Topological Torsion and Topological Spin

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ABSTRACT. Affine Torsion is often treated as a differential *geometric* property of equilibrium systems that can be expressed in terms of certain exterior differential 2-forms. Of greater importance are the 3-forms of Topological Torsion and Topological spin, which are artifacts of non-equilibrium thermodynamic systems and thermodynamically irreversible processes that describe topological change.

1 Introduction

1.1 Affine Torsion 2-forms

Many of the studies of torsion focus attention on Affine Torsion. The classic tensor definition of Affine Torsion describes a differential property of assymetry which can be related to certain exterior differential 2-forms. The concept of Affine Torsion arises through the consideration of a vector space defined in terms of a matrix, $[\mathbb{B}(\xi)]$, of basis vectors, viewed as an array of c2 functions over a set of base variables, ξ. The concept of a vector space requires that the inverse matrix of basis functions exists, and it follows that the differential of the Basis Frame can be explicitly evaluated[1]. The procedure demonstrates the concept of Differential Closure, such that the differential of any basis vector in the Frame is a

[1] A matrix formalism will be used (instead of tensor analysis) in order to get rid of the debauche des indices.

linear combination of the basis vectors of the original set.

$$d[\mathbb{B}] = [\mathbb{B}] \circ \{-[\mathbb{B}]^{-1} \circ [d\mathbb{B}]\} = [\mathbb{B}] \circ \{[\mathbb{B}]^{-1} \circ [d\mathbb{B}]\}, \tag{1}$$

$$= [\mathbb{B}] \circ [\mathbb{C}_{right}] = [\mathbb{B}] \circ [\mathbb{C}(\xi, d\xi)] \tag{2}$$

The matrix of differentials, $[\mathbb{C}_{right}]$, is defined as the Right Cartan Connection matrix of exterior differential 1-forms [1].

The Basis Frame, $[\mathbb{B}]$, need not be global. In fact a more important class of Basis Frames are those that map infinitesimals of perfect differentials $|d\xi\rangle$ into systems of exterior forms $|A\rangle$:

$$[\mathbb{B}] \circ |d\xi\rangle = |A\rangle\,, \tag{3}$$

$$d\{[\mathbb{B}] \circ |d\xi\rangle\} = [\mathbb{B}] \circ [\mathbb{C}_{right}]\hat{}\,|d\xi\rangle = |dA\rangle = |F\rangle \tag{4}$$

The Cartan matrix can have both Christoffel $[\Gamma]$ components, and residue components that can have symmetric and anti-symmetric , $[\mathbb{T}]$. The matrix-exterior product of $[\mathbb{C}(\xi, d\xi)]\hat{}\,|d\xi\rangle$, defines the vector of Closure 2-forms,

$$\text{Vector of Closure 2-forms:} \quad [\mathbb{C}(\xi, d\xi)]\hat{}\,|d\xi\rangle = \{[|\Gamma\rangle] + [\mathbb{T}]\}\hat{}\,|d\xi\rangle\,. \tag{5}$$

It is only the anti-symmetric component of the connection, $[\mathbb{T}]$, that is to be associated with Affine Torsion. The matrix-exterior product, $[\mathbb{T}_A(\xi, d\xi)]\hat{}\,|d\xi\rangle$, defines the vector of Affine Torsion 2-forms, $|\mathfrak{T}\rangle$.

Vector of Affine Torsion 2-forms :

$$[\mathbb{T}_A(\xi, d\xi)]\hat{}\,|d\xi\rangle = |\mathfrak{T}\rangle = \left| C^a_{[bc]} d\xi^c \hat{}\, d\xi^b \right\rangle, \tag{6}$$

$$[\mathbb{B}] \circ [\mathbb{C}_{right}]\hat{}\,|d\xi\rangle = [\mathbb{B}] \circ \hat{}\,|\mathfrak{T}\rangle = |F\rangle\,. \tag{7}$$

Both the symmetric and antisymmetric features of the Basis Frame can induce Affine Torsion 2-forms, but if the Basis Frame is constructed as a congruent correlation with a well defined Christoffel connection, then the antisymmetric features of the Cartan connection vanish. The continuum field vector space is free from Affine Torsion.

In this article the focus will be on the concept of "Topological Torsion" (an exterior differential 3-form) and its relationship to non-equilibrium thermodynamic systems and thermodynamically irreversible processes. Be aware, the Frenet Torsion (a scalar 0-form) is not the same as Affine Torsion (2-forms), nor is it the same as Topological Torsion (3-forms). The vector of Topological Torsion 3-forms is defined as:

$$\text{Vector of Topological Torsion 3-forms} \ : \quad |A\hat{}F\rangle\,, \tag{8}$$

$$\text{Topological Torsion vector,} \quad i(\mathbf{T}_4)\Omega_4 = A\hat{}F. \tag{9}$$

The ubiquitous Frenet theory develops the concept of torsion of a (Newtonian) space curve, in terms of an immersion of single parameter into 3D space: $t \Rightarrow [x(t), y(t), z(t)]$. The Basis Frame is constructed in terms of two differential and one algebraic processes producing the matrix of column vectors, $[\mathbb{B}] = [\mathbf{V}, \mathbf{A}, \mathbf{W} = \mathbf{V} \times \mathbf{A}]$. This frame can be orthonormalised by choice of a differential arclength parameter, $ds = \sqrt{(V \circ V)}dt$, to yield the classic Frenet Frame, $[\mathbb{B}_{Frenet}] = [\mathbf{T}, \mathbf{N}, \mathbf{B}]$ and its associated completely anti-symmetric matrix of Cartan 1-forms in terms κds (Frenet differential curvature) and τds (Frenet differential torsion).

$$d[\mathbb{B}] = [\mathbb{B}] \circ [\mathbb{C}_{right}] = [\mathbb{B}] \circ [\mathbb{T}] = [\mathbb{B}] \circ \begin{bmatrix} 0 & \kappa ds & 0 \\ -\kappa ds & 0 & \tau ds \\ 0 & -\tau ds & 0 \end{bmatrix}. \tag{10}$$

Note that the classic Frenet Framing requires two differential processes; other framing methods (based on angular momentum, and not acceleration) require only one differential process (See Chapter 2.2 in Vol 3 [11]).

The Frenet method can be extended to immersions of 3 parameters (or more) which are somewhat useful for fluid systems. These methods are have been detailed in Chapter 8 of [11]). The important difference is that the concept of differential arclength is no longer an exact differential, but is, indeed, a 1-form that may not admit unique integrability: $ds \Rightarrow \sigma$. In 3D, non-integrability of the differential arclength, σ, is related to topological torsion of the 1-form: $\sigma\hat{}d\sigma \neq 0$. In simple 3D cases, the Topological Torsion is a 3-form that has a functional coefficient equal to what has been called Helicity density in fluid systems:

$$\begin{aligned} &\text{Helicity density (Topological Torsion of arc length)} \\ &\qquad = \{\mathbf{V} \circ Curl\mathbf{V}/(\mathbf{V} \circ \mathbf{V})\}dx\hat{}dy\hat{}dz\ . \end{aligned}$$

1.2 Torsion of elastic deformation:

In 3D elasticity theory, the concept of torsion can be related to elastic energy stored in "twisted" deformations. For example, the elastic deformation of a rubber tube into an untwisted torus will created a torus that can reside as a planar (2-dimensional) object - without Affine Torsion. On the other hand, if the rubber tube is deformed into a twisted torus, the induced torsional energy will not permit the twisted torus to lie flat on a plane. The fact that the twisted elastic torus will not lie flat on the plane indicates that the system is irreducibly 3 dimensional. This idea will be extended to systems that are irreducibly 3 dimensional in a topological (not a geometrical) sense, with applications to non-equilibirum thermodynamic systems, thermodynamically irreversible processes, and possible chirality effects. Note that if the closed elastic torus is cut to become open, torsion effects will cause it to untwist, and curvature effects will cause it to straighten out. Pictures of such twisted torus structures (which are associated with DNA studies) can be found at: http://www22.pair.com/csdc/download/torsionpics.pdf.

1.3 Linear versus Non-linear Basis Frames

The elementary discussion of a connection considers the class of Basis Matrix of functions which define *linear* mappings of the components of any given vector array of functions into another vector array of functions, such that:

$$[\mathbb{B}(x)] \circ \left|X^k(x)\right\rangle \Rightarrow \left|Y^j(x)\right\rangle . \tag{11}$$

Such Basis Frames can be used in eq. (7) to yield (possibly) the non-zero components of the Affine Torsion 2-forms. In this article, the concept of Topological Torsion is treated as a differential *topological* property, which can be related to certain exterior differential 3-forms, not 2-forms. The Basis Matrix is no longer constrained to the class of linear mappings of vector functions to vector functions, but is extended to include *non-linear* differential mappings, for which only the differentials obey the feature of linearity, mapping differentials into differential 1-forms:

$$[\mathbb{B}(\xi)] \circ \left|d\xi^k\right\rangle \Rightarrow \left|\sigma^{(j)}(\xi, d\xi)\right\rangle, \tag{12}$$

$$d[\mathbb{B}(\xi)] \circ \left|d\xi^k\right\rangle = [\mathbb{B}(\xi)] \circ \{[\Gamma] + [\mathbb{T}]\}\hat{} \left|d\xi^k\right\rangle \tag{13}$$

$$\Rightarrow \left|dg^{(j)}(\xi, d\xi)\right\rangle + \left|dA^{(j)}(\xi, d\xi)\right\rangle, \tag{14}$$

$$[\mathbb{B}] \circ [\Gamma]\hat{} \left|d\xi^k\right\rangle = 0 = \left|dg^{(j)}\right\rangle, \tag{15}$$

$$[\mathbb{B}] \circ [\mathbb{T}]\hat{} \left|d\xi^k\right\rangle = [\mathbb{B}] \circ \left|G^k\right\rangle = \left|dA^{(j)}\right\rangle \text{ , det } [\mathbb{B}] < 0. \tag{16}$$

$$[\mathbb{B}] \circ [\mathbb{T}]\hat{} \left|d\xi^k\right\rangle = [\mathbb{B}] \circ \left|\mathfrak{T}^k\right\rangle = \left|dA^{(j)}\right\rangle \text{ , det } [\mathbb{B}] > 0. \tag{17}$$

Note that the if the 2-forms of the Cartan connection are divided into the sum of two parts, the first part created by an arbitrary symmetric connection and the second part created by the residue, it is only the residue which contributes to the generation of Affine Torsion 2-forms. It is tempting to associate the anti-symmetric parts with electromagnetic-like systems, such that $\left|dA^{(j)}\right\rangle \Rightarrow \left|F^{(j)}\right\rangle$, the 2-forms of electromagnetic intensities. In such cases, the 2-forms of Affine Torsion must be sensitive to orientation, in order to yield charges as period integrals that have both plus and minus values. If the Affine Torsion 2-forms are independent of orientation, their topological period integrals are positive definite, a property that can be associated with the positive definite quality of mass. The notation $\left|G^k\right\rangle$ will be used for the pseudoscalar affine torsion 2-forms (charge) in EM theory, and the notation $\left|\mathfrak{T}^k\right\rangle$ will be used for the scalar affine torsion 2-forms (mass) in hydrodynamic theory

The non-linear mapping of differentials may not admit unique integrability. Solutions can exist, but they are singular or multi-valued. The classic multi-valued example is given by the Huygen envelope. For any 1-form, A, the question of unique integrability is decided by the Frobenius theorem, $A\hat{}dA = 0$. Non-zero values of the 3-form, $A\hat{}dA$, indicate that the system is not uniquely integrable. There do not exist integrating factors, λ, such that $d(\lambda A) = 0$.

Non-zero values of a 3-form, $A\hat{}dA$, are defined as exhibitions of "Topological Torsion". The existence of Topological Torsion implies that the system is of irreducible topological dimension 3 (not necessarily geometrical dimension 3) or greater. Remarkably, the property of non-zero Topological Torsion implies that the system under study is a closed non-equilibrium system in a thermodynamic sense. The topological theory of *non-equilibrium* thermodynamics (see Vol 1 [11]), the theory of *topological chirality* [2], the existence of thermodynamically irreversible processes that admit (unlike tensor diffeomorphisms) topological change (that can describe the decay of turbulence), the concept of envelopes and edges of regression, all require a non-zero value of Topological Torsion.

On the other hand, all classic integrable definitions of Affine Torsion imply that the dynamics is thermodynamically reversible.

1.4 Topological Torsion and Spin 3-forms

The major part of this article emphasizes the *topological* properties of torsion, especially those properties defined in terms of exterior differential 3-forms of,

$$\text{Topological Torsion} = A\hat{}dA = A\hat{}F \quad (18)$$

$$\text{Topological Spin} = A\hat{}G. \quad (19)$$

These concepts come from a topological perspective of non-equilibrium electromagnetism (see Vol 4 [11]). The 2-form of field intensities, F, and the 2-form of field excitations, G, are topologically distinct, as $dF = 0$, but dG need not be zero. Given any exterior 1-form, A, the properties of Topological Torsion are well defined. Remarkably, as demonstrated below, if the 3-forms are non-zero, they are indicators of non-equilibrium processes and thermodynamic systems of irreducible topological dimension (not geometrical dimension) of 3 or more. Note that the existence of field excitations, G, is necessary, but not sufficient, to insure that Topological Spin, $A\hat{}G$, is not zero. Moreover, if the Topological Spin is to be quantized, topologically, the 3-form, $A\hat{}G$, must be closed, $d(A\hat{}G) = 0$. This argument will be used below to demonstrate that the claim that Affine Torsion is the source of Spin, or Spin is the source of Affine Torsion, should be re-evaluated.

To study Topological Torsion, the class of Basis Matrices considered are extended to include *non-linear* mappings as described in eq. (12). The RHS of eq. (12) consists of a vector array of (j) 1-forms, which can be described as the vector of 1-form (Closure) potentials. The derived 2-forms can be separated into components created by the symmetric and anti-symmetric components of the connection. If the "electromagnetic" assumption is made (such that the 2-forms $|F\rangle$ represent field intensities, and the 2-forms $|G\rangle$ represent field excitations), then in terms of the vector of Affine Torsion 2-forms, $|G\rangle$:

$$[\mathbb{B}] \circ |G\rangle = |F\rangle\,, \quad |G\rangle = [\mathbb{B}]^{-1} \circ |F\rangle \ . \tag{20}$$

The vector of Affine Torsion 2-forms plays the role of electromagnetic excitation 2-forms, $|G\rangle$. Using the language of electromagnetic theory, the inverse Basis Matrix plays the role of a "constitutive tensor" relating the components of field excitations, $|G\rangle$, and field intensities, $|F\rangle$. Note that a zero value of $|G\rangle$ (or $|F\rangle$) implies a zero value of $|F\rangle$, (or $|G\rangle$). Further note that the topological Spin 3-form, $\langle A| \circ |G\rangle$, requires that the vector of Affine Torsion 2-forms must not be zero, if the Spin 3-form is not to be zero; but, the Spin 3-form can be zero, even if the vector of Affine Torsion 2-forms does not vanish. Non-zero Topological Torsion implies non-zero Affine Torsion, but non-zero Affine Torsion does not imply non-zero Topological Torsion.

1.5 A non-linear Lorentz map

The symmetric components of the Basis Frame can generate Affine Torsion. As an example, consider the non-linear (conformal) Lorentz transformation (a collineation) , $[\mathbf{B}(\theta)_{xt}] = [\mathbb{L}(\theta)_{xt}]\,/(\cos\theta)$, in terms of an "angular parameter", θ, - which preserves the null eikonal equation - as a suitable basis:

$$[\eta]\,/\cos^2\theta = [\mathbf{B}(\theta)_{xt}]^T \circ [\eta] \circ [\mathbf{B}(\theta)_{xt}] \tag{21}$$

$$[\mathbb{B}(\theta)_{xt}] = \begin{bmatrix} \frac{1}{\cos^2\theta} & 0 0 \pm \frac{\sin\theta}{\cos^2\theta} \\ 0 & 0 0 & 0 \\ 0 & 0 0 & 0 \\ \pm\frac{\sin\theta}{\cos^2\theta} & 0 0 & \frac{1}{\cos^2\theta} \end{bmatrix} . \tag{22}$$

The Basis Frame is symmetric, with a determinant which is not negative, and the vector, $|\mathfrak{T}\rangle$, of Affine Torsion coefficients is not zero.

The matrix of Cartan 1-forms, $[\mathbb{C}^a_{m,n}]$, is symmetric in a matrix sense, $[\mathbb{C}^m_{a,n}]$, but not in the differential form sense. $[\mathbb{C}^a_{m,n}] \neq [\mathbb{C}^a_{n,m}]$!:

$$[\mathbb{C}(\theta)_{xt}] = \begin{bmatrix} \frac{\sin(\theta)d\theta}{\cos\theta} & 0 & 0 & \frac{d\theta}{\cos\theta} \\ 0 & \frac{\sin(\theta)d\theta}{\cos\theta} & 0 & 0 \\ 0 & 0 & \frac{\sin(\theta)d\theta}{\cos\theta} & 0 \\ \frac{d\theta}{\cos\theta} & 0 & 0 & \frac{\sin(\theta)d\theta}{\cos\theta} \end{bmatrix} . \tag{23}$$

Yet the vector of Affine Torsion 2-forms generated by the symmetric component is not zero:

$$\left|\mathfrak{T}^k\right\rangle = \left|\begin{array}{c} \{\sin(\theta)d\theta\hat{}dx + d\theta\hat{}dt\}/\cos\theta \\ \{\sin(\theta)d\theta\hat{}dy/\cos\theta \\ \{\sin(\theta)d\theta\hat{}dz/\cos\theta \\ \{\sin(\theta)d\theta\hat{}dt + d\theta\hat{}dx\}/\cos\theta \end{array}\right\rangle \tag{24}$$

It is remarkable that the matrix elements of the differential, $[\mathbb{B}(\theta)_{xt}]$, are precisely those that represent the diffraction wakes found in non-equilibirum fluid flow and plasmas. The $1/\cos^2\theta$ term generates the Kelvin-Helmholtz instability and the $\pm\sin(\theta)/\cos^2(\theta)$ term generates the Rayleigh-Taylor instability. The details with many examples are given in Chapter 8 of [11].

1.6 Affine Torsion is not a topological invariant

It is significant that the concept of "Affine Torsion" can be made equal to zero, if integrating factors, λ^j, can be found for each of the 1-forms that make up the elements of $\left|A^{(j)}\right\rangle$. Although the symbols $C^a_{[bc]}$ define the components of the vector of "Affine Torsion 2-forms" (a tensor with respect to diffeomorphisms), it is possible to eradicate the Affine torsion by changing to a new reference Basis Frame. The requirement for such a possibility is that the 3-forms $A^{(j)}\hat{}dA(j)$ (of Topological Torsion) vanish. Then the original Basis Frame (with non-zero Affine Torsion) can be modified by pre-multiplication with a diagonal matrix, $[\lambda^{(j)}_{diag}]$, of integrating factors:

$$\text{If } [\widehat{\mathbb{B}}(\xi)] = [\lambda^{(j)}_{diag}] \circ [B^j_a(\xi)] \text{ such that} \tag{25}$$

$$[\widehat{\mathbb{B}}(\xi)] \circ |d\xi^a\rangle = [\lambda^{(j)}_{diag}] \circ [B^j_a(\xi)] \circ |d\xi^a\rangle = [\lambda^{(j)}_{diag}] \circ \left|A^{(j)}\right\rangle = \left|\lambda^{(j)}A^{(j)}\right\rangle, \tag{26}$$

$$\text{then } d\left|\lambda^{(k)}A^k\right\rangle \Rightarrow 0, \text{ if } \lambda^{(k)} \text{ is an integrating factor.} \tag{27}$$

The new Basis Frame, $[\widehat{\mathbb{B}}(\xi)]$, does not exhibit Affine Torsion! Hence the concept of Affine Torsion is not intrinsically irreducible, unless the Topological Torsion 3-forms, $A\hat{}dA$, of the 1-forms, $\left|A^{(j)}\right\rangle$, are not zero.

The domain of invertability of the new frame can be significantly different from that of the old frame, as it depends on the determinant of the diagonal array of integrating factors, which are functions of the base variables. The affine torsion is reducible, but a class of defect structures appear, compensating for the (perhaps) smaller neighborhoods of definition for the Frame matrix.

Conclusion 1 *These results indicate that Affine Torsion cannot be the source of Topological Spin, for then the concept of Spin would not be intrinsic and irreducible when the Affine Torsion was reducible. Moreover, Spin cannot be the source of Affine Torsion, for when the Spin is zero, the Affine Torsion need not be zero.*

2 Topological Torsion and Non-Equilibrium Thermodynamics

2.1 The Axioms of Topological Thermodynamics

The topological methods used herein are based upon Cartan's theory of exterior differential forms [11]. The thermodynamic view assumes that the physical systems to be studied can be encoded in terms of a 1-form of Action Potentials (per unit source, or, per mole), A, on a four-dimensional variety of ordered independent variables, $\{\xi^1, \xi^2, \xi^3, \xi^4\}$. The variety supports a differential volume element $\Omega_4 = d\xi^1\hat{}d\xi^2\hat{}d\xi^3\hat{}d\xi^4$. This statement implies that the differentials of the $\mu = 4$ base variables are functionally independent. No metric, no connection, no constraint of gauge symmetry is imposed upon the four-dimensional pre-geometric variety. Topological constraints can be expressed in terms of exterior differential systems placed upon this set of base variables [3].

In order to make the equations more suggestive to the reader, the symbolism for the variety of independent variables will be changed to the format $\{x, y, z, t\}$, but be aware that no constraints of metric or connection are imposed upon this variety, at this, thermodynamic, level. For instance, it is NOT assumed that the variety is spatially Euclidean.

With this notation, the Axioms of Topological Thermodynamics can be summarized as:

> **Axiom 1**. *Thermodynamic physical systems can be encoded in terms of a 1-form of covariant Action Potentials, $A_\mu(x, y, z, t...)$, on a four-dimensional abstract variety of ordered independent variables, $\{x, y, z, t\}$. The variety supports differential volume element $\Omega_4 = dx\hat{}dy\hat{}dz\hat{}dt$.*

Axiom 2. *Thermodynamic processes are assumed to be encoded, to within a factor, $\rho(x, y, z, t...)$, in terms of a contravariant Vector and/or complex Spinor directionfields, symbolized as $V_4(x, y, z, t)$.*

Axiom 3. *Continuous Topological Evolution of the thermodynamic system can be encoded in terms of Cartan's magic formula (see p. 122 in [4]). The Lie differential with respect to the process, ρV_4, when applied to an exterior differential 1-form of Action, $A = A_\mu dx^\mu$, is equivalent, abstractly, to the first law of thermodynamics.*

$$\text{Cartan's Magic Formula} \quad L_{(\rho\mathbf{V}_4)}A = i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A), \tag{28}$$

$$\text{First Law} \quad : W + dU = Q, \tag{29}$$

$$\text{Inexact Heat 1-form} \quad Q = W + dU = L_{(\rho\mathbf{V}_4)}A, \tag{30}$$

$$\text{Inexact Work 1-form} \quad W = i(\rho\mathbf{V}_4)dA, \tag{31}$$

$$\text{Internal Energy} \quad U = i(\rho\mathbf{V}_4)A. \tag{32}$$

Axiom 4. *Equivalence classes of systems can be defined in terms of the Pfaff Topological Dimension and topological structure generated by of the 1-forms of Action, A. Continuous processes can be defined in terms of the Pfaff Topological Dimension and topological structure generated by of the 1-forms of* Work, W, and Heat, Q.

Axiom 5. If $Q\hat{}dQ \neq 0$, *then the thermodynamic process is irreversible.*

2.2 Cartan's Magic Formula ≈ First Law of Thermodynamics

The Lie differential (not Lie derivative) is the fundamental generator of Continuous Topological Evolution. When acting on an exterior differential 1-form of Action, $A = A_\mu dx^\mu$, Cartan's magic (algebraic) formula is equivalent *abstractly* to the first law of thermodynamics:

$$L_{(\rho\mathbf{V}_4)}A = i(\rho\mathbf{V}_4)dA + d(i(\rho\mathbf{V}_4)A), \tag{33}$$

$$= W + dU = Q. \tag{34}$$

In effect, Cartan's magic formula leads to a topological basis of thermodynamics, where the thermodynamic Work, W, thermodynamic Heat, Q, and the thermodynamic internal energy, U, are defined *dynamically* in terms of Continuous Topological Evolution. In effect, the First Law is a statement of Continuous Topological Evolution in terms of deRham cohomology theory; the difference between two non-exact differential forms is equal to an exact differential, $Q - W = dU$.

My recognition (some 30 years ago) of this correspondence between the Lie *differential* and the First Law of thermodynamics has been the corner stone of my research efforts in applied topology.

It is important to realize that the Cartan formula is to be interpreted algebraically. Many textbook presentations of the Cartan-Lie differential formula presume a dynamic constraint, such that the vector field $\mathbf{V}_4(x, y, z, t)$ be the generator of a single parameter group. If true, then the topological constraint of Kinematic Perfection can be established as an exterior differential system of the format:

$$\textbf{Kinematic Perfection :} \qquad dx^k - \mathbf{V}^k dt \Rightarrow 0. \tag{35}$$

The topological constraint of Kinematic Perfection, in effect, defines (or presumes) a limit process. This constraint leads to the concept of the Lie *derivative* of the 1-form A. Professor Zbigniew Oziewicz told me that Slebodzinsky was the first to formulate the idea of the Lie derivative in his thesis (in Polish). The evolution then is represented by the infinitesimal propagation of the 1-form, A, down the flow lines generated by the 1-parameter group. Cartan called this set of flow lines "the tube of trajectories".

However, such a topological, kinematic constraint is *not* imposed in the presentation found in this essay; the directionfield, $\mathbf{V}_4$, may have multiple parameters. This observation leads to the important concept of topological fluctuations (about Kinematic Perfection), such as given by the expressions:

Fluctuations : (Topological)

$$(dx^k - \mathbf{V}^k dt) = \Delta\mathbf{x}^k \neq 0, \qquad (\sim \text{ Pressure}) \tag{36}$$

$$(dV^k - \mathbf{A}^k dt) = (\Delta\mathbf{V}^k) \neq 0, \quad (\sim \text{ Temperature}) \tag{37}$$

$$d(\Delta\mathbf{x}^k) = -(d\mathbf{V}^k - \mathbf{A}^k dt)\hat{}dt = -(\Delta\mathbf{V}^k)\hat{}dt, \tag{38}$$

In this context it is interesting to note that in Felix Klein's discussions [5] of the development of calculus, he says

> *"The primary thing for him (Leibniz) was not the differential quotient (the derivative) thought of as a limit. The differential, dx, of the variable x had for him (Leibniz) actual existence..."*

The Leibniz concept is followed throughout this presentation. It is important for the reader to remember that the concept of a differential form is different from the concept of a derivative, where a (topological) limit has been defined, thereby constraining the topological evolution.

The topological methods to be described below go beyond the notion of processes which are confined to equilibrium systems of kinematic perfection. Non-equilibrium systems and processes which are thermodynamically irreversible, as well as many other classical thermodynamic ideas, can be formulated in precise mathematical terms using the topological structure and refinements generated by the three thermodynamic 1-forms, A, W, and Q.

2.3 The Pfaff Sequence and the Pfaff Topological Dimension

2.3.1 The Pfaff Topological Dimension of the System 1-form, A

It is important to realize that the Pfaff Topological Dimension of the system 1-form of Action, A, determines whether the thermodynamic system is Open, Closed, Isolated or Equilibrium. Also, it is important to realize that the Pfaff Topological Dimension of the thermodynamic Work 1-form, W, determines a specific category of reversible and/or irreversible processes. It is therefore of some importance to understand the meaning of the Pfaff Topological Dimension of a 1-form. Given the functional format of a general 1-form, A, on a 4D variety it is an easy step to compute the Pfaff Sequence, using one exterior differential operation, and several algebraic exterior products. For a differential 1-form, A, defined on a geometric domain of 4 base variables, the Pfaff Sequence is defined as:

$$\textbf{Pfaff Sequence} \quad \{A, dA, A\hat{}dA, dA\hat{}dA...\} \tag{39}$$

It is possible that over some domains, as the elements of the sequence are computed, one of the elements (and subsequent elements) of the Pfaff Sequence will vanish. The number of non-zero elements in the Pfaff Sequence (PS) defines the Pfaff Topological Dimension (PTD) of the specified 1-form. The Pfaff Topological dimension has been called the "class" of a 1-form in the old literature. I prefer the more suggestive name. Modulo singularities, the Pfaff Topological Dimension determines the minimum number M of N functions of base variables $(N \geq M)$ required to define the topological properties of the connected component of the 1-form A.

The Pfaff Topological Dimension of the 1-form of Action, A, can be put into correspondence with the four classic topological structures of thermodynamics. Equilibrium, Isolated, Closed, and Open systems. The classic thermodynamic interpretation is that the first two structures do not exchange mass (mole numbers) or radiation with their environment. The Closed structure can exchange radiation with its environment but not mass (mole numbers). The Open structure can exchange both mass and radiation with its environment. The following table summarizes these properties. For reference purposes, I have given the various elements of the Pfaff sequence specific names:

Topological p-form name	**PS** element	**Nulls**	**PTD**	Thermodynamic system
Action	A	$dA = 0$	1	**Equilibrium**
Vorticity	dA	$A\hat{}dA = 0$	2	**Isolated**
Torsion	$A\hat{}dA$	$dA\hat{}dA = 0$	3	**Closed**
Parity	$dA\hat{}dA$	$-$	4	**Open**

Table 1 Applications of the Pfaff Topological Dimension.

The four thermodynamic systems can be placed into two topological categories. If the Pfaff Topological Dimension of A is 2 or less, the first category is determined by the closure (or differential ideal) of the 1-form of Action, $A \cup dA$. This Cartan topology is a connected topology. In the case that the Pfaff Topological Dimension is greater than 2, the Cartan topology is based on the union of two closures, $\{(A \cup dA) \cup (A\hat{}dA \cup dA\hat{}dA)\}$, and is therefore a disconnected topology.

It is a topological fact that there exists a (topologically) continuous C2 process from a disconnected topology to a connected topology, but there does not exist a C2 continuous process from a connected topology to a disconnected topology. This fact implies that topological change can occur continuously by a "pasting" processes representing (for example) the *decay* of turbulence by "condensations" from non-equilibrium to equilibrium systems. On the other hand, the *creation* of Turbulence involves a discontinuous (non C2) process of "cutting" into parts. This warning was given long ago [6] to prove that computer analyses that smoothly match value and slope will not replicate the *creation* of turbulence, but can faithfully replicate the *decay* of turbulence.

2.3.2 The Pfaff Topological Dimension of the Thermodynamic Work 1-form, W

The topological structure of the thermodynamic Work 1-form, W, can be used to refine the topology of the physical system; recall that the physical system is encoded by the Action 1-form, A.

Claim 2 *The PDE's that represent the system dynamics are determined by the Pfaff Topological Dimension of the 1-form of Work, W, and the 1-form of Action, A, that encodes the physical system.*

The Pfaff Topological Dimension of the thermodynamic Work 1-form depends upon both the physical system, A, and the process, $\mathbf{V}_4$. In particular if the Pfaff Dimension of the thermodynamic Work 1-form is zero, ($W = 0$), then system dynamics is generated by an extremal vector field which admits a Hamiltonian realization. However, such extremal direction fields can occur only when the Pfaff Topological Dimension of the system encoded by A is odd, and equal or less than the geometric dimension of the base variables.

For example, if the geometric dimension is 3, and the Pfaff Topological Dimension of A is 3, then there exists a unique extremal field on the Contact manifold defined by dA. This unique directionfield is the unique eigen directionfield of the 3x3 antisymmetric matrix (created by the 2-form $F = dA$) with eigenvalue equal to zero.

If the geometric dimension is 4, and the Pfaff Topological Dimension of A is 3, then there exists a two extremal fields on the geometric manifold. These directionfields are those generated as the eigen directionfields of the 4x4 antisymmetric matrix (created by the 2-form $F = dA$) with eigenvalue equal to zero.

If the geometric dimension is 4, and the Pfaff Topological Dimension of A is 4, then there do not exist extremal fields on the Symplectic manifold defined by dA. All of the eigen directionfields of the 4x4 antisymmetric matrix (created by the 2-form $F = dA$) are complex isotropic spinors with pure imaginary eigenvalues not equal to zero.

2.4 Topological Torsion and other Continuous Processes

2.4.1 Reversible Processes

Physical Processes are determined by directionfields (both vector and spinor fields) with the symbol, $\mathbf{V}_4$, to within an arbitrary function, ρ. There are several classes of direction fields that are defined as follows [7]:

$$\begin{aligned} \textbf{Associated Class} &: \quad i(\rho\mathbf{V}_4)A = 0, && (40)\\ \textbf{Extremal Class} &: \quad i(\rho\mathbf{V}_4)dA = 0, && (41)\\ \textbf{Characteristic Class} &: \quad i(\rho\mathbf{V}_4)A = 0, && (42)\\ \text{and} & \quad i(\rho\mathbf{V}_4)dA = 0, && (43)\\ \textbf{Helmholtz Class} &: \quad d(i(\rho\mathbf{V}_4)dA) = 0, && (44) \end{aligned}$$

Extremal Vectors (relative to the 1-form of Action, A) produce zero thermodynamic work, $W = i(\rho\mathbf{V}_4)dA = 0$, and admit a Hamiltonian representation. Associated Vectors (relative to the 1-form of Action, A) can be adiabatic if the process remains orthogonal to the 1-form, A. Helmholtz processes (which include Hamiltonian processes, Bernoulli processes and Stokes flow) conserve the 2-form of Topological vorticity, dA. All such processes are thermodynamically reversible. The energy of the process might appear to decay, but the exchange of t to $-t$ merely retraces the evolution. Thermodynamic irreversibility implies a lack of time reversal invariance. Many examples of these systems are detailed in the reference monographs.

2.4.2 Irreversible Processes

There is one directionfield that is uniquely defined by the coefficient functions of the 1-form, A, that encodes the thermodynamic system on a 4D geometric variety. This vector exists only in non-equilibrium systems, for which the Pfaff Topological Dimension of A is 3 or 4. This 4 vector is defined herein as the topological Torsion vector, $\mathbf{T}_4$. To within a factor, this directionfield has the four coefficients of the 3-form $A\hat{}dA$, with the following properties:

$$\textbf{Properties of} \;:\textbf{Topological Torsion}\;\mathbf{T}_4\;\text{ on }\;\Omega_4 \tag{45}$$

$$i(\mathbf{T}_4)\Omega_4 = i(\mathbf{T}_4)dx\hat{}dy\hat{}dz\hat{}dt = A\hat{}dA, \tag{46}$$

$$W = i(\mathbf{T}_4)dA = \sigma\ A, \tag{47}$$

$$dW = d\sigma\hat{}A + \sigma dA = dQ \tag{48}$$

$$U = i(\mathbf{T}_4)A = 0, \quad \mathbf{T}_4 \text{ is associative} \tag{49}$$

$$i(\mathbf{T}_4)dU = 0 \tag{50}$$

$$i(\mathbf{T}_4)Q = 0 \qquad \mathbf{T}_4 \text{ is adiabatic} \tag{51}$$

$$L_{(\mathbf{T}_4)}A = \sigma\ A, \quad \mathbf{T}_4 \text{ is homogeneous} \tag{52}$$

$$L_{(\mathbf{T}_4)}dA = d\sigma\hat{}A + \sigma dA = dQ, \tag{53}$$

$$Q\hat{}dQ = L_{(\mathbf{T}_4)}A\hat{}L_{(\mathbf{T}_4)}dA = \sigma^2 A\hat{}dA \neq 0, \tag{54}$$

$$dA\hat{}dA = d(A\hat{}dA) = d\{(i(\mathbf{T}_4)\Omega_4\} = (div_4\mathbf{T}_4)\Omega_4, \tag{55}$$

$$L_{(\mathbf{T}_4)}\Omega_4 = d\{(i(\mathbf{T}_4)\Omega_4\} = (2\sigma)\Omega_4, \tag{56}$$

If the Pfaff Topological Dimension of A is 4 (an Open thermodynamic system), then $\mathbf{T}_4$ has a non-zero 4 divergence, (2σ), representing an expansion or a contraction of the 4D volume element Ω_4. The Heat 1-form, Q, generated by the process, $\mathbf{T}_4$, is NOT integrable. Q is of Pfaff Topological Dimension greater that 2, when $\sigma \neq 0$. Furthermore the $\mathbf{T}_4$ process is locally adiabatic as the change of internal energy in the direction of the process path is zero. Therefore, in the Pfaff Topological Dimension 4 case, where $dA\hat{}dA \neq 0$, the $\mathbf{T}_4$ direction field[2] represents an *irreversible, adiabatic process.*

When σ is zero *and* $d\sigma = 0$, but $A\hat{}dA \neq 0$, the Pfaff Topological Dimension of the system is 3 (a Closed thermodynamic system). In this case, the $\mathbf{T}_4$ direction field becomes a characteristic vector field which is both extremal and associative, and induces a Hamilton-Jacobi representation (the ground state of the system for which $dQ = 0$).

For any process and any system, equation (54) can be used as a test for irreversibility.

[2] A direction field is defined by the components of a vector field which establish the "line of action" of the vector in a projective sense. An arbitrary factor times the direction field defines the same projective line of action, just reparameterized. In metric based situations, the arbitrary factor can be interpreted as a renormalization factor.

It seems a pity, that the concept of the Topological Torsion vector and its association with non-equilibrium systems, where it can be used to establish design criteria to minimize energy dissipation, has been ignored by the engineering community.

2.5 The Spinor class

It is rather remarkable (and only fully appreciated by me in February, 2005) that there is a large class of direction fields useful to the topological dynamics of thermodynamic systems (given herein the symbol $\rho\mathbf{S}_4$) that do not behave as vectors (with respect to rotations). They are isotropic complex vectors of zero length, defined as Spinors by E. Cartan [8], but which are most easily recognized as the eigen directionfields relative to the antisymmetric matrix, $[F]$, generated by the component of the 2-form $F = dA$:

$$\textbf{The Spinor Class} \qquad [F] \circ |\rho\mathbf{S}_4\rangle = \lambda\, |\rho\mathbf{S}_4\rangle \neq 0, \tag{57}$$

$$\langle\rho\mathbf{S}_4| \circ |\rho\mathbf{S}_4\rangle = 0, \quad \lambda \neq 0 \tag{58}$$

In the language of exterior differential forms, if the Work 1-form is not zero, the process must contain Spinor components:

$$W = i(\rho\mathbf{S}_4)dA \neq 0 \tag{59}$$

As mentioned above, Spinors have metric properties, behave as vectors with respect to transitive maps, but do not behave as vectors with respect to rotations (see p. 3, [8]). Spinors generate harmonic forms and are related to conjugate pairs of minimal surfaces. The notation that a Spinor is a complex isotropic *directionfield* is preferred over the names "complex isotropic *vector*", or "null *vector*" that appear in the literature. As shown below, the familiar formats of Hamiltonian mechanical systems exclude the concept of Spinor process directionfields, for the processes permitted are restricted to be represented by direction fields of the extremal class, which have zero eigenvalues.

Remark 3 *Spinors are normal consequences of antisymmetric matrices, and, as topological artifacts, they are not restricted to physical microscopic or quantum constraints. According to the topological thermodynamic arguments, Spinors are implicitly involved in all processes for which the 1-form of thermodynamic Work is not zero. Spinors play a role in topological fluctuations and irreversible processes.*

The thermodynamic Work 1-form, W, is generated by a completely antisymmetric 2-form, F, and therefore, if not zero, must have Spinor components. In the odd dimensional Contact manifold case there is one eigen Vector, with eigenvalue zero, which generates the extremal processes that can be associated with a Hamiltonian representation. The other two eigendirection fields are Spinors. In the even dimensional Symplectic manifold case, any non-zero component of work requires that the evolutionary directionfields must contain Spinor components. All eigen directionfields on symplectic spaces are Spinors.

The fundamental problem of Spinor components is that there can be more than one Spinor direction field that generates the same geometric path. For example, there can be Spinors of left or right handed polarizations and Spinors of expansion or contraction that produce the same optical (null congruence) path. This result does not fit with the classic arguments of mechanics, which require unique initial data to yield unique paths. Furthermore, the concept of Spinor processes can annihilate the concept of time reversal symmetry, inherent in classical hydrodynamics. The requirement of uniqueness is not a requirement of non-equilibrium thermodynamics, where Spinor "entanglement" has to be taken into account.

2.6 Emergent Topological Defects

Suppose an evolutionary process starts in a domain of Pfaff Topological Dimension 4, for which a process in the direction of the Topological Torsion vector, $\mathbf{T}_4$, is known to represent an irreversible process. Examples can demonstrate that the irreversible process can proceed to a domain of the geometric variety for which the dissipation coefficient, σ, becomes zero. Physical examples (see Vol 2, [11]) such as the skidding bowling ball proceed with irreversible dissipation ($PTD = 6$) until the "no-slip" condition is reached ($PTD = 5$). In fluid systems the topological defects can emerge as long lived states far from equilibrium. The process is most simply visualized as a "condensation" from a turbulent gas, such as the creation of a star in the model which presumes the universe is a very dilute, turbulent van der Waals gas near its critical point. The red spot of Jupiter, a hurricane, the ionized plasma ring in a nuclear explosion, Falaco Solitons, the wake behind an aircraft are all exhibitions of the emergence process to long lived topological structures far from equilibrium. It is most remarkable that the emergence of these experimental defect structures occurs in finite time.

The idea is that a subdomain of the original system of Pfaff Topological Dimension 4 can evolve continuously with a change of topology to a region of Pfaff Topological Dimension 3. The emergent subdomain of Pfaff Topological Dimension 3 is a topological defect, with topological coherence, and often with an extended lifetime (as a soliton structure with a dominant Hamiltonian evolutionary path), embedded in the Pfaff dimension 4 turbulent background.

The Topological Torsion vector in a region of Pfaff Topological Dimension 3 is an extremal vector direction field in systems of Pfaff Topological Dimension 3; it then has a zero 4D divergence, and leaves the volume element invariant. Moreover the existence of an extremal direction field implies that the 1-form of Action can be given a Hamiltonian representation, $P_k dq^k + H(P, q, t)dt$. In the domain of Pfaff dimension 3 for the Action, A, the subsequent continuous evolution of the system, A, relative to the process $\mathbf{T}_4$, can proceed in an energy conserving, Hamiltonian manner, representing a "stationary" or "excited" state far from equilibrium (the ground state). This argument is based on the assumption that the Hamiltonian component of the direction field is dominant, and any Spinor components in the $PTD = 3$ domain, representing topological fluctuations, can be ignored. These excited states, far from equilibrium, can be interpreted as the evolutionary topological defects that emerge and self-organize due to irreversible processes in the turbulent dissipative system of Pfaff dimension 4.

The descriptive words of self-organized states far from equilibrium have been abstracted from the intuition and conjectures of I. Prigogine [9]. The methods of Continuous Topological Evolution correct the Prigogine conjecture that "dissipative structures" can be caused by dissipative processes and fluctuations. The long-lived excited state structures created by irreversible processes are non-equilibrium, deformable topological defects almost void of irreversible dissipation. The topological theory presented herein presents for the first time a solid, formal, mathematical justification (with examples) for the Prigogine conjectures. Precise definitions of equilibrium and non-equilibrium systems, as well as reversible and irreversible processes can be made in terms of the topological features of Cartan's exterior calculus. Using Cartan's methods of exterior differential systems, thermodynamic irreversibility and the arrow of time can be well defined in a topological sense, a technique that goes beyond (and without) statistical analysis [10]. Thermodynamic irreversibility and the arrow of time requires that the evolutionary process produce topological change.

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