A New Proof of Bell’s Theorem Based on Fourier Series Analysis

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ABSTRACT. We want to prove Bell’s theorem using Fourier series expansion analysis. Comparing to already known algebraic methods, this is a new calculus-based model. Although the notation and procedure we use here is based on the Clauser-Horne model, the corresponding Fourier series method can be simply applied to different versions of Bell’s theorem..

KEYWORDS. Bell’s theorem, Fourier series.

1 Introduction

The main idea leading to Bell's Theorem originates in Einstein's attempt to find an objective meaning to the local properties of a quantum system. Einstein, along with Podolski and Rosen (EPR) [1], proposed a special type of experiment (in which an initially correlated system is separated into two parts) to show the incompleteness of the standard quantum theory for the description of the concept of local realism. After about two decades, Bohm introduced the "spin" version (today known as Bohm version) of EPR that involves, compared to the EPR argument on the position/momentum of the system, the spin components of the correlated system [2].

Three decades after EPR, Bell introduced some supplementary parameters and worked on them algebraically to find an inequality that cannot be generally satisfied by quantum mechanical predictions [3]. Today we know Bell's theorem as: there isn't a full consistency between quantum theory and local realism. The advantage of Bell's work was that the non-classical correlations between pairs of particles could be experimentally tested. Such experiments have been repeated many times and come down firmly in favor of quantum mechanics [4].

After the work of Bell in 1964, a number of generalizations/models of Bell's theorem have been introduced. Some examples are: Clauser and Horne model [5] that is a stochastic (free of the assumption of determinism) and well proposed model for real experiments, and the work introduced by H. Stapp [6] who has claimed Bell's theorem can be proved even without considering the assumption of hidden variables.

Recently, Bell's theorem, in addition to be known as a theorem on the fundamental concepts concerning modern physics, it has taken the attention of engineers and mathematician to its applications in the hot subjects of quantum information and quantum computing [7].

Almost, all different versions of Bell's theorem, including the original work of Bell and the Clauser-Horne model, deal with some algebraic inequalities, and the other ones involve simple algebra of probability and/or mean value functions. Here, we want to prove Bell's theorem using Fourier series expansion analysis. In other words, we want to introduce a calculus-based version of Bell's theorem.

2 Bell’s theorem based on Fourier series analysis

Consider a composite system with total zero angular momentum (J=0) disintegrating spontaneously into two spin 1/2 particles with no relative orbital angular momentum (e.g. proton-proton scattering at low kinetic energies that is a process in which the interacting protons are in singlet state (zero orbital angular momentum) and the spin states of the scattered protons are correlated even after they get separated by a macroscopic distance).

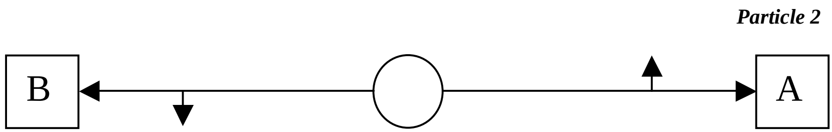

Figure 1 : spin correlation in a singlet state

Each particle goes through a Stern-Gerlach apparatus and is then observed by one of the observers **A** or **B** (see fig. 1). The Stern-Gerlach apparatus receiving particle 1 can take the arbitrary orientation $\hat{a}$, and the one receiving particle 2 can take arbitrary orientation $\hat{b}$. Denote by $P_1(\hat{a},\lambda)$ and $P_2(\hat{b},\lambda)$ the probability for the observation of particles 1 and 2 respectively, and by $P_{12}(\hat{a},\hat{b},\lambda)$ the correlation probability that both particles are observed. Here λ denotes the collection of (hidden) variables characterizing the state of each particle with a normalized probability distribution $\rho(\lambda)$.

$$\int d\lambda \rho(\lambda) = 1 \tag{1}$$

The probability functions $P_1(\hat{a})$, $P_2(\hat{b})$ and $P_{12}(\hat{a},\hat{b})$ are the probabilities, after averaging over probability distribution $\rho(\lambda)$, of an observation by **A**, **B**, and a coincident observation at both **A** and **B** respectively:

$$P_1(\hat{a}) = \int d\lambda \rho(\lambda)\, P_1(\hat{a},\lambda) \tag{2}$$

$$P_2(\hat{b}) = \int d\lambda \rho(\lambda)\, P_2(\hat{b},\lambda) \tag{3}$$

$$P_{12}(\hat{a},\hat{b}) = \int d\lambda \rho(\lambda)\, P_{12}(\hat{a},\hat{b},\lambda) \tag{4}$$

Since the experiments involve dichotomic (±) parameters (spin 1/2 states), the probability functions $P_1(\hat{a},\lambda)$, $P_2(\hat{b},\lambda)$, and $P_{12}(\hat{a},\hat{b},\lambda)$ are short hand notations for the probability functions $P_1(\vec{\sigma}_1 \cdot \hat{a} = +1,\lambda)$ as the probability for the observation of particle 1 in its up (+) state, $P_2(\vec{\sigma}_2 \cdot \hat{b} = +1,\lambda)$ as the observation probability for particle 2 in its up (+) state, and $P_{12}(\vec{\sigma}_1 \cdot \hat{a} = +1, \vec{\sigma}_2 \cdot \hat{b} = +1,\lambda)$ as the probability for simultaneous observation of both particles in their up (+) states respectively.

To ensure that there is no action at a distance between observers **A** and **B**, the following locality condition is used

$$P_{12}(\hat{a},\hat{b},\lambda) = P_1(\hat{a},\lambda) P_2(\hat{b},\lambda) \tag{5}$$

Applying the locality condition (5) to the relation (4), the joint probability function $P_{12}(\hat{a},\hat{b},\lambda)$ is found as the following integral expression

$$P_{12}(\hat{a},\hat{b}) = \int d\lambda\rho(\lambda)\; P_1(\hat{a},\lambda)\; P_2(\hat{b},\lambda) \tag{6}$$

Assuming P_1 and P_2 are continuous functions of the angle parameters, we can write the following Fourier series expansions in the interval $[0,2\pi)$ [8]

$$\begin{aligned} P_1(\hat{a},\lambda) &= A_0(\lambda) + \sum_{n=1}^{\infty}[A_n(\lambda)\cos n\theta_a + B_n(\lambda)\sin n\theta_a] \\ P_2(\hat{b},\lambda) &= C_0(\lambda) + \sum_{n=1}^{\infty}[C_n(\lambda)\cos n\theta_b + D_n(\lambda)\sin n\theta_b] \end{aligned} \tag{7}$$

where θ_a and θ_b are the corresponding angles to the directions $\hat{a}$ and $\hat{b}$ respectively.

From the experimental configuration of the model, it is clear that we have a full symmetry under the following (simultaneous) interchanges

$$\begin{aligned} 1 \to 2 \,\&\, 2 \to 1 \\ \theta_a \to \theta_b \& \theta_b \to \theta_a \end{aligned} \tag{8}$$

This helps us to write the Fourier series expansions (7) as

$$\begin{aligned} P_1(\hat{a},\lambda) &= A_0(\lambda) + \sum_{n=1}^{\infty}[A_n(\lambda)\cos n\theta_a + B_n(\lambda)\sin n\theta_a] \\ P_2(\hat{b},\lambda) &= A_0(\lambda) + \sum_{n=1}^{\infty}[A_n(\lambda)\cos n\theta_b + B_n(\lambda)\sin n\theta_b] \end{aligned} \tag{9}$$

Using (6) and (9), the joint probability function $P_{12}(\hat{a},\hat{b},\lambda)$ is found as

$$P_{12}(\hat{a},\hat{b}) = \int d\lambda\rho(\lambda)\{[A_0(\lambda) + \sum_{n=1}^{\infty} A_n(\lambda)\cos n\theta_a + B_n(\lambda)\sin n\theta_a][A_0(\lambda) + \sum_{m=1}^{\infty} A_m(\lambda)\cos m\theta_b + B_m(\lambda)\sin m\theta_b]\} \tag{10}$$

Now, let see what happens if we consider quantum mechanical results. In quantum theory we have [9]

$$P_{12}(\hat{a},\hat{b}) = P_{12}(\vec{\sigma}_1\cdot\hat{a} = +1, \vec{\sigma}_2\cdot\vec{b} = +1) = \frac{1}{2}\sin^2(\frac{\theta_{ab}}{2}) \tag{11}$$

where $\theta_{ab} = \theta_a - \theta_b$.

We can also write (11) in the following form

$$P_{12}(\hat{a},\hat{b})\Big|_{Q.M.} = \frac{1}{2}\sin^2(\frac{\theta_{ab}}{2}) = \frac{1}{4} - \frac{1}{4}\cos\theta_{ab} = \frac{1}{4} - \frac{1}{4}\cos\theta_a\cos\theta_b - \frac{1}{4}\sin\theta_a\sin\theta_b$$

(12)

Is quantum theory in full consistency with the above-introduced local realistic model (i.e. can quantum theory reproduce all the predictions of the above-introduced local realistic model)? To answer this question, we should check if the right hand sides of the relations (12) and (10) can be set equal to each other for all possible configurations. Since the complete set of the functions $(\cos n\theta, \sin n\theta)$ are linearly independent (orthogonal) [8], the right hand sides of the relations (12) and (10) are generally equal if

$$\int d\lambda\rho(\lambda)[A_0(\lambda)]^2 = \frac{1}{4} \tag{13}$$

$$\int d\lambda\rho(\lambda)A_n(\lambda)A_m(\lambda) = 0$$

(for all values of n and m except when $n = m = 0,1$) 14

$$\int d\lambda \rho(\lambda) A_n(\lambda) B_m(\lambda) = 0 \text{ (for all values of } n \text{ and } m) \tag{15}$$

$$\int d\lambda \rho(\lambda) B_n(\lambda) B_m(\lambda) = 0$$
$$\text{(for all values of } n \text{ and } m \text{ except when } n = m = 1) \tag{16}$$

$$\int d\lambda \rho(\lambda) [A_1(\lambda)]^2 = -\frac{1}{4} \tag{17}$$

$$\int d\lambda \rho(\lambda) [B_1(\lambda)]^2 = -\frac{1}{4} \tag{18}$$

The particular cases ($n = m \neq 0,1$) of the relation (14) and ($n = m \neq 1$) of the relation (15) are

$$\int d\lambda \rho(\lambda) [A_n(\lambda)]^2 = 0 \text{ (for } n \geq 2) \tag{19}$$

$$\int d\lambda \rho(\lambda) [B_n(\lambda)]^2 = 0 \text{ (for } n \geq 2) \tag{20}$$

Since all the integrands in the integral equations (17) to (20) are positive definite[1], these equations cannot be satisfied unless

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[1] The distribution function $\rho(\lambda)$ is a weighting function ($\rho(\lambda) > 0$). Note: $\rho(\lambda) = 0$ corresponds to trivial cases that can be excluded just at the first phase of calculations. If $\rho(\lambda)$ vanishes for all values of λ then we shall result in this fact that the model is trivial (it isn't a realistic model).

$$A_n(\lambda) = B_n(\lambda) = 0 \text{ (for } n \geq 1) \tag{21}$$

This means that

$$P_1(\hat{a},\lambda) = P_2(\hat{b},\lambda) = A_0(\lambda) \tag{22}$$

which is clearly incorrect; therefore, the primary assumption of the equality of the right hand sides of the joint probabilities (10) and (12) is not possible (Bell's theorem is proved).

Finally, we should mention that although we have used the subject of linear independency (orthogonality) to show that the right hand sides of the joint probability functions (10) and (12) cannot be generally set equal to each other, this doesn't mean that one cannot find particular configurations of equality. As we know, the well-known algebraic methods of proving Bell's theorem involve some inequalities that can be satisfied by a wide range of configurations; but, they aren't satisfied generally.

3 Conclusion

We have proved Bell's theorem using Fourier series expansion analysis. Comparing to already known algebraic methods, this is a new calculus-based model. Although the notation and procedure we have used here is based on the Clauser-Horne model which is one of the best (both theoretically and experimentally) models of Bell's theorem, the corresponding Fourier series method can be simply applied to a wide variety of other models. This is because different models of Bell's theorem deal with some discrete and/or continuous functions (e.g. probability functions, mean value functions, or dichotomic functions) that have angle parameters of detection/observation as their arguments; thus, one can always expand these functions based on Fourier series expansions of the angles. Occasionally, one of the advantages of the method introduced here, in addition to being based on a new analytical/calculus procedure, is in its applicability to different versions of Bell's theorem based on the above-mentioned Fourier series expansion; because, to the best of our knowledge, other well- known models involve some mathematical procedures that are not simply/necessarily applicable to each other.

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