

On the Born rule and the Everett programme

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RÉSUMÉ. Certains auteurs, favorables à l'interprétation d'Everett de la théorie quantique, ont fait des efforts pour démontrer que pour un observateur dans une branche, tout ce passe comme si le postulat de projection était vrai sans postuler explicitement celui-ci. Dans ce papier, nous indiquons qu'il ne soit possible de 'déduire' dit postulat qu'en acceptant une hypothèse supplémentaire. Nous faisons cela en changeant le postulat de projection en un postulat alternatif, en gardant la partie unitaire de la théorie quantique, et en démontrant que ceci produit une théorie — certes, étrange — mais logiquement cohérente.

ABSTRACT. Some proponents of the Everett interpretation of quantum theory have made efforts to show that to an observer in a branch, everything happens as if the projection postulate were true without postulating it. In this paper, we will indicate that it is only possible to 'deduce' this rule by introducing an extra assumption. We do this by examining the consequences of changing the projection postulate into an alternative one, while keeping the unitary part of quantum theory, and indicate that this is a consistent (although strange) physical theory.

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1 Introduction

An important part in the programme of 'the Everett interpretation' (first proposed in Everett (1957) [4]) or the relative state interpretation of quantum theory is to avoid postulating the Projection Postulate (PP) which von Neumann (1955) [12] entitled 'process (1)', so that we can assume that only unitary evolution is necessary, and that this induces, for

an observer, a measurement history *as if* the Projection Postulate were true (Dewitt & Graham, 1973 [3]). This emergence is essential because the Projection Postulate (with the Born probability rule) is needed in some form or another in order to find a link to experimental observation. The relative state view on quantum theory needs to address two issues in order to produce the effects of the Born rule: (i) it needs to show *in what basis* an effective Born rule will emerge and (ii) it needs to show that the *correct probabilities* will emerge for an observer. A good recent overview of the history of the subject can for example be found in the work of Rubin (2003) [10].

Much progress has been made on (i), mainly through the decoherence programme, thoroughly described in a book by Joos et al. (2005) [9] and references therein. Its relevance for the Everett programme is for instance described in work by Zurek (1998) [15]. However, (ii) seems to be much more problematic.

As an example of recent work on (ii), Deutsch (1999) [2] has proven an interesting theorem, which states, under additional 'reasonable' assumptions, that the only way a rational decider can assign probabilities to outcomes of future quantum measurements, is through the Born rule. Several papers then argued on the validity of this proof (Hanson 2003 [8]; Wallace 2002, 2003 [13] [14]; Gill 2003 [6] ; Greaves 2004 [7]). Finkelstein (2000) [5] claims that Gleason's theorem solves the issue ; but this theorem also contains an additional assumption.

We will argue in this paper that the extra assumptions, in all these cases, are logically equivalent to introducing the PP/Born rule. This doesn't affect the value of Deutsch's and other's work, which allow us to reformulate the PP in other terms, and thus to understand better what exactly are its essential ingredients and its physical interpretation. But it means that there is no hope of *deriving* the PP/Born rule directly from the rest of the machinery of quantum theory, and hence puts that part of the Everett programme to an end. We would like to emphasize here that the conclusion is certainly not that an Everett-like view is somehow wrong, on the contrary. But rather, that in any case, assigning (subjective or objective) observational probabilities to the different components of the wave function needs an extra postulate, outside of the scope of strictly unitary quantum theory, and that all attempts to try to deduce it logically from the unitary part alone, are in vain.

The situation is in certain ways reminiscent of attempts, during more than 2 millennia, of deducing Euclid's Fifth Postulate (Trudeau, 1987

[11] gives a marvellous account on that history) from the other postulates of Euclidean geometry, until it was resolved by Gauss and Bolyai and independently by Lobachevsky, by showing that there was a consistent way of building Non-Euclidean geometry by explicitly introducing an alternative Fifth Postulate. We will try to apply the same strategy: we will postulate an Alternative Projection Postulate (APP) and see that this gives rise to a consistent theory on the same level as the standard theory — even if the theory is experimentally of course completely wrong. The very logical existence of this theory then indicates the independence of the PP from the unitary part of quantum theory.

Apart from proving the logical independence of the PP from the unitary part of quantum theory, constructing such a logical alternative has another practical advantage: it allows one more easily to find out where "proofs" of the PP make hidden (or explicit) extra assumptions. We will then examine where exactly it is in disagreement with Deutsch's 'reasonable assumptions', or with Gleason's theorem.

2 The Alternative Projection Postulate.

Let us propose a quantum theory à la von Neumann (1955) [12], except for the projection postulate (which he calls process (1)), which we replace by the Alternative Projection Postulate (APP). It has to be said that the APP can seem slightly more limited in scope than the original PP, in that only measurements with a finite number of different outcomes are handled. However, this is not a physical shortcoming, because any true measurement can result only in a finite amount of information, and hence in a finite number of discrete outcomes. We propose the APP:

Let $\{\hat{X}_k\}$ be a set¹ of commuting self-adjoint operators with a finite, common discrete spectrum (the different possible outcomes of the measurement). This finite spectrum is given by a finite series of sets of eigenvalues $\{x^k\}_i$. The full set of $\{\hat{X}_k\}$ defines the measurement to be performed. To each different set of eigenvalues $\{x^k\}_i$ of $\{\hat{X}_k\}$ corresponds a projector P_i on the space of common eigenvectors belonging to $\{x^k\}_i$. There are by hypothesis only a finite number of such projectors, which form a complete, orthogonal set². Let N

¹We need a set, only because we want to be able to label the outcomes with several different real numbers. As long as there are only a finite number of different outcomes, one single operator could in principle be sufficient.

²This means that $\sum_i P_i = 1$ and that $P_i P_j = 0$ if $i \neq j$.

be the (finite) number of projectors P_i . Let $|\psi\rangle$ be the state of the system in Hilbert space before the measurement. Let n_ψ be the number of projectors for which $P_i|\psi\rangle$ is different from 0. We obviously have: $1 \leq n_\psi \leq N$. If the system is in state $|\psi\rangle$, each of the set of values $\{x^k\}_i$ corresponding to such a projector has a probability $1/n_\psi$ to be realized. The other sets of values have probability 0 to be realized. If the outcome of the measurement equals $\{x^k\}_u$, then the state after measurement equals $P_u|\psi\rangle$, properly normalized.

One notices the difference with the original PP as found in most standard texts on quantum mechanics, such as Cohen-Tannoudji (1997) [1]: the probability equals $1/n_\psi$ instead of $\langle\psi|P_i|\psi\rangle^3$.

We should point out that the APP is in fact the most natural probability rule that goes with the Everett interpretation: on each "branching" of an observer due to a measurement, all of its alternative 'worlds' receive an equal probability.

3 Consistency of Quantum Theory based upon the APP.

The alternative quantum theory (which is normal quantum theory, with the PP replaced by the APP, for short AQT) will turn out to be a physical theory which is completely different from standard quantum theory (SQT) and also experimentally totally wrong. However, we will try to show that it is a consistent theory on the same level as SQT. It is a priori very difficult to prove that a physical theory is consistent. However, the bulk of the mathematical machinery of SQT and AQT is the same (the unitary evolution). The intervention of the APP on the mathematical machinery is the same as the PP (indeed, it is a projection of the state vector on an eigenspace, followed by a normalization of the projection, in both cases). So on the purely mathematical side, both theories are identical concerning the evolution of the state vector.

The subtler aspects are related the physical interpretation. Indeed, the PP is the only link to experimental quantities, and this is replaced by the APP. We have to ensure that through the APP, we arrive at an operational definition of the mathematical entities which is consistent. We will show that it is in fact exactly the same as in SQT. Furthermore,

³We could even go further and postulate the probability to be equal to $\alpha/n_\psi + (1 - \alpha)\langle\psi|P_i|\psi\rangle$ with α a free parameter and real number between 0 and 1, which defines a generalized APP for each choice for the value of α .

we have to prove that different mathematical descriptions describing the same physical situation give identical results. This means invariance under unitary transformations, and invariance under different ways of formulating the same measurement process.

3.1 *Respect of unitary transformations.*

The representation of the state space, and all of the unitary evolution machinery, can undergo a unitary transformation without changing their interpretation. We have to ensure that our AQT gives identical results when such an isomorphism is applied. So we need to show:

Any unitary transformation of the Hilbert space of states, such that $|\psi\rangle$ is mapped upon $U|\psi\rangle$ and every observable O is mapped upon UOU^\dagger , leaves the results and effects of measurements, such as they are introduced by the APP, invariant.

The proof is straightforward. First of all, the projectors P_i are transformed into UP_iU^\dagger , so that the projections of $U|\psi\rangle$ are transformed into $UP_i|\psi\rangle$. This projection is zero if and only if $P_i|\psi\rangle = 0$, so the number of non-zero projections n is conserved, as well as the eigenvalues $\{x^k\}_i$ which belong to such projections. The probabilities of the measurement results are hence the same before and after the transformation. Also any further evolution, after the measurement, is equivalent to the evolution before transformation, given that the state after the measurement (with result $\{x^k\}_u$) is now $UP_u|\psi\rangle$, properly normalized, which is nothing else but the transformation, under U , of the state we would have obtained under the same circumstances.

3.2 *Measurement results predicted with certainty in AQT and SQT are the same.*

In SQT, the interpretation of the mathematical entities (state vector, observable etc...) is completely fixed by the experimental results predicted with certainty. We will show that this interpretation is exactly the same in AQT.

If $|\psi\rangle$ is an eigenvector of the different \hat{X}^k with respective eigenvalues $\{x^k\}_i$, then the measurement will give with certainty the result $\{x^k\}_i$ for this observable, and the state after measurement will still be $|\psi\rangle$.

The proof is trivial and based upon the fact that $P_i|\psi\rangle = |\psi\rangle$ and $P_j|\psi\rangle = 0$ for $i \neq j$.

We also have:

If $|\psi\rangle$ doesn't have any component with eigenvalues $\{x^k\}_i$, then the measurement will give a result which is different from $\{x^k\}_i$, with certainty.

This is a direct consequence of the APP.

3.3 *Equivalence of physically identical measurements.*

Two measurements are physically identical if exactly the same information is extracted by either of both measurements. We will show that two mathematically different sets of observables, $\{X_k\}$ and $\{Y_k\}$, which correspond to physically identical measurements, result in operationally identical results.

If $\{X_k\}$ and $\{Y_k\}$ extract the same information, this means that to each distinct set of eigenvalues $\{x^k\}_i$ corresponds exactly one set of distinct eigenvalues $\{y^k\}_i$ and vice versa ; and that for each case where the result of measurement of $\{X_k\}$ gives with certainty $\{x^k\}_i$, then the result of the measurement if we measure $\{Y_k\}$ should give with certainty $\{y^k\}_i$. But this means that $P_i^X = P_i^Y$. As only the projectors play a role in the APP, the two measurements with equivalent sets of observables yield exactly the same results under the APP.

4 **Strange properties of AQT.**

We will discuss some strange properties of AQT, which immediately disqualify it as a possible candidate for a physical theory of our world. However, we want to emphasize that such a world is a logical possibility including the unitary part of quantum theory, even if it is a very strange one to our standards. By examining some very strange and 'unreasonable' properties, we also show how easy it is to eliminate AQT by introducing 'reasonable assumptions'.

If we take the topology induced by the Hilbert in product, then an arbitrary small change of $|\psi\rangle$ (by adding a small component of an eigenspace that was orthogonal to $|\psi\rangle$) can induce a discrete change in probabilities of outcomes. But this, as such, is not an internal contradiction of the theory. Note also that in general, the state of a system is

never strictly orthogonal to an eigenspace of a set of observables (except immediately after measurement), so one can usually assume that $N = n$, except immediately after a measurement.

This also means that if there is a small time lapse between two identical measurements, that the two results are uncorrelated except in the case where all the $\{X^k\}$ commute with the full Hamiltonian of the system. Although this result seems very strange indeed, it does not necessarily indicate an internal inconsistency, but just means that measurements incompatible with the full H are a waste of time, because the information is immediately lost. As we usually don't know the full H , this means that most measurements are a waste of time. AQT describes a very random world indeed, in which, most of the time, the outcomes of measurements are independent of the state the system is in!

Another strange property of AQT is the following. Imagine that we consider two different, commuting observables, measuring the same quantity. We have a course-grained one, X , and a fine-grained one, Y . Let us assume that Y has 5 distinct eigenvalues, namely 1,2,3,4, and 5. Let us assume that X has two eigenvalues, 10 and 20, and that X takes on value 10 when Y takes on value 1, and that X takes on the value 20 when Y takes on the values 2,3,4, and 5. For a general state $|\psi\rangle$ which isn't 'particular' with respect to X or Y (meaning, has non-zero components for all of their eigenstates), a measurement consisting purely of X will result in a probability 0.5 for value 10, and 0.5 for value 20. A measurement consisting purely of Y will give a probability of 0.2 for each of (1,2,3,4 and 5), and hence, if we calculate X from it, a probability of 0.2 for finding 10 and a probability 0.8 for finding 20. So, depending on whether we measure also Y or not, the result $x = 10$ has a probability of 0.5 or 0.2. At first sight, this is a strange result ; however it is not an inconsistency, and just extends the "strangeness" already present in SQT. In SQT, incompatible measurements influence each other's outcome probabilities ; in AQT, even compatible, but different, measurements influence each other's outcome probabilities. Indeed, the measurement consisting purely of X is not physically equivalent with the measurement consisting of X, Y , because a different amount of information is extracted. On the other hand, the measurement X, Y and the measurement Y are identical, because the measurement of Y is also a measurement of X ; this is an illustration of the equivalence of measurements. It is a property of AQT that changing the resolution of a measurement can change the probabilities of the outcome of the crude

measurement, which is not the case under SQT. Note that this property of SQT is the 'non-contextuality' needed in the application of Gleason's theorem. Indeed, in AQT, the fact that the probability of measuring 10 for X depends on whether we have measured Y or not (and which corresponds physically to two different measurement situations), means that AQT is not non-contextual. So an axiom of non-contextuality can be seen as an axiom, equivalent to the Born rule.

We now see where some of Deutsch's 'reasonable' assumptions (made more explicit in Wallace (2003) [14]) explicitly rule out AQT, and hence indicate where an extra assumption, apart from unitary quantum theory, is made. One instance is requiring identical probabilities under Payoff Equivalence, when the function $f(x_i)$ is not invertible (meaning, for $x_i \neq x_j$ we can have $f(x_i) = f(x_j)$). Indeed, in AQT, the measurement $f(X)$ and X are not considered equivalent because $f(X)$ extracts less information from the system than X . Using payoff equivalence with f non-bijective is a crucial point in the proof of Deutsch's theorem, as made clear in steps (35) and (36) in Wallace (2003) [14].

We want again to emphasize that these strange results are a logical possibility of a theory evolving according to "unitary quantum theory". They are simply different from those given by SQT, in the same way that geometrical results in hyperbolic geometry are different from the geometrical results by Euclidean geometry, and are "strange" as compared to everyday "geometrical measurements".

5 Discussion

In this paper we tried to show that quantum theory, with the projection postulate replaced by an alternative one, gives rise to a consistent physical theory, at least at the same level as standard quantum theory. This theory has very strange consequences and can certainly not describe our world, but its consistency (in relation to standard theory) proves that it is not possible to deduce an emergent or apparent projection postulate (with Born rule probabilities) from the 'unitary' part of quantum theory alone, in the same way it is not possible to deduce Euclid's Fifth Axiom from the four other ones. All attempts to do so need the introduction of extra assumptions.

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