $0 \leq a, b, c... < \infty$. Since the action itself does not create gravity, it is possible to form linear superpositions of $|\hbar a>= (a!)^{-1/2}(\hat{\eta}^+)^a|\hbar 0>$ constituting $SU(\infty)$ multiplets of the Planck's action quanta operator $\hat{S} = \hbar\hat{\eta}^+\hat{\eta}$ with the spectrum $S_a = \hbar a$ in the separable Hilbert space $\mathcal{H}$. Therefore, we shall primarily quantize the action, and not the energy. The relative (local) vacuum of some problem is not necessarily the state with minimal energy, it is a state with an extremal of some action functional.

The space-time representation of these states and their coherent superposition is postponed to the dynamical stage as it will be described below. We shall construct non-linear field equations describing energy (frequency) distributions between EAS's $|\hbar a>$, the soliton-like solution of which provide the quantization of the dynamical variables. Presumably, the stationary processes are represented by stable particles and quasi-stationary processes are represented by unstable resonances.

Generally the coherent superposition

$$|F>=\sum_{a=0}^{\infty} f^a|\hbar a>, \tag{1}$$

may represent a ground state or a "vacuum" of some quantum system with the action operator

$$\hat{S} = \hbar A(\hat{\eta}^+\hat{\eta}). \tag{2}$$

Then one can define the action functional

$$S[|F>] = \frac{<F|\hat{S}|F>}{<F|F>}, \tag{3}$$

which has the eigen-value $S[|\hbar a>] = \hbar a$ on the eigen-vector $|\hbar a>$ of the operator $\hbar A(\hat{\eta}^+\hat{\eta}) = \hbar\hat{\eta}^+\hat{\eta}$ and that deviates in general from this value on superposed states $|F>$ and of course under a different choice of $\hat{S} = \hbar A(\hat{\eta}^+\hat{\eta}) \neq \hbar\hat{\eta}^+\hat{\eta}$. In order to study the variation of the action functional on superposed states one needs more details on geometry of their superposition.

In fact only a finite member N of elementary quantum states (EQS's) $(|\hbar 0>, |\hbar 1>, ..., |\hbar(N-1)>)$ may be involved in the coherent superposition $|F>$. Then $\mathcal{H} = \mathcal{C}^{\mathcal{N}}$ and the ray space $CP(\infty)$ will be restricted to finite dimensional $CP(N-1)$. Hereafter we will use the indices as follows: $0 \leq a, b \leq N$, and $1 \leq i, k, m, n, s \leq N-1$. This superposition physically corresponds to the complete amplitude of some quantum motion. Sometimes it may be interpreted as a extremum of the action functional of some classical variational problem.

The global vacuum $|\hbar 0>$ corresponds to the zero number of action quanta in the places of the Universe far enough from stars with pseudo-Euclidean metric in accordance with the

POSTULATE 2.

'Mach's quantum principle': the Universe generates the omnipresent average self-consistent cosmic potential coinciding with the fundamental constant $g_{00} = \Phi_U = c^2$.

Matter exists in the motion of a finite number of the action quanta. The mass of some quantum particle gives the rate of variation of the Universe potential Φ_U in accordance with the de Broglie frequency $\omega = \frac{mc^2}{\hbar}$. Therefore omnipresent $\Phi_U = c^2$ serves as a "spring" of the "local internal clock" showing the state-dependent time τ instead of the "world time" of Newton-Stueckelberg-Horwitz-Piron [18].

Since any ray of the action amplitude has isotropy group $H = U(1) \times U(N)$ only the coset transformations $G/H = SU(N)/S[U(1) \times U(N-1)] = CP(N-1)$ effectively act in $\mathcal{H}$. Therefore the ray representation of $SU(N)$ in C^N, in particular, the embedding of H and G/H in G, is a state-dependent parametrization. Hence, there is a diffeomorphism between the space of the rays marked by the local coordinates in the map $U_j : \{|G>, |g^j| \neq 0\}, j > 0$

$$\pi^i_{(j)} = \begin{cases} \frac{g^i}{g^j}, & \text{if } 1 \leq i < j \\ \\ \frac{g^{i+1}}{g^j} & \text{if } j \leq i < N-1 \end{cases} \tag{4}$$

and the group manifold of the coset transformations $G/H = SU(N)/S[U(1) \times U(N-1)] = CP(N-1)$. This diffeomorphism is provided by the coefficient functions Φ^i_α of the local generators (see below). The choice of the map U_j means, that the comparison of quantum amplitudes refers to the amplitude with the action $\hbar j$. The breakdown of $SU(N)$ symmetry on each action amplitude to the isotropy group $H = U(1) \times U(N-1)$ contracts the full dynamics down to $CP(N-1)$. The physical interpretation of these transformations is given by the

POSTULATE 3.

The unitary transformations of the action amplitudes may be identified with physical fields; i.e., transformations of the form $U(\tau) = \exp(i\Omega^\alpha \hat{\lambda}_\alpha \tau)$, where the field functions Ω^α are the parameters of the adjoint representations of $SU(N)$. *The coset transformation* $G/H = SU(N)/S[U(1) \times U(N-1)] = CP(N-1)$ *is the quantum analog of a classical force; its action is equivalent to some physically distinguishable variation of generalized coherent states (GCS) in* $CP(N-1)$.

Thus the quantum dynamics in the $CP(N-1)$ manifold is similar to general relativity dynamics, where due to the equivalence principle, gravity is locally non-distinguishable from an accelerated reference frame [19]. But in general relativity one has the distinction (by definition) between gravity (curvature) and its 'matter' source. In quantum physics, however, all physical fields are 'matter' and variation of these fields leads to the variation of the basis in the state space.

3 Non-linear treatment of the eigen-problem

The quantum mechanics assumes the priority of the Hamiltonian given by some classical model which henceforth should be "quantized". It is known that this procedure is ambiguous. In order to avoid the ambiguity, I intend to use a *quantum state* itself and the invariant conditions of its conservation and perturbation. These invariant conditions are rooted in the global geometry of the dynamical group manifold. Namely, the geometry of $G = SU(N)$, the isotropy group $H = U(1) \times U(N-1)$ of the pure quantum state, and the coset $G/H = SU(N)/U(1) \times U(N-1)$ geometry, play an essential role in the quantum state evolution (the super-relativity principle [20]). The stationary states (some eigen-states of the action operator, i.e. the states of motion with the least action) may be treated as *initial conditions* for GCS evolution. Particulary they may represent a local minimum of energy (vacuum).

Let me assume that $\{|\hbar a>\}_0^{N-1}$ is the basis in Hilbert space $\mathcal{H}$. Then a typical vector $|F> \in \mathcal{H}$ may be represented as a superposition $|F>= \sum_0^{N-1} f^a|\hbar a>$. The eigen-problem may be formulated for some hermitian dynamical variable $\hat{D}$ on these typical vectors

$\hat{D}|F>=\lambda_D|F>$. This equation may be written in components as follows: $\sum_0^{N-1} D_b^a f^b=\lambda_D f^a$, where $D_b^a=<a|\hat{D}|b>=F_D^\alpha\hat{\lambda}_{\alpha,(ab)}$,

$$\hat{D}=\sum_{a,b\geq 0}<a|\hat{D}|b>\hat{P}_{ab}=\sum_{a,b\geq 0}D_{ab}\hat{P}_{ab}=\sum_{a,b\geq 0}F_D^\alpha\hat{\lambda}_{\alpha,(ab)}\hat{P}_{ab}, \qquad (5)$$

where $\hat{P}_{ab}$ is projector. In particular, the Hamiltonian has a similar representation with $F_H^\alpha=\hbar\Omega^\alpha$ [21].

One has the spectrum of $\lambda_D : \{\lambda_0,...,\lambda_{N-1}\}$ from the equation $Det(\hat{D}-\lambda_D\hat{E})=0$, and then one has the set of equations $\hat{D}|D_p>=\lambda_p|D_p>$, where $p=0,...,N-1$ and $|D_p>=\sum_0^{N-1} g_p^a|\hbar a>$ are eigen-vectors. It is worthwhile to note that the solution of this problem gives rays and not vectors, since eigen-vectors are defined up to the complex factor. In other words we deal with rays or points of the non-linear complex projective space $CP(N-1)$ for a $N\times N$ matrix of the linear operator acting on C^N. The Hilbert spaces of the infinite dimension will be discussed later.

For each eigen-vector $|D_p>$ corresponding λ_p it is possible to chose at least one such component g_p^j of the $|D_p>$, that $|g_p^j|\neq 0$. This choice defines in fact the map $U_{j(p)}$ of the local projective coordinates

$$\pi_{j(p)}^i=\begin{cases}\frac{g_p^i}{g_p^j}, & \text{if } 1\leq i<j\\ \frac{g_p^{i+1}}{g_p^j} & \text{if } j\leq i<N-1\end{cases} \qquad (6)$$

in $CP(N-1)$ for each eigen-vector $|D_p>$ of the ray. Note, if all $\pi_{j(p)}^i=0$ it means that one has the "pure" state $|D_p>=g_p^j|j>$ (without summation in j). Any different points of the $CP(N-1)$ corresponds to the GCS's. They will be treated as self-rays of some deformed action operator. Beside this I will treat the superposition state $|G>=\sum_{a=0}^{N-1}g^a|a\hbar>$ as "analytic continuation" of the of eigen-vector for an arbitrary set of the local coordinates.

People frequently omit the index p, assuming that $\lambda:=\lambda_p$, for $j=0$. Then they have, say, for the $N\times N$ Hamiltonian matrix $\hat{H}$ the eigen-problem $(\hat{H}-\lambda)|\psi>=0$ where I put $\psi^a:=g_0^a$.

In accordance with our assumption the λ is such that $\psi^0\neq 0$. Let then divide all equations by ψ^0. Introducing local coordinates $\pi^i=\frac{\psi^i}{\psi^0}$, we get the system of the non-homogeneous equations

$$(H_{11}-\lambda)\pi^1+...+H_{1i}\pi^i+...+H_{1N-1}\pi^{N-1}=-H_{10}$$

$$H_{21}\pi^1 + (H_{22} - \lambda)\pi^2 + ... + H_{2i}\pi^i + ... + H_{2N-1}\pi^{N-1} = -H_{20}$$
$$\vdots$$
$$H_{N-11}\pi^1 + ... + H_{N-1i}\pi^i + ... + (H_{N-1N-1} - \lambda)\pi^{N-1} = -H_{N-10}, \quad (7)$$

where the first equation

$$H_{01}\pi^1 + ... + H_{0i}\pi^i + ... + H_{0N-1}\pi^{N-1} = -(H_{00} - \lambda) \quad (8)$$

is omitted. If $D = det(H_{ik} - \lambda\delta_{ik}) \neq 0, i \neq 0, k \neq 0$ then the single defined solutions of this system may be expressed through Cramer's rule

$$\pi^1 = \frac{D_1}{D}, ..., \pi^{N-1} = \frac{D_{N-1}}{D}. \quad (9)$$

It is easy to see that these solutions being substituted into the first omitted equation give us simply the reformulated characteristic equation of the eigen-problem. Therefore one has the single valued ray solution of the eigen-problem expressed in local coordinates instead of the vector solution with additional freedom of a complex scale multiplication.

This approach does not offer an essential advantage for a single operator, it only shows that the formulation in local coordinates is quite natural. But if one tries to understand the multi-dimensional variation of a hermitian operator included in a parameterized family, the local formulation is inevitable. First of all it is interesting to know the invariants of such variations. In particular, the quantum measurement of a dynamical variable represented by a hermitian $N \times N$ matrix should be described in the spirit of typical polarization measurement of the coherent photons [22].

The initial state of the coherent photons $|x>$ is modulated passing through an optically active medium (for instance the Faraday effect in YIG film magnetized along the main axes in the z-direction by a harmonic magnetic field with frequency Ω and the angle amplitude β). Formally this process may be described by the action of the unitary matrix $\hat{h}_{os_3}$ belonging to the isotropy group of $|R>$ [15]. Then the coherence vector will oscillate along the equator of the Poincaré sphere. The next step is the dragging of the oscillating state $|x'(t)>= \hat{h}_{os_3}|x>$ with frequency ω up to the "north pole" corresponding to the state $|R>$. In fact this is the motion of the coherence vector. This may be achieved by the variation of the azimuth of the linear polarized state from $\frac{\theta}{2} = -\frac{\pi}{4}$

up to $\frac{\theta}{2} = \frac{\pi}{4}$ with help of the dense flint of appropriate length embedded into the sweeping magnetic field. Further this beam should pass the $\lambda/4$ plate. This process of variation of the ellipticity of the polarization ellipse may be described by the unitary matrix $\hat{b}_{os'_1}$ belonging to the coset homogeneous sub-manifold $U(2)/[U(1) \times U(1)] = CP(1)$ of the dynamical group $U(2)$ [22]. This dragging without modulation leads to the evolution of the initial state along the geodesic of $CP(1)$ and the trace of the coherent vector is the meridian of the Poincaré sphere between the equator and one of the poles. The modulation deforms both the geodesic and the corresponding trace of the coherence vector on the Poincaré sphere during such unitary evolution.

The action of the $\lambda/4$ plate depends upon the state of the incoming beam (the relative orientation of the fast axes of the plate and the polarization of the beam). Furthermore, only relative phases and amplitudes of photons in the beam have a physical meaning for the $\lambda/4$ plate. Neither the absolute amplitude (intensity of the beam), nor the general phase affect the polarization character of the outgoing state. It means that the device action depends only upon the local coordinates $\pi^1 = \frac{\Psi^1}{\Psi^0} \in CP(1)$. Small relative re-orientation of the $\lambda/4$ plate and the incoming beam leads to a small variation of the outgoing state. This means that the $\lambda/4$ plate re-orientation generates the tangent vector to $CP(1)$. It is natural to discuss the two components of such a vector: velocities of the variations of the ellipticity and of the azimuth (inclination) angle of the polarization ellipse. They are examples of local dynamical variable (LDV). The comparison of such dynamical variables for different coherent states requires that the affine parallel transport agrees with the Fubini-Study metric.

As far as I know the generalized problem of the quantum measurement of an arbitrary hermitian dynamical variable $\hat{H} = E^\alpha \hat{\lambda}_\alpha, \quad \hat{\lambda}_\alpha \in AlgSU(N)$ in the operational manner given above was never done. It is solved here by the exact analytical diagonalization of a hermitian matrix. Previously this problem was partly solved in the works [23, 24, 25]. Geometrically it looks like embedding "the ellipsoid of polarizations" into the iso-space of the adjoint representation of $SU(N)$. This ellipsoid is associated with the quadric form $< F|\hat{H}|F >= \sum_1^{N^2-1} E^\alpha < F|\hat{\lambda}_\alpha|F >= H_{ab}(E^\alpha) f^{a*} f^b$ depending on $N^2 - 1$ real parameters E^α. The shape of this ellipsoid with N main axes is given by the $2(N-1)$ parameters of the coset transformations $G/H = SU(N)/S[U(1) \times U(N-1)] = CP(N-1)$ which are related to the $(N-1)$ complex local coordinates of the eigen-state of $\hat{H}$ in $CP(N-1)$. Its orientation in iso-space R^{N^2-1} is much more complicated than in the case of R^3. It is given by generators of the isotropy group containing $N-1 = rank(AlgSU(N))$ independent parameters of "rotations" about commutative operators $\hat{\lambda}_3, \hat{\lambda}_8, \hat{\lambda}_{15}, ...$ and $(N-1)(N-2)$ parameters of rotations about non-commutative operators. All these $(N-1)^2 = (N-1) + (N-1)(N-2)$ gauge angles of the isotropy group $H = S[U(1) \times U(N-1)]$ of the eigen-state giving orientation of this ellipsoid in iso-space R^{N^2-1} will be calculated now during the process of analytical diagonalization of the hermitian matrix $H_{ab} =< a|\hat{H}|b >$ corresponding to some dynamical variable $\hat{H}$.

Stage 1. Reduction of the general Hermitian Matrix to three-diagonal form. Let me start from general hermitian $N \times N$ matrix $\hat{H}$. One should choose some basis in C^N. I will take the standard basis

$$|1>= \begin{pmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, \quad |2>= \begin{pmatrix} 0 \\ 1 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}, ..., |N>= \begin{pmatrix} 0 \\ 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{pmatrix}. \tag{10}$$

For instance for $N = 3$ the set of Gell-Mann $\hat{\lambda}$ matrices can be decomposed into the two sets with respect to the state $|1>$: B-set $\hat{\lambda_1}, \hat{\lambda_2}, \hat{\lambda_4}, \hat{\lambda_5}$ the exponents of which act effectively on the $|1>$, and the H-set $\hat{\lambda_3}, \hat{\lambda_8}, \hat{\lambda_6}, \hat{\lambda_7}$, the exponents of which leave $|1>$ intact. For any finite dimension N one may define the "I-spin" $(1 \leq I \leq N)$ as an analog of the well known "T-,U-,V- spins" of the $SU(3)$ theory using the invariant character of the commutation relations of B-and H-sets

$$[B,B] \in H, \quad [H,H] \in H, \quad [B,H] \in B. \tag{11}$$

Let me to represent our hermitian matrix in following manner

$$\hat{H} = \begin{pmatrix} & H_{01} & ...H_{0i} & ...H_{0N-1} \\ H_{10} & 0 & ...0 & ...0 \\ H_{20} & 0 & ...0 & ...0 \\ . & & & \\ . & & & \\ . & & & \\ H_{N-10} & 0 & ...0 & ...0 \end{pmatrix}_B$$

$$
+\begin{pmatrix}
H_{00} & 0 & ...0 & ...0 \\
0 & H_{11} & ...H_{1i} & ...H_{1N-1} \\
0 & H_{21} & ...H_{2i} & ...H_{2N-1} \\
. & & & \\
. & & & \\
. & & & \\
0 & H_{N-11} & ...H_{N-1i} & ...H_{N-1N-1}
\end{pmatrix}_H . \quad (12)
$$

With respect to the ket $|1>$ one may to classify the first matrix as $B-type$ and the second one as a matrix of the $H-type$. I will apply now the "squeezing ansatz" [20, 25]. The first "squeezing" unitary matrix is

$$
\hat{U}_1 = \begin{pmatrix}
1 & 0 & 0 & . & . & . & 0 \\
0 & 1 & 0 & . & . & . & 0 \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
0 & . & . & . & 1 & 0 & 0 \\
. & . & . & . & 0 & \cos\phi_1 & e^{i\psi_1}\sin\phi_1 \\
0 & 0 & . & . & 0 & -e^{-i\psi_1}\sin\phi_1 & \cos\phi_1
\end{pmatrix} . \quad (13)
$$

The transformation of similarity being applied to our matrix gives $\hat{H}_1 = \hat{U}_1^+ \hat{H} \hat{U}_1$ with the result for $\hat{H}_B$ shown for simplicity in the case $N = 4$

$$
\hat{H}_{B1} = \begin{pmatrix}
0 & H_{01} & \tilde{H_{02}} & \tilde{H_{03}} \\
H^*_{01} & 0 & 0 & 0 \\
\tilde{H^*_{02}} & 0 & 0 & 0 \\
\tilde{H^*_{03}} & 0 & 0 & 0
\end{pmatrix}, \quad (14)
$$

where $\tilde{H_{02}} = H_{02}\cos\phi - H_{03}\sin\phi e^{-i\psi}$ and $\tilde{H_{03}} = H_{02}\sin\phi e^{i\psi} + H_{03}\cos\phi$. Now one has solve two "equations of annihilation" of $\Re(H_{02}\sin(\phi)e^{i\psi} + H_{03}\cos(\phi)) = 0$ and $\Im(H_{02}\sin(\phi)e^{i\psi} + H_{03}\cos(\phi)) = 0$ in order to eliminate the last element of the first row and its hermitian conjugate [20, 25]. This gives us ϕ'_1 and ψ'_1. I will put $H_{02} = \alpha_{02} + i\beta_{02}$ and $H_{03} = \alpha_{03} + i\beta_{03}$, then the solution of the "equations of annihilation" is as follows:

$$
\phi'_1 = \arctan\sqrt{\frac{\alpha_{03}^2 + \beta_{03}^2}{\alpha_{02}^2 + \beta_{02}^2}},
$$
$$
\psi'_1 = \arctan\frac{\alpha_{03}\beta_{02} - \alpha_{02}\beta_{03}}{\sqrt{(\alpha_{02}^2 + \beta_{02}^2)(\alpha_{03}^2 + \beta_{03}^2)}}. \quad (15)
$$

This transformation acts of course on the second matrix $\hat{H}_H$ too but it is easy to see that its structure remains intact. The next step is the similarity transformations given by the matrix with the diagonally shifted transformation block

$$\hat{U}_2 = \begin{pmatrix} 1 & 0 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ 0 & . & . & . & 1 & 0 & 0 \\ . & . & . & . & 0 & cos\phi_2 & e^{i\psi_2} sin\phi_2 \\ 0 & 0 & . & . & 0 & -e^{-i\psi_2} sin\phi_2 & cos\phi_2 \\ 0 & . & . & . & 0 & 0 & 1 \end{pmatrix} \tag{16}$$

and the similar evaluation of ψ_2', ϕ_2'. Generally one should make $N-2$ steps in order to remove the $N-2$ elements of the first row. The next step is to represent our transformed $\hat{H}_1 = \hat{U}_1^+ \hat{H} \hat{U}_1$ as follows:

$$\hat{H}_1 = \begin{pmatrix} 0 & \tilde{H}_{01} & 0 & ...0 \\ \tilde{H}_{10} & 0 & \tilde{H}_{12} & ...\tilde{H}_{1N-1} \\ 0 & \tilde{H}_{21} & 0 & ...0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & \tilde{H}_{N-1,1} & ...0 & ...0 \end{pmatrix}_B + \begin{pmatrix} H_{00} & 0 & ...0 & ...0 \\ 0 & \tilde{H}_{11} & ...0 & ...0 \\ 0 & 0 & ...\tilde{H}_{2i} & ...\tilde{H}_{2N-1} \\ . & & & \\ . & & & \\ . & & & \\ 0 & 0 & ...\tilde{H}_{N-1,i} & ...\tilde{H}_{N-1,N-1} \end{pmatrix}_H . \tag{17}$$

Now one should applied the squeezing ansatz in $N-3$ steps for second row, etc., generally one has $(N-1)(N-2)$ orientation angles. Thereby we come to the three-diagonal form of the our matrix.

Stage 2. Diagonalization of the three-diagonal form. The eigenvalue problem for the three-diagonal hermitian matrix is well known, but I will do it for the sake of completeness. The eigen- problem

$(\hat{\tilde{H}} - \lambda \hat{E})|\xi >= 0$ for the three-diagonal matrix has the following form

$$\begin{pmatrix} \tilde{H}_{00}\xi^0 & \tilde{H}_{01}\xi^1 & 0 & . & .\ . & 0 \\ \tilde{H}^*_{01}\xi^0 & \tilde{H}_{11}\xi^1 & \tilde{H}_{12}\xi^2 & 0 & .\ . & 0 \\ 0 & \tilde{H}^*_{12}\xi^1 & \tilde{H}_{22}\xi^2 & \tilde{H}_{23}\xi^3 & .\ . & 0 \\ 0 & 0 & \tilde{H}^*_{23}\xi^2 & . & .\ . & 0 \\ 0 & 0 & . & . & .\ . & 0 \\ 0 & 0 & . & . & .\ . & \tilde{H}_{N-1,N-2}\xi^{N-1} \\ 0 & 0 & . & . & .\ . & \tilde{H}_{N-1,N-1}\xi^{N-1} \end{pmatrix} = \begin{pmatrix} \lambda\xi^0 \\ \lambda\xi^1 \\ \lambda\xi^2 \\ . \\ . \\ . \\ \lambda\xi^{N-1} \end{pmatrix} \quad (18)$$

Since $\xi^1 = \frac{\lambda - \tilde{H}_{00}}{\tilde{H}_{01}}\xi^0$, etc., one has the recurrent relations between all components of the eigen-vector corresponding to given λ. Thereby only $N-1$ complex local coordinates $(\pi^1 = \frac{\xi^1}{\xi^0}, ..., \pi^{N-1} = \frac{\xi^{N-1}}{\xi^0})$ which characterize the shape of the ellipsoid of polarization are relevant.

Stage 3. The coset "force" acting during a measurement The real measurement assumes some interaction of the measurement device and incoming state. If we assume for simplicity that the incoming state is $|1>$ (modulation, etc. are neglected), then all transformations from H-subalgebra will leave it intact. Only the coset unitary transformations

$$\hat{T}(\tau, g) =$$
$$\begin{pmatrix} \cos g\tau & \frac{-p^{1*}}{g}\sin g\tau & \frac{-p^{2*}}{g}\sin g\tau & . & \frac{-p^{N-1*}}{g}\sin g\tau \\ \frac{p^1}{g}\sin g\tau & 1+[\frac{|p^1|}{g}]^2(\cos g\tau - 1) & [\frac{p^1 p^{2*}}{g}]^2(\cos g\tau - 1) & . & [\frac{p^1 p^{N-1*}}{g}]^2(\cos g\tau - 1) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \frac{p^{N-1}}{g}\sin g\tau & [\frac{p^{1*}p^{N-1}}{g}]^2(\cos g\tau - 1) & . & . & 1+[\frac{|p^{N-1}|}{g}]^2(\cos g\tau - 1) \end{pmatrix}, \quad (19)$$

with $g = \sqrt{|p^1|^2 +, ..., +|p^{N-1}|^2}$ will effectively influence this state dragging it along one of the geodesic in $CP(N-1)$ [20]. This matrix describes the process of the transition from one pure state to another, in particular between two eigen-states connected by the geodesic. This means that these transformations deform the ellipsoid. All possible shapes of these ellipsoids are distributed along a single geodesic.

Generally, in the dynamical situation this "stationary" global procedure is not applicable and one should apply the local analog of $\hat{\lambda}$-matrices, i.e. $SU(N)$ generators and related dynamical variables should be parameterized by the local quantum states coordinates $(\pi^1, ..., \pi^{N-1})$.

4 Local dynamical variables

The state space $\mathcal{H}$ with finite action quanta is a stationary construction. We introduce dynamics *by the velocities of the GCS variation* representing some "elementary excitations" (quantum particles). Their dynamics is specified by the Hamiltonian, giving time variation velocities of the action quantum numbers in different directions of the tangent Hilbert space $T_{(\pi^1,...,\pi^{N-1})}CP(N-1)$ which takes the place of the ordinary linear quantum state space as will be explained below. The rate of the action variation gives the energy of the "particles" whose expression should be established by some wave equations.

The local dynamical variables correspond to the internal $SU(N)$ group of the GCS and the breakdown of this group should be expressed now in terms of the local coordinates π^k. The Fubini-Study metric

$$G_{ik^*} = [(1+\sum |\pi^s|^2)\delta_{ik} - \pi^{i^*}\pi^k](1+\sum |\pi^s|^2)^{-2} \tag{20}$$

and the affine connection

$$\Gamma^i_{mn} = \frac{1}{2}G^{ip^*}(\frac{\partial G_{mp^*}}{\partial \pi^n} + \frac{\partial G_{p^*n}}{\partial \pi^m}) = -\frac{\delta^i_m \pi^{n^*} + \delta^i_n \pi^{m^*}}{1+\sum |\pi^s|^2} \tag{21}$$

in these coordinates will be used. Hence the internal dynamical variables and their norms should be state-dependent, i.e. local in the state space [20].

Without the application of (20) the local dynamical variables realize a non-linear representation of the unitary global $SU(N)$ group in the Hilbert state space C^N. Namely, N^2-1 generators of $G = SU(N)$ may be divided in accordance with the Cartan decomposition: $[H,H] \in H, [B,H] \in B, [B,B] \in H$. The $(N-1)^2$ generators

$$\Phi^i_h \frac{\partial}{\partial \pi^i} + c.c. \in H, \quad 1 \le h \le (N-1)^2 \tag{22}$$

of the isotropy group $H = U(1) \times U(N-1)$ of the ray (Cartan subalgebra) and $2(N-1)$ generators

$$\Phi^i_b \frac{\partial}{\partial \pi^i} + c.c. \in B, \quad 1 \le b \le 2(N-1) \tag{23}$$

are the coset $G/H = SU(N)/S[U(1) \times U(N-1)]$ generators realizing the breakdown of the $G = SU(N)$ symmetry of the GCS. Furthermore, the

$(N-1)^2$ generators of the Cartan sub-algebra may be divided into the two sets of operators: $1 \leq c \leq N-1$ ($N-1$ is the rank of $AlgSU(N)$) Abelian operators, and $1 \leq q \leq (N-1)(N-2)$ non-Abelian operators corresponding to the non-commutative part of the Cartan sub-algebra of the isotropy (gauge) group. Here $\Phi^i_\sigma, \quad 1 \leq \sigma \leq N^2-1$ are the coefficient functions of the generators of the non-linear $SU(N)$ realization. They give the infinitesimal shift of the i-component of the coherent state driven by the σ-component of the unitary multipole field Ω^α rotating the generators of $AlgSU(N)$ and they are defined as follows:

$$\Phi^i_\sigma = \lim_{\epsilon \to 0} \epsilon^{-1} \left\{ \frac{[\exp(i\epsilon\lambda_\sigma)]^i_m g^m}{[\exp(i\epsilon\lambda_\sigma)]^j_m g^m} - \frac{g^i}{g^j} \right\} = \lim_{\epsilon \to 0} \epsilon^{-1} \{\pi^i(\epsilon\lambda_\sigma) - \pi^i\}, \quad (24)$$

[20, 25]. Then the sum of the N^2-1 energies associated with the intensity of deformations of the GCS is represented by the local Hamiltonian vector field $\vec{H}$ which is linear in the partial derivatives $\frac{\partial}{\partial \pi^i} = \frac{1}{2}(\frac{\partial}{\partial \Re \pi^i} - i\frac{\partial}{\partial \Im \pi^i})$ and $\frac{\partial}{\partial \pi^{*i}} = \frac{1}{2}(\frac{\partial}{\partial \Re \pi^i} + i\frac{\partial}{\partial \Im \pi^i})$. In other words it is the tangent vector to $CP(N-1)$

$$\vec{H} = \vec{T_c} + \vec{T_q} + \vec{V_b} = \hbar\Omega^c\Phi^i_c\frac{\partial}{\partial \pi^i} + \hbar\Omega^q\Phi^i_q\frac{\partial}{\partial \pi^i} + \hbar\Omega^b\Phi^i_b\frac{\partial}{\partial \pi^i} + c.c. \quad (25)$$

In order to express some eigen-vectors in the local coordinates, I put

$$|D_p(\pi^1_{j(p)}, ..., \pi^{N-1}_{j(p)}) >= \sum_0^{N-1} g^a(\pi^1_{j(p)}, ..., \pi^{N-1}_{j(p)})|\hbar a>, \quad (26)$$

where $\sum_{a=0}^{N-1} |g^a|^2 = R^2$, and

$$g^0(\pi^1_{j(p)}, ..., \pi^{N-1}_{j(p)}) = \frac{R^2}{\sqrt{R^2 + \sum_{s=1}^{N-1} |\pi^s_{j(p)}|^2}}. \quad (27)$$

For $1 \leq i \leq N-1$ one has

$$g^i(\pi^1_{j(p)}, ..., \pi^{N-1}_{j(p)}) = \frac{R\pi^i_{j(p)}}{\sqrt{R^2 + \sum_{s=1}^{N-1} |\pi^s_{j(p)}|^2}}, \quad (28)$$

i.e. $CP(N-1)$ is embedded in the Hilbert space $\mathcal{H} = C^N$. Hereafter I will suppose $R = 1$.

Now we see that all eigen-vectors corresponding to different eigen-values (even under the degeneration) are applied to different points $(\pi^1_{j(p)},...,\pi^{N-1}_{j(p)})$ of the $CP(N-1)$. Nevertheless the eigen-vectors $|D_p(\pi^1_{j(p)},...,\pi^{N-1}_{j(p)})>$ are mutually orthogonal in $\mathcal{H}=C^N$ if $\hat{H}$ is a hermitian Hamiltonian. Therefore one has the "splitting" or delocalization of degenerated eigen-states in $CP(N-1)$. Thus the local coordinates π^i give a convenient parametrization of the $SU(N)$ action as one will see below.

Let me assume that $|G>=\sum_{a=0}^{N-1} g^a|a\hbar>$ is a "ground state" of some least action problem. Then the velocity of the ground state evolution referred to the world time is given by the formula

$$|H>=\frac{d|G>}{d\tau}=\frac{\partial g^a}{\partial \pi^i}\frac{d\pi^i}{d\tau}|a\hbar>=|T_i>\frac{d\pi^i}{d\tau}=H^i|T_i>, \tag{29}$$

where

$$|T_i>=\frac{\partial g^a}{\partial \pi^i}|a\hbar>=T^a_i|a\hbar> \tag{30}$$

is the tangent vector to the evolution curve $\pi^i=\pi^i(\tau)$, and

$$\begin{aligned} T^0_i=\frac{\partial g^0}{\partial \pi^i} &= -\frac{1}{2}\frac{\pi^{*i}}{\left(\sqrt{\sum_{s=1}^{N-1}|\pi^s|^2+1}\right)^3}, \\ T^m_i=\frac{\partial g^m}{\partial \pi^i} &= \left(\frac{\delta^m_i}{\sqrt{\sum_{s=1}^{N-1}|\pi^s|^2+1}}-\frac{1}{2}\frac{\pi^m\pi^{*i}}{\left(\sqrt{\sum_{s=1}^{N-1}|\pi^s|^2+1}\right)^3}\right) \end{aligned} \tag{31}$$

Then the "acceleration" is as follows

$$\begin{aligned} |A>=\frac{d^2|G>}{d\tau^2}=|g_{ik}>\frac{d\pi^i}{d\tau}\frac{d\pi^k}{d\tau}+|T_i>\frac{d^2\pi^i}{d\tau^2}=|N_{ik}>\frac{d\pi^i}{d\tau}\frac{d\pi^k}{d\tau} \\ +(\frac{d^2\pi^s}{d\tau^2}+\Gamma^s_{ik}\frac{d\pi^i}{d\tau}\frac{d\pi^k}{d\tau})|T_s>, \end{aligned} \tag{32}$$

where

$$|g_{ik}>=\frac{\partial^2 g^a}{\partial \pi^i \partial \pi^k}|a\hbar>=|N_{ik}>+\Gamma^s_{ik}|T_s> \tag{33}$$

and the state

$$|N>=N^a|a\hbar>=(\frac{\partial^2 g^a}{\partial \pi^i \partial \pi^k}-\Gamma^s_{ik}\frac{\partial g^a}{\partial \pi^s})\frac{d\pi^i}{d\tau}\frac{d\pi^k}{d\tau}|a\hbar> \qquad (34)$$

is the normal to the "hypersurface" of the ground states. Then the minimization of this "acceleration" under the transition from point τ to $\tau+d\tau$ may be achieved by the annihilation of the tangential component

$$(\frac{d^2\pi^s}{d\tau^2}+\Gamma^s_{ik}\frac{d\pi^i}{d\tau}\frac{d\pi^k}{d\tau})|T_s>=0, \qquad (35)$$

i.e. under the condition of the affine parallel transport of the Hamiltonian vector field

$$dH^s+\Gamma^s_{ik}H^i d\pi^k=0. \qquad (36)$$

We saw that $SU(N)$ geometry gives the shape and the orientation of the ellipsoid associated with the "average" of dynamical variable given by a quadric form $<F|\hat{D}|F>$, i.e. this form constitutes an ordinary eigenvalue problem. But if one rises the question about the real operational sense of the quantum measurement of this dynamical variable or the process of the transition from one eigen-state to another, one sees that the corresponding quantum state and its dynamical variables are in a in much more complicated relation than in the orthodox quantum scheme. The simple reason for this is that the decomposition (representation) of the state vector of a quantum system strongly depends on the spectrum and eigen-vectors of its dynamical variable. Overloaded system of the GCS's supplies us by enough big "reserve" of functions but their superposition should be local and they span a tangent space at any specific point of $CP(N-1)$ marked by the local coordinates.

The "probability" may be introduced now in a pure geometric way by $cos^2\phi$ in tangent state space according to the following argument.

For any two tangent vectors $D^i_1=<D_1|T_i>, D^i_2=<D_2|T_i>$ one can define the scalar product

$$(D_1,D_2)=\Re G_{ik^*}D^i_1 D^{k^*}_2=\cos\phi_{1,2}(D_1,D_1)^{1/2}(D_2,D_2)^{1/2}. \qquad (37)$$

Then the value

$$P_{1,2}(\pi^1_{j(p)},...,\pi^{N-1}_{j(p)})=\cos^2\phi_{1,2}=\frac{(D_1,D_2)^2}{(D_1,D_1)(D_2,D_2)} \qquad (38)$$

may be treated as a relative probability of the appearance of two states during the measurements of two different dynamical variables.

Some LDV $\vec{\Psi} = \Psi^i \frac{\partial}{\partial \pi^i} + c.c.$ may be associated with the "state vector" $|\Psi> \in \mathcal{H}$ which has tangent components $\Psi^i =< T_i|\Psi>$ in $T_\pi CP(N-1)$. Thus the scalar product

$$(\Psi, D) = \Re G_{ik^*} \Psi^i D^{k^*} = \cos\phi_{\Psi,D} (\Psi,\Psi)^{1/2} (D,D)^{1/2} \tag{39}$$

gives the local correlation between two LDV's at same GCS. The cosines of directions

$$P_{\Psi,i}(\pi^1_{j(p)}, ..., \pi^{N-1}_{j(p)}) = \cos^2\phi_{\Psi,i} = \frac{(\Psi, D^i)^2}{(\Psi,\Psi)(D^i,D^i)} \tag{40}$$

may be identified with "probabilities" in each tangent direction of $T_\pi CP(N-1)$. The conservation law of "probability" is given by the simple identity

$$\sum_{i=1}^{N-1} P_{\Psi,i} = \sum_{i=1}^{N-1} \cos^2\phi_{\Psi,i} = 1. \tag{41}$$

The notion of the "probability" is of course justified by our experience since different kinds of fluctuations prevent the exact knowledge of any quantum dynamical variable. That is not only because the uncertainty relation between *two* canonically conjugated dynamical variables puts the limit of accuracy, but also because any real measurement of a *single* dynamical variable or the process of preparation of some state are not absolutely exact. It is easy to see from the relation between the velocity $V^i = \frac{d\pi^i}{d\tau}$ in $CP(N-1)$ and the energy variance $(\Delta H)^2$ through the Aharonov-Anandan relationship $\frac{dS}{d\tau} = \frac{2\Delta H}{\hbar}$ [26], where $\Delta H = \sqrt{<\hat{H}^2> - <\hat{H}>^2}$ is the uncertainty of the Hamiltonian $\hat{H}$. Indeed, the quadric form in the local coordinates is as follows: $dS^2 = G_{ik^*} d\pi^i d\pi^{k*} = \frac{4(\Delta H)^2}{\hbar^2} d\tau^2$ and, therefore,

$$(\Delta H)^2 = \frac{\hbar^2}{4} G_{ik^*} \frac{d\pi^i}{d\tau} \frac{d\pi^{k*}}{d\tau}, \tag{42}$$

i.e. velocity V^i in $CP(N-1)$ defines the variance of the Hamiltonian.

But it is not the reason to deny a possibility to know any dynamical variable with an acceptable accuracy.

5 Objective Quantum Measurement

The $CP(N-1)$ manifold takes the place of the "classical phase space" since its points, corresponding to the GCS, are most close to classical states of motion. These points may be interpreted as the "Schrödinger's lump" [27]. It is important that in this case the "Schrödinger's lump" has the exact mathematical description and clear physical interpretation: points of $CP(N-1)$ are the axis of the ellipsoid of the action operator $\hat{S}$, i.e. extremals of the action functional $S[|F>]$. Then the velocities of variation of these axis correspond to local Hamiltonian or different local dynamical variables.

Let me assume that GCS described by local coordinates $(\pi^1, ..., \pi^{N-1})$ corresponds to the original lump, and the coordinates $(\pi^1+\delta\pi^1, ..., \pi^{N-1}+\delta\pi^{N-1})$ correspond to the lump displaced due to measurement. I will use a GCS $(\pi^1_{j(p)}, ..., \pi^{N-1}_{j(p)})$ of some action operator $\hat{S} = \hbar A(\hat{\eta^+ \eta})$ representing physically distinguishable states. This means that any two points of $CP(N-1)$ define two ellipsoids which differ at least by the orientations, if not by the shape, as it was discussed above. As such, they may be used as "yes/no" states of some two-level detector.

Local coordinates of the lump give a firm geometric tool for the description of quantum dynamics during interaction which used for a measuring process. The question that I would like to raise is as follows: *which "classical field", i.e. field in space-time, corresponds to the transition from the original to the displaced lump?* In other words I would like to find the measurable physical manifestation of the lump , which I called the "field shell", its space-time shape and its dynamics. The lump's dynamics will be represented by energy (frequencies) distributions that are not a priori given, but are defined by some field equations where the latter should be established by means of a new variation problem. Before its formulation, we wish to introduce a differential geometric construction.

I assume that there is a *expectation state* $|D>$: $\hat{D}|D>= \lambda_p|D>$, associated with the "measuring device" tuned for the measurement of the dynamical variable $\hat{D}$ at some eigen-state $(\pi^1_{j(p)},...,\pi^{N-1}_{j(p)})$

$$|D>=|D_p(\pi^1_{j(p)},...,\pi^{N-1}_{j(p)})>=\sum_{a=0}^{N-1}g^a(\pi^1_{j(p)},...,\pi^{N-1}_{j(p)})|\hbar a> \\ =\sum_{a=0}^{N-1}g^a|\hbar a>. \tag{43}$$

Hereafter I will omit indices $j(p)$ for a simplicity. Now one should build the spinor of the "logical spin 1/2" in the local basis $(|N>,|\widetilde{D}>)$ for the quantum question with respect to the measurement of the local dynamical variable $\vec{D}$ at corresponding GCS which may be marked by the local normal state

$$|N>=N^a|\hbar a>=(\frac{\partial^2 g^a}{\partial\pi^i\partial\pi^k}-\Gamma^s_{ik}\frac{\partial g^a}{\partial\pi^s})\frac{d\pi^i}{d\tau}\frac{d\pi^k}{d\tau}|\hbar a>. \tag{44}$$

Since in general $|D>$ it is not a tangent vector to $CP(N-1)$, the deviation from GCS during the measurement of $\hat{D}$ will be represented by tangent vector

$$|\widetilde{D}>=|D>-<Norm|D>|Norm>=|D>-<N|D>\frac{|N>}{<N|N>} \tag{45}$$

defined as the covariant derivative on $CP(N-1)$. This operation is the orthogonal projector $\hat{Q}$. Indeed,

$$\begin{aligned} \widetilde{|\widetilde{D}>} &= (|D>-\widetilde{<Norm|D>}|Norm>) \\ &= |D>-<Norm|D>|Norm> \\ -<Norm|(|D>-<Norm|D>|Norm>)|Norm> \\ &= |D>-<Norm|D>|Norm>=|\widetilde{D}>. \end{aligned} \tag{46}$$

This projector $\hat{Q}$ takes the place of dichotomic dynamical variable (quantum question) for the discrimination of the normal state $|N>$ (it represents the eigen-state at GCS) and the orthogonal tangent state $|\widetilde{D}>$ that represents the velocity of deviation form GCS. The coherent superposition of two eigen-vectors of $\hat{Q}$ at the point $(\pi^1,...,\pi^{N-1})$ forms the spinor η with the components

$$\alpha_{(\pi^1,...,\pi^{N-1})}=\frac{<N|D>}{<N|N>}$$

$$\beta_{(\pi^1,...,\pi^{N-1})} = \frac{< \widetilde{D}|D >}{< \widetilde{D}|\widetilde{D} >}. \tag{47}$$

Then from the infinitesimally close GCS $(\pi^1+\delta^1, ..., \pi^{N-1}+\delta^{N-1})$, whose shift is induced by the interaction used for a measurement, one gets a close spinor $\eta + \delta\eta$ with the components

$$\alpha_{(\pi^1+\delta^1,...,\pi^{N-1}+\delta^{N-1})} = \frac{< N'|D >}{< N'|N' >}$$
$$\beta_{(\pi^1+\delta^1,...,\pi^{N-1}+\delta^{N-1})} = \frac{< \widetilde{D}'|D >}{< \widetilde{D}'|\widetilde{D}' >}, \tag{48}$$

where the basis $(|N' >, |\widetilde{D}' >)$ is the lift of the parallel transported $(|N >, |\widetilde{D} >)$ from the infinitesimally close point $(\pi^1 + \delta^1, ..., \pi^{N-1} + \delta^{N-1})$ back to the $(\pi^1, ..., \pi^{N-1})$. It is clear that such parallel transport should be somehow connected with the variation of the coefficients Ω^α in the dynamical space-time.

The covariant relative transition from one GCS to another

$$(\pi^1_{j(p)}, ..., \pi^{N-1}_{j(p)}) \to (\pi^1_{j'(q)}, ..., \pi^{N-1}_{j'(q)}) \tag{49}$$

and the covariant differentiation (relative Fubini-Study metric) of vector fields provides the objective character of the "quantum question" $\hat{Q}$ and, hence, the quantum measurement. This serves as a basis for the construction of the dynamical space-time.

Two infinitesimally close spinors may be expressed as functions of θ, ϕ, ψ, R and $\theta + \epsilon_1, \phi + \epsilon_2, \psi + \epsilon_3, R + \epsilon_4$, and represented as follows

$$\eta = R \begin{pmatrix} \cos\frac{\theta}{2}(\cos\frac{\phi-\psi}{2} - i\sin\frac{\phi-\psi}{2}) \\ \sin\frac{\theta}{2}(\cos\frac{\phi+\psi}{2} + i\sin\frac{\phi+\psi}{2}) \end{pmatrix} = R \begin{pmatrix} C(c - is) \\ S(c_1 + is_1) \end{pmatrix} \tag{50}$$

and

$$\eta + \delta\eta = R \begin{pmatrix} C(c - is) \\ S(c_1 + is_1) \end{pmatrix}$$
$$+R \begin{pmatrix} S(is - c)\epsilon_1 - C(s + ic)\epsilon_2 + C(s + ic)\epsilon_3 + C(c - is)\frac{\epsilon_4}{R} \\ C(c_1 + is_1)\epsilon_1 + S(ic_1 - s_1)\epsilon_2 - S(s_1 - ic_1)\epsilon_3 + S(c_1 + is_1)\frac{\epsilon_4}{R} \end{pmatrix}. \tag{51}$$

They may be connected by the infinitesimal "Lorentz spin transformations matrix" [28]

$$L = \begin{pmatrix} 1-\frac{i}{2}\tau(\omega_3+ia_3) & -\frac{i}{2}\tau(\omega_1+ia_1-i(\omega_2+ia_2)) \\ -\frac{i}{2}\tau(\omega_1+ia_1+i(\omega_2+ia_2)) & 1-\frac{i}{2}\tau(-\omega_3-ia_3) \end{pmatrix} \quad (52)$$

Then accelerations a_1, a_2, a_3 and angle velocities $\omega_1, \omega_2, \omega_3$ may be found in the linear approximation from the equation

$$\eta + \delta\eta = L\eta \quad (53)$$

as functions of the "logical spin 1/2" spinor components depending on local coordinates $(\pi^1, ..., \pi^{N-1})$.

Hence the infinitesimal Lorentz transformations define small "space-time" coordinates variations. It is convenient to take Lorentz transformations in the following form $ct' = ct + (\vec{x}\vec{a})d\tau, \quad \vec{x'} = \vec{x} + ct\vec{a}d\tau + (\vec{\omega} \times \vec{x})d\tau$, where I put $\vec{a} = (a_1/c, a_2/c, a_3/c), \quad \vec{\omega} = (\omega_1, \omega_2, \omega_3)$ [28] in order to have for τ the physical dimension of time. The expression for the "4-velocity" v^μ is as follows

$$v^\mu = \frac{\delta x^\mu}{\delta\tau} = (\vec{x}\vec{a}, ct\vec{a} + \vec{\omega} \times \vec{x}). \quad (54)$$

The coordinates x^μ of points in dynamical space-time serve in fact merely for the parametrization of deformations of the "field shell" arising under its motion according to non-linear field equations [13, 14].

6 Field equations in the dynamical space-time

Now our aim is to find field equations for Ω^α included in the local Hamiltonian vector field $\vec{H} = \hbar\Omega^\alpha\Phi^i_\alpha\frac{\partial}{\partial\pi^i} + c.c.$ These field equations should be established in the dynamical space-time intrinsically connected with the objective quantum measurement of the "elementary lump" associated with a quantum particle. At each point $(\pi^1, ..., \pi^{N-1})$ of the $CP(N-1)$ one has an "expectation value" of the $\vec{H}$ defined by a measuring device. But a displaced GCS may by reached along one of the continuum paths. Therefore the comparison of two vector fields and their "expectation values" at neighboring points requires some natural rule. The comparison for the same "particle" may be realized by "field shell" dynamics along some path in $CP(N-1)$. For this reason one should have an identification procedure. The affine parallel transport in $CP(N-1)$

of vector fields is a natural and the simplest rule for the comparison of corresponding "field shells".

The dynamical space-time coordinates x^μ may be introduced as the state-dependent quantities, transforming in accordance with the local Lorentz transformations $x^\mu + \delta x^\mu = (\delta^\mu_\nu + \Lambda^\mu_\nu \delta\tau)x^\nu$ depend on the transformations of local reference frame in $CP(N-1)$ as it was described in the previous paragraph.

Let us discuss now the self-consistent problem

$$v^\mu \frac{\partial \Omega^\alpha}{\partial x^\mu} = -(\Gamma^m_{mn}\Phi^n_\beta + \frac{\partial \Phi^n_\beta}{\partial \pi^n})\Omega^\alpha\Omega^\beta, \quad \frac{d\pi^k}{d\tau} = \Phi^k_\beta \Omega^\beta \tag{55}$$

arising under the condition of the affine parallel transport

$$\frac{\delta H^k}{\delta \tau} = \hbar \frac{\delta(\Phi^k_\alpha \Omega^\alpha)}{\delta \tau} = 0 \tag{56}$$

of the Hamiltonian field. I will discuss the simplest case of $CP(1)$ dynamics when $1 \leq \alpha, \beta \leq 3, \quad i, k, n = 1$. This system in the case of the spherical symmetry being split into the real and imaginary parts takes the form

$$\begin{aligned} (r/c)\omega_t + ct\omega_r &= -2\omega\gamma F(u,v), \\ (r/c)\gamma_t + ct\gamma_r &= (\omega^2 - \gamma^2)F(u,v), \\ u_t &= \beta U(u,v,\omega,\gamma), \\ v_t &= \beta V(u,v,\omega,\gamma), \end{aligned} \tag{57}$$

It is impossible of course to solve this self-consistent problem analytically even in this simplest case of the two state system, but it is reasonable to develop a numerical approximation in the vicinity of the following exact solution. Let me put $\omega = \rho \cos\psi, \quad \gamma = \rho \sin\psi$, then, assuming for simplicity that $\omega^2 + \gamma^2 = \rho^2 = constant$, the two first PDE's may be rewritten as follows:

$$\frac{r}{c}\psi_t + ct\psi_r = F(u,v)\rho\cos\psi. \tag{58}$$

The one of the exact solutions of this quasi-linear PDE is

$$\psi_{exact}(t,r) = \arctan \frac{\exp(2c\rho F(u,v) f(r^2 - c^2t^2))(ct+r)^{2F(u,v)} - 1}{\exp(2c\rho F(u,v) f(r^2 - c^2t^2))(ct+r)^{2F(u,v)} + 1}, \tag{59}$$

where $f(r^2 - c^2t^2)$ is an arbitrary function of the interval.

In order to obtain the physical interpretation of these equations I will find the stationary solution for (58). Let me put $\xi = r - ct$. Then one will get the ordinary differential equation

$$\frac{d\Psi(\xi)}{d\xi} = -F(u,v)\rho\frac{\cos\Psi(\xi)}{\xi}. \tag{60}$$

Two solutions

$$\Psi(\xi) = arctan(\frac{\xi^{-2M}e^{-2CM} - 1}{\xi^{-2M}e^{-2CM} + 1}, \frac{2\xi^{-M}e^{-2CM}}{\xi^{-2M}e^{-2CM} - 1}), \tag{61}$$

where $M = F(u,v)\rho$ are concentrated in the vicinity of the light-cone look like solitary waves, see Fig.1. Hence one may treat them as some "potentials" for the local coordinates of GCS ($u = \Re\pi^1, v = \Im\pi^1$). The character of these solutions should be discussed elsewhere.

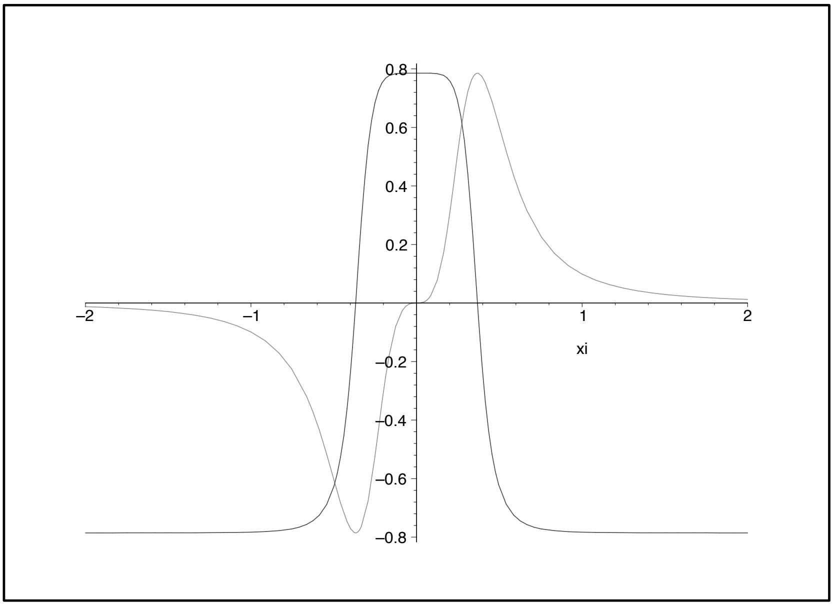

Figure 1: Two solutions of (60) in the light-cone vicinity.

Conclusion

1. The generalized (in comparison with "2-level" case [22]) geometric scheme of the quantum measurement of an arbitrary Hermitian "N-level" dynamical variable has been proposed. The interaction arising from the breakdown of the $G = SU(N)$ symmetry is used for such measurement and it is represented by the affine gauge "field shell" propagated in the dynamical state-dependent space-time.

2. The concept of "super-relativity" [20] is in fact a different kind of attempts of "hybridization" of internal and space-time symmetries. In contrast to supersymmetry where a priori the extended space-time - "super-space" is introduced, in my approach the dynamical space-time arises under "yes/no" quantum measurement of $SU(N)$ local dynamical variables.

3. The pure local formulation of quantum theory in $CP(N-1)$ leads seemingly to the decoherence [22]. We may, of course, make mentally the concatenation of any two quantum systems living in direct product of their state spaces. The variation of one of them during a measurement may lead formally to some variations in the second one. Unavoidable fluctuations in our devices may even confirm predictable correlations. But the introduction of the state-dependent dynamical space-time evokes a necessity to reformulate the Bell's inequalities which may lead then to a different condition for the coincidences.

4. The locality in the quantum phase space $CP(N-1)$ leads to extended quantum particles - "field shell" that obey the quasi-linear PDE [13, 14]. The physical status of their solutions is the open question. But if they are somehow really connected with "elementary particles", say, electrons, then the plane waves of de Broglie should not be literally refer to the state vector of the electron itself but rather to covector (1-form) realized, say, by electrons in a periodic cristall lattice. The fact that the condition for diffraction is in nice agreement with experiments may be explained that for this agreement it is important only *relative velocity* of electron and the lattice.

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