

# Duffin-Kemmer-Petiau formalism reexamined: non-relativistic approximation for spin 0 and spin 1 particles in a Riemannian space-time

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**ABSTRACT.** It is shown that the generally covariant Duffin-Kemmer-Petiau equation, formulated in the frame of the Tetrode-Weyl-Fock-Ivanenko tetrad formalism, allows a non-relativistic approximation if the metric tensor has a special form  $dS^2 = dx^0 dx^0 - g_{ij}(x^0, x^k) dx^i dx^j$ . The Pauli equation for a vector particle involves the Riemann curvature tensor explicitly. In analogous way, the procedure of the non-relativistic approximation in the theory of a scalar particle, charged and neutral, is investigated in the background of Riemannian space-time. A generalized covariant Schrödinger equation is derived when taking into account non-minimal interaction term through scalar curvature  $R(x)$ , it substantially differs from the conventional generally covariant Schrödinger equation produced when  $R(x) = 0$ . It is demonstrated that the non-relativistic wave function is always complex-valued irrespective of the type of relativistic scalar particle, charged or neutral, taken initially. The theory of vector particle proves the same property: even if the wave function of the relativistic particle of spin 1 is taken real, the corresponding wave function in the non-relativistic approximation is complex-valued.

## 1 Introduction

Matrix Duffin-Kemmer-Petiau formalism, for boson fields has a long and rich history inseparably linked with description of photons and mesons:

Louis de Broglie [1-9], A. Mercier [10], G. Petiau [11], A. Proca [12], Duffin [13], N. Kemmer [14, 15], H.J. Bhabha [16, 17], F.J. Belinfante [18, 19], S. Sakata – M. Taketani [20], M.A. Tonnelat [21], H.A.S. Erikson [22], E. Schrödinger [23-25], W. Heitler [26, 27], Yarish-Chandra [28-30], B. Hoffmann [31], R. Utiyama [32], I.M. Gel'fand – A.M. Yaglom [33],

J.A. Schouten [34], S.N. Gupta [35], K. Bleuler [36], I. Fujiwara [37], K.M. Case [38], H. Umezawa [39], A.A. Borgardt [40-43], F.I. Fedorov [44], T. Kuohsien [45], S. Hjalmaris [46], A.A. Bogush – F.I. Fedorov [47], N.A. Chernikov [48], J. Beckers [49], P. Roman [50], F.I. Fedorov – A.I. Bolsun [51], L. Oliver [52], J. Beckers, C. Pirotte [53], G. Casanova [54], I.Yu. Krivski – G.D. Romamenko – V.I. Fushchych [55-57], A.A. Bogush et al . [58, 59], T. Goldman et al [60], R.A. Krajcik – M. M. Nieto [61], J.D. Jenkins [62, 63], E. Fischbach et al [64], K. Karpenko. [65].

Usually description of interaction between a quantum mechanical particle and an external classical gravitational field looks basically differently for fermions and bosons (S. Weinberg [66]). For a fermion, starting from the Dirac equation

$$(i\gamma^a \partial_a - \frac{mc}{\hbar}) \Psi(x) = 0$$

we have to generalize through the use of the tetrad formalism [66]. For a vector boson, a totally different approach is generally used: it consists in ordinary formal changing all involved tensors and usual derivative  $\partial_a$  into general relativity ones. For example, in case of a vector particle, the flat space Proca equations

$$\partial_a \Psi_b - \partial_b \Psi_a = \frac{mc}{\hbar} \Psi_{ab}, \quad \partial^b \Psi_{ab} = \frac{mc}{\hbar} \Psi_a$$

being subjected to the formal change  $\partial_a \rightarrow \nabla_\alpha$ ,  $\Psi_a \rightarrow \Psi_\alpha$ ,  $\Psi_{ab} \rightarrow \Psi_{\alpha\beta}$  result in

$$\nabla_\alpha \Psi_\beta - \nabla_\beta \Psi_\alpha = \frac{mc}{\hbar} \Psi_{\alpha\beta}, \quad \nabla^\beta \Psi_{\alpha\beta} = \frac{mc}{\hbar} \Psi_\alpha. \quad (1)$$

However, the known Duffin-Kemmer-Petiau formalism in the curved space-time till recent time was not used, though that possibility is known (S. Weinberg [66]). The situation is changing now:

J.T. Lunardi – B.M. Pimentel – R.G. Teixeira – J.S. Valverde – L.A. Manzoni [67-69], V.Ya. Fainberg – B.M. Pimentel [70], M. de Montigny – F.C. Khanna – A.E. Santana – E.S. Santos – J.D.M. Vianna. [71], I.V. Kanatchikov[72], R. Casana – V.Ya. Fainberg – B.M. Pimentel – J.T. Lunardi – R.G. Teixeira. [73, 74], Taylan Yetkin – Ali Havare [75], S. Gonen, A. Havare, N. Unal [76], A. Okninski [77], Mustafa Salti, Ali Havare [78], A.A. Bogush – V.S. Otchik – V.V. Kisel – N.G. Tokarevskaya – V.M. Red'kov [79-89].

## 2 Duffin-Kemmer-Petiau equation in gravitational field

We start from a flat space equation in its matrix DKP-form

$$\left( i \beta^a \partial_a - \frac{mc}{\hbar} \right) \Phi(x) = 0 ; \quad (2)$$

where

$$\begin{aligned} \Phi &= (\Phi_0, \Phi_1, \Phi_2, \Phi_3; \Phi_{01}, \Phi_{02}, \Phi_{03}, \Phi_{23}, \Phi_{31}, \Phi_{12}) , \\ \beta^a &= \begin{vmatrix} 0 & \kappa^a \\ \lambda^a & 0 \end{vmatrix} = \kappa^a \oplus \lambda^a , \quad (\kappa^a)_j^{[mn]} = -i (\delta_j^m g^{na} - \delta_j^n g^{ma}) , \\ (\lambda^a)_j^{[mn]} &= -i (\delta_m^a \delta_n^j - \delta_n^a \delta_m^j) = -i \delta_{mn}^{aj} ; \quad (3) \end{aligned}$$

$(g^{na}) = \text{diag}(+1, -1, -1, -1)$ . The basic properties of  $\beta^a$  are

$$\begin{aligned} \beta^c \beta^a \beta^b &= \begin{vmatrix} 0 & \kappa^c \lambda^a \kappa^b \\ \lambda^c \kappa^a \lambda^b & 0 \end{vmatrix} , \quad (\lambda^c \kappa^a \lambda^b)_j^{[mn]} = i (\delta_{mn}^{cb} g^{aj} - \delta_{mn}^{cj} g^{ab}) , \\ (\kappa^c \lambda^a \kappa^b)_j^{[mn]} &= i [\delta_j^a (g^{cm} g^{bn} - g^{cn} g^{bm}) + g^{ac} (\delta_j^n g^{mb} - \delta_j^m g^{nb})] , \quad (4) \end{aligned}$$

and

$$\begin{aligned} \beta^c \beta^a \beta^b + \beta^b \beta^a \beta^c &= \beta^c g^{ab} + \beta^b g^{ac} , \\ [\beta^c, j^{ab}] &= g^{ca} \beta^b - g^{cb} \beta^a , \quad j^{ab} = \beta^a \beta^b - \beta^b \beta^a , \\ [j^{mn}, j^{ab}] &= (g^{na} j^{mb} - g^{nb} j^{ma}) - (g^{ma} j^{nb} - g^{mb} j^{na}) . \quad (5) \end{aligned}$$

In accordance with tetrad recipe one should generalize the DKP-equation as follows

$$\begin{aligned} [ i \beta^\alpha(x) (\partial_\alpha + B_\alpha(x)) - \frac{mc}{\hbar} ] \Phi(x) &= 0 , \\ \beta^\alpha(x) &= \beta^a e_{(a)}^\alpha(x) , \quad B_\alpha(x) = \frac{1}{2} j^{ab} e_{(a)}^\beta \nabla_\alpha (e_{(b)\beta}) . \quad (6) \end{aligned}$$

This equation contains the tetrad  $e_{(a)}^\alpha(x)$  explicitly. Therefore, there must exist a possibility to demonstrate the equivalence of any variants of this equation associated with various tetrads:

$$e_{(a)}^\alpha(x) , \quad e'_{(b)}^\alpha(x) = L_a{}^b(x) e_{(b)}^\alpha(x) , \quad (7)$$

$L_a{}^b(x)$  is a local Lorentz transformation. We will show that two such equations

$$\begin{aligned} [i\beta^\alpha(x) (\partial_\alpha + B_\alpha(x)) - \frac{mc}{\hbar}] \Phi(x) &= 0, \\ [i\beta'^\alpha(x) (\partial_\alpha + B'_\alpha(x)) - \frac{mc}{\hbar}] \Phi'(x) &= 0, \end{aligned} \quad (8)$$

generated in tetrads  $e_{(a)}^\alpha(x)$  and  $e'_{(b)}^\alpha(x)$  respectively, can be converted into each other through a local gauge transformation:

$$\Phi'(x) = \begin{vmatrix} \phi'_a(x) \\ \phi'_{[ab]}(x) \end{vmatrix} = \begin{vmatrix} L_a{}^l & 0 \\ 0 & L_a{}^m L_b{}^n \end{vmatrix} \begin{vmatrix} \phi_l(x) \\ \phi_{[mn]}(x) \end{vmatrix}. \quad (9)$$

Starting from the first equation in (8), let us obtain an equation for  $\Phi'(x)$ . Allowing for  $\Phi'(x) = S(x) \Phi(x)$ , we get

$$[i S \beta^\alpha S^{-1} (\partial_\alpha + S B_\alpha S^{-1} + S \partial_\alpha S^{-1}) - \frac{mc}{\hbar}] \Phi'(x) = 0.$$

One should verify relationships

$$S(x) \beta^\alpha(x) S^{-1}(x) = \beta'^\alpha(x), \quad (10)$$

$$S(x) B_\alpha(x) S^{-1}(x) + S(x) \partial_\alpha S^{-1}(x) = B'_\alpha(x). \quad (11)$$

The first one can be rewritten as

$$S(x) \beta^a e_{(a)}^\alpha(x) S^{-1}(x) = \beta^b e'_{(b)}^\alpha(x);$$

from where we come to

$$S(x) \beta^a S^{-1}(x) = \beta^b L_b{}^a(x). \quad (12)$$

The latter condition is well-known in *DKP*-theory; one can verify it through the use of the block structure of  $\beta^a$ , which provides two relations:

$$\begin{aligned} L(x) \kappa^a [L^{-1}(x) \otimes L(x)^{-1}] &= \kappa^b L_b{}^a(x), \\ [L(x) \otimes L(x)] \lambda^a L(x)^{-1} &= \lambda^b L_b{}^a(x). \end{aligned}$$

They will be satisfied identically, after we take explicit form of  $\kappa^a$  and  $\lambda^a$  and allow for the pseudo orthogonality condition:  $g^{al} (L^{-1})_l{}^k(x) =$

$g^{kb} L_b^a(x)$ . Now, let us pass to the proof of the relationship (11). By using the determining relation for *DKP*-connection we readily produce

$$S(x) \partial_\alpha S^{-1}(x) = B'_\alpha(x) - \frac{1}{2} j^{mn} L_m^n(x) g_{ab} \partial_\alpha L_n^b(x)$$

therefore eq. (11) results in

$$S(x) \partial_\alpha S^{-1}(x) = \frac{1}{2} L_m^a(x) g_{ab} (\partial_\alpha L_n^b(x)).$$

The latter condition is an identity readily verified through the use of block structure of all involved matrices. Thus, the equations from (8) are translated into each other. In other words, they manifest a gauge symmetry under local Lorentz transformations in complete analogy with more familiar Dirac particle case. In the same time, the wave function from this equation represents scalar quantity relative to general coordinate transformations: if  $x^\alpha \rightarrow x'^\alpha = f^\alpha(x)$ , then  $\Phi'(x) = \Phi(x)$ .

It remains to demonstrate that this *DKP* formulation can be inverted into the Proca formalism in terms of general relativity tensors. To this end, as a first step, let us allow for the sectional structure of  $\beta^\alpha$ ,  $J^{ab}$  and  $\Phi(x)$  in the *DKP*-equation; then instead of (6) we get

$$\begin{aligned} i [ \kappa^c e_{(c)}^\alpha (\partial_\alpha + \kappa^a \lambda^b e_{(a)}^\beta \nabla_\alpha e_{(b)\beta}) ]_{[mn]}^l \Phi_l &= \frac{mc}{\hbar} \Phi_{[mn]}, \\ i [ \kappa^c e_{(c)}^\alpha (\partial_\alpha + \lambda^a \kappa^b e_{(a)}^\beta \nabla_\alpha e_{(b)\beta}) ]_l^{[mn]} \Phi_{[mn]} &= \frac{mc}{\hbar} \Phi_l, \end{aligned} \quad (13)$$

which lead to

$$\begin{aligned} (e_{(a)}^\alpha \partial_\alpha \Phi_b - e_{(b)}^\alpha \partial_\alpha \Phi_a) + (\gamma^c_{ab} - \gamma^c_{ba}) \Phi_c &= \frac{mc}{\hbar} \Phi_{ab}, \\ e^{(b)\alpha} \partial_\alpha \Phi_{ab} + \gamma^{nb}{}_n \Phi_{ab} + \gamma_a{}^{mn} \Phi_{mn} &= \frac{mc}{\hbar} \Phi_a; \end{aligned} \quad (14)$$

the symbol  $\gamma_{abc}(x)$  is used to designate Ricci coefficients:  $\gamma_{abc}(x) = -e_{(a)\alpha;\beta} e_{(b)}^\alpha e_{(c)}^\beta$ . In turn, (14) will look as the Proca equations (1) after they are rewritten in terms of tetrad components

$$\Phi_a(x) = e_{(a)}^\alpha(x) \Phi_\alpha(x), \quad \Phi_{ab}(x) = e_{(a)}^\alpha(x) e_{(b)}^\beta(x) \Phi_{\alpha\beta}(x). \quad (15)$$

So, as evidenced by the above, the manner of introducing the interaction between a spin 1 particle and external classical gravitational

field can be successfully unified with the approach that occurred with regard to a spin 1/2 particle and was first developed by Tetrode, Weyl, Fock, Ivanenko. One should attach great significance to that possibility of unification. Moreover, its absence would be a very strange fact. Let us add some more details.

The manner of extending the flat space Dirac equation to general relativity case indicates clearly that the Lorentz group underlies equally both these theories. In other words, the Lorentz group retains its importance and significance at changing the Minkowski space model to an arbitrary curved space-time. In contrast to this, at generalizing the Proca formulation, we automatically destroy any relations to the Lorentz group, although the definition itself for a spin 1 particle as an elementary object was based on this group. Such a gravity sensitiveness to the fermion-boson division might appear rather strange and unattractive asymmetry, being subjected to the criticism. Moreover, just this feature has brought about a plenty of speculation about this matter. In any case, this peculiarity of particle-gravity field interaction is recorded almost in every handbook.

### 3 Non-relativistic approximation, 10-component formalism

The first who was interested in a non-relativistic equation for a particle with spin 1 was A. Proca [12]. Let us consider such a problem in presence of external gravitational fields. To have a non-relativistic approximation, we must use the limitation on space-time models:

$$dS^2 = (dx^0)^2 + g_{ij}(x)dx^i dx^j, \quad e_{(a)\alpha}(x) = \begin{vmatrix} 1 & 0 \\ 0 & e_{(i)k}(x) \end{vmatrix}. \quad (16)$$

DKP-equation in presence both of curved space background and electromagnetic field is

$$\begin{aligned} [i\beta^0 D_0 + i\beta^k(x) D_k - \frac{mc}{\hbar}] \Psi &= 0, \\ D_\alpha &= \partial_\alpha + B_\alpha(x) - i\frac{e}{\hbar c} A_\alpha(x). \end{aligned} \quad (17)$$

In the metric (16), expressions for vector connections  $D_0, D_l$  become much simpler:

$$B_0 = \frac{1}{2} J^{ik} e_{(i)}^m (\nabla_0 e_{(k)m}), \quad B_l = \frac{1}{2} J^{ik} e_{(i)}^m (\nabla_l e_{(k)m}), \quad (18)$$

so there is no contribution from  $J^{0k}$  generators. Because of identities

$$\beta^0 \beta^0 J^{kl} = J^{kl} \beta^0 \beta^0 \implies \beta^0 \beta^0 B_\alpha(x) = B_\alpha(x) \beta^0 \beta^0 \quad (19)$$

the operator  $D_k$  commutes with  $(\beta^0)^2$ . Multiplying eq. (17) by projective  $(\beta^0)^2$  and  $1 - (\beta^0)^2$ , and taking into account relations

$$\begin{aligned} (\beta^0)^2 \beta^l(x) &= \beta^l(x) [1 - (\beta^0)^2], \\ [1 - (\beta^0)^2] \beta^l(x) &= \beta^l(x) (\beta^0)^2, \quad (\beta^0)^3 = \beta^0, \end{aligned} \quad (20)$$

we get to equations for  $\chi$  and  $\varphi$ :

$$\begin{aligned} \chi &= (\beta^0)^2 \Psi, \quad \varphi = (1 - (\beta^0)^2) \Psi, \quad \Psi = \chi + \varphi, \\ i\beta^0 D_0 \chi + i\beta^k(x) D_k \varphi &= \frac{mc}{\hbar} \chi, \quad i\beta^k(x) D_k \chi = \frac{mc}{\hbar} \varphi. \end{aligned} \quad (21)$$

Excluding a non-dynamical part  $\varphi$ , we arrive at

$$i\beta^0 D_0 \chi - \frac{\hbar}{mc} \beta^k(x) \beta^l(x) D_k D_l \chi = \frac{mc}{\hbar} \chi. \quad (22)$$

Now let us introduce two operators

$$\Pi_\pm = \frac{1}{2} \beta^0 (1 \pm \beta^0), \quad \Pi_+ \beta^0 = + \Pi_+, \quad \Pi_- \beta^0 = - \Pi_-.$$

From (22) it follows

$$iD_0 \Pi_+ \chi - \frac{\hbar}{mc} \Pi_+ \beta^k(x) \beta^l(x) D_k D_l \chi - \frac{mc}{\hbar} \Pi_+ \chi = 0; \quad (23)$$

with the help of

$$\Pi_+ \beta^k(x) \beta^l(x) = \frac{1}{2} [ (-\beta^l(x) \beta^k(x) + g^{lk}(x)) \beta^0 + \beta^k(x) \beta^l(x) (\beta^0)^2 ],$$

eq. (23) leads to

$$\begin{aligned} iD_0 (\Pi_+ \chi) - \frac{\hbar}{2mc} [ (-\beta^l(x) \beta^k(x) + g^{lk}(x)) \beta^0 + \\ + \beta^k(x) \beta^l(x) (\beta^0)^2 ] D_k D_l \chi = \frac{mc}{\hbar} (\Pi_+ \chi). \end{aligned} \quad (24)$$

In the same manner, starting from

$$-iD_0 \Pi_- \chi - \frac{\hbar}{mc} \Pi_- \beta^k(x) \beta^l(x) D_k D_l \chi = \frac{mc}{\hbar} \Pi_- \chi,$$

with the help of

$$\Pi_- \beta^k(x) \beta^l(x) = \frac{1}{2} [ (-\beta^l(x) \beta^k(x) + g^{kl}(x)) \beta^0 - \beta^k(x) \beta^l(x) (\beta^0)^2 ],$$

we get

$$\begin{aligned} -iD_0 \Pi_- \chi - \frac{\hbar}{2mc} [ (-\beta^l(x) \beta^k(x) + g^{kl}(x)) \beta^0 - \\ -\beta^k(x) \beta^l(x) (\beta^0)^2 ] D_k D_l \chi = \frac{mc}{\hbar} \Pi_- \chi. \end{aligned} \quad (25)$$

Changing matrices  $\beta^0$  and  $(\beta^0)^2$  according to

$$\beta^0 = \Pi_+ + \Pi_-, \quad (\beta^0)^2 = \Pi_+ - \Pi_-,$$

and using the notation

$$\begin{aligned} \Pi_- \chi &= \chi_-, & \Pi_+ \chi &= \chi_+, \\ J^{[kl]}(x) &= \beta^k(x) \beta^l(x) - \beta^l(x) \beta^k(x), \\ J^{(kl)}(x) &= \beta^k(x) \beta^l(x) + \beta^l(x) \beta^k(x), \end{aligned}$$

reduce eqs. (24) and (25) to the form

$$\begin{aligned} iD_0 \chi_+ - \frac{\hbar}{2mc} [ J^{[kl]}(x) D_k D_l \chi_+ - J^{(kl)}(x) D_k D_l \chi_- \\ + D^l D_l (\chi_+ + \chi_-) ] = \frac{mc}{\hbar} \chi_+, \\ -iD_0 \chi_- - \frac{\hbar}{2mc} [ J^{[kl]}(x) D_k D_l \chi_- - J^{(kl)}(x) D_k D_l \chi_+ \\ + D^l D_l (\chi_+ + \chi_-) ] = \frac{mc}{\hbar} \chi_-. \end{aligned} \quad (26)$$

Now, one should separate a special factor depending on the rest-energy, so that in eq. (26) one should make one formal change:

$$\chi \implies \exp(-i \frac{mc^2}{\hbar} t) \chi, \quad iD_0 \implies iD_0 + \frac{mc}{\hbar}. \quad (27)$$



As a result, eq. (26) gives

$$\begin{aligned}
 & iD_0 \chi_+ - \frac{\hbar}{2mc} [ (J^{[kl]}(x) D_k D_l \chi_+ \\
 & - J^{(kl)}(x) D_k D_l \chi_-) + D^l D_l (\chi_+ + \chi_-) ] = 0 , \\
 & -iD_0 \chi_- - \frac{\hbar}{2mc} [ (J^{[kl]}(x) D_k D_l \chi_- \\
 & - J^{(kl)}(x) D_k D_l \chi_+) + D^l D_l (\chi_+ + \chi_-) ] = 2 \frac{mc}{\hbar} \chi_- . \quad (28)
 \end{aligned}$$

Now, taking  $\chi_-$  as small and ignoring the term  $-iD_0\chi_-$  compared with  $\frac{mc}{\hbar}\chi_-$  we arrive at

$$\begin{aligned}
 & (J^{(kl)} D_k D_l - D^l D_l) \chi_+ = \frac{4m^2 c^2}{\hbar^2} \chi_- , \\
 & i\hbar D_t \chi_+ = \frac{\hbar^2}{2m} (D^l D_l + J^{[kl]} D_k D_l) \chi_+ . \quad (29)
 \end{aligned}$$

The second equation in (29) should be considered as a non-relativistic Pauli equation for spin 1 particle in DKP-approach.

It is interesting to see what is the form of the non-relativistic approximation in tensor form? At first, let us restrict ourselves to the case of the flat space. From (21) it follows

$$\beta^0 \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_{01} \\ \Phi_{02} \\ \Phi_{03} \\ \Phi_{23} \\ \Phi_{31} \\ \Phi_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ i\Phi_{01} \\ i\Phi_{02} \\ i\Phi_{03} \\ -i\Phi_1 \\ -i\Phi_2 \\ -i\Phi_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \quad \chi = (\beta^0)^2 \Psi = \begin{pmatrix} 0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_{01} \\ \Phi_{02} \\ \Phi_{03} \\ 0 \\ 0 \\ 0 \end{pmatrix} , \quad \varphi = (1 - (\beta^0)^2) \Psi = \begin{pmatrix} \Phi_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \Phi_{23} \\ \Phi_{31} \\ \Phi_{12} \end{pmatrix} ,$$

so that

$$\chi_+ = \frac{1}{2} \begin{vmatrix} 0 \\ (\Phi_1 + i \Phi_{01}) \\ (\Phi_2 + i \Phi_{02}) \\ (\Phi_3 + i \Phi_{03}) \\ -i(\Phi_1 + i \Phi_{01}) \\ -i(\Phi_2 + i \Phi_{02}) \\ -i(\Phi_3 + i \Phi_{03}) \\ 0 \\ 0 \\ 0 \end{vmatrix}, \quad \chi_- = \frac{1}{2} \begin{vmatrix} 0 \\ -(\Phi_1 - i \Phi_{01}) \\ -(\Phi_2 - i \Phi_{02}) \\ -(\Phi_3 - i \Phi_{03}) \\ -i(\Phi_1 - i \Phi_{01}) \\ -i(\Phi_2 - i \Phi_{02}) \\ -i(\Phi_3 - i \Phi_{03}) \\ 0 \\ 0 \\ 0 \end{vmatrix}.$$

Instead of  $\Phi_k$  and  $\Phi_{0k}$ , let us introduce new field variables:

$$\begin{aligned} \frac{1}{2} (\Phi_k - i \Phi_{0k}) &= M_k, & \frac{1}{2} (\Phi_k + i \Phi_{0k}) &= B_k, \\ \Phi_k &= B_k + M_k, & \Phi_{0k} &= -i (B_k - M_k); \end{aligned} \quad (30)$$

that is

$$\chi_+ = \begin{vmatrix} 0 \\ \vec{B} \\ -i\vec{B} \\ 0 \\ 0 \\ 0 \end{vmatrix}, \quad \chi_- = \begin{vmatrix} 0 \\ -\vec{M} \\ -i\vec{M} \\ 0 \\ 0 \\ 0 \end{vmatrix}. \quad (31)$$

Thus, in tensor representation the big and small components coincides with 3-vectors  $\vec{B}$  and  $\vec{M}$  respectively. Now it is a matter of simple calculation to repeat the limiting procedure in tensor basis. Indeed, starting from Proca equations (it is convenient to change the notation  $mc/\hbar \Rightarrow m$ )

$$\begin{aligned} D_0 \Phi_k - D_k \Phi_0 &= m \Phi_{0k}, & D_k \Phi_l - D_l \Phi_k &= m \Phi_{kl}, \\ D^l \Phi_{0l} &= m \Phi_0, & D^0 \Phi_{k0} + D^l \Phi_{kl} &= m \Phi_k, \end{aligned} \quad (32)$$

and excluding the non-dynamical components  $\Phi_0, \Phi_{kl}$ ,

$$\begin{aligned} D_0 \Phi_k - \frac{1}{m} D_k D^l \Phi_{0l} &= m \Phi_{0k}, \\ D^0 \Phi_{k0} + \frac{1}{m} D^l (D_k \Phi_l - D_l \Phi_k) &= m \Phi_k \end{aligned} \quad (33)$$

we get

$$\begin{aligned}
 m (\Phi_k \pm i \Phi_{0k}) &= ( D^0 \Phi_{k0} + \frac{1}{m} D^l D_k \Phi_l - \\
 &- \frac{1}{m} D^l D_l \Phi_k ) \pm i ( D_0 \Phi_k - \frac{1}{m} D_k D^l \Phi_{0l} ). \quad (34)
 \end{aligned}$$

From (34), with the help of (30), it follow

$$\begin{aligned}
 2m B_k &= +2i D_0 B_k - \frac{1}{m} D^l D_l (B_k + M_k) + \\
 &+ \frac{1}{m} [ (D^l D_k - D_k D^l) B_l + (D^l D_k + D_k D^l) M_l ], \\
 2m M_k &= -2i D_0 M_k - \frac{1}{m} D^l D_l (B_k + M_k) + \\
 &+ \frac{1}{m} [ (D^l D_k + D_k D^l) B_l + (D^l D_k - D_k D^l) M_l ]. \quad (35)
 \end{aligned}$$

After separating the rest-energy term

$$i D_0 B_k \implies (i D_0 + m) B_k, \quad i D_0 M_k \implies (i D_0 + m) M_k,$$

from (35) we arrive at

$$\begin{aligned}
 &+i D_0 B_k - \frac{1}{2m} \{ D^l D_l (B_k + M_k) + \\
 &+ (D^l D_k - D_k D^l) B_l + (D^l D_k + D_k D^l) M_l \} = 0, \\
 &-i D_0 M_k - \frac{1}{2m} \{ D^l D_l (B_k + M_k) + \\
 &+ (D^l D_k + D_k D^l) B_l + (D^l D_k - D_k D^l) M_l \} = 4m M_k. \quad (36)
 \end{aligned}$$

Therefore, a non-relativistic wave equation for the big component  $B_k(x)$  has the form (let us change the notation:  $B_k(x) \implies \psi_k(x)$ )

$$+i D_0 \psi_k = \frac{1}{2m} [ -D_l D_l \psi_k - (D_k D_l - D_l D_k) \psi_l ]. \quad (37)$$

#### 4 Tetrad 3-dimensional non-relativistic equation

The non-relativistic equation (29) in DKP-formalism is symbolical in a sense, because it is written formally for a 10-component function though

in fact the non-relativistic function is a 3-vector. Let us turn to the above limiting procedure again (for shortness  $e/\hbar c \implies e$ ,  $mc/\hbar \implies m$ )

$$[i\beta^0 D_0 + i\beta^l(x) D_l - m] \Psi = 0, \quad (38)$$

where

$$\begin{aligned} \beta^l(x) &= \beta^k e_{(k)}^l, \quad D_l = \partial_l + B_l - ieA_l, \quad D_0 = \partial_0 + B_0 - ieA_0, \\ B_0 &= \frac{1}{2} J^{ik} e_{(i)}^m (\nabla_0 e_{(k)m}), \quad B_l = \frac{1}{2} J^{ik} e_{(i)}^m (\nabla_l e_{(k)m}). \end{aligned} \quad (39)$$

With the use of block-forms for DKP-matrices

$$\beta^0 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i_3 & 0 \\ 0 & -i_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad \beta^k = \begin{vmatrix} 0 & 0 & w_k & 0 \\ 0 & 0 & 0 & \tau_k \\ \tilde{w}_k & 0 & 0 & 0 \\ 0 & -\tau_k & 0 & 0 \end{vmatrix}, \quad (40)$$

where

$$\begin{aligned} w_1 &= (i, 0, 0), \quad w_2 = (0, i, 0), \quad w_3 = (0, 0, i), \\ \tilde{w}_1 &= \begin{vmatrix} i \\ 0 \\ 0 \end{vmatrix}, \quad \tilde{w}_2 = \begin{vmatrix} 0 \\ i \\ 0 \end{vmatrix}, \quad \tilde{w}_3 = \begin{vmatrix} 0 \\ 0 \\ i \end{vmatrix}, \quad i_3 = \begin{vmatrix} i00 \\ 0i0 \\ 00i \end{vmatrix}, \\ \tau_1 &= \begin{vmatrix} 00 & 0 \\ 00 & -i \\ 0i & 0 \end{vmatrix}, \quad \tau_2 = \begin{vmatrix} 00 & i \\ 000 \\ -i00 \end{vmatrix}, \quad \tau_3 = \begin{vmatrix} 0 & -i0 \\ i & 00 \\ 0 & 00 \end{vmatrix} \end{aligned} \quad (41)$$

and taking explicit form of generators  $J^{kl}$ , for connections  $B_0(x)$  and  $B_l(x)$  we have expressions

$$B_0(x) = \begin{vmatrix} 00 & 0 & 0 \\ 0b_0 & 0 & 0 \\ 00 & b_0 & 0 \\ 00 & 0 & b_0 \end{vmatrix}, \quad B_l(x) = \begin{vmatrix} 00 & 0 & 0 \\ 0b_l & 0 & 0 \\ 00 & b_l & 0 \\ 00 & 0 & b_l \end{vmatrix}, \quad (42)$$

where

$$\begin{aligned} b_0(x) &= -i [ \tau_1 e_{(2)}^k \partial_0 e_{(3)k} + \tau_2 e_{(3)}^k \partial_0 e_{(1)k} + \tau_3 e_{(1)}^k \partial_0 e_{(2)k} ], \\ b_l(x) &= -i [ \tau_1 e_{(2)}^k \nabla_l e_{(3)k} + \tau_2 e_{(3)}^k \nabla_l e_{(1)k} + \tau_3 e_{(1)}^k \nabla_l e_{(2)k} ]. \end{aligned} \quad (43)$$

Eq. (38) can be rewritten as a system for constituents of  $\Psi = (\Phi_0, \Phi, E, H)$ :

$$\begin{aligned}
 iw^l(x) (\nabla_l + b_l - ieA_l) E &= m \Phi_0 , \\
 -(\nabla_0 + b_0 - ieA_0) E + i\tau^l(x) (\nabla_l + b_l - ieA_l) H &= m \Phi , \\
 i\tilde{w}^l(x) (\nabla_l - ieA_l) \Phi_0 + (\nabla_0 + b_0 - ieA_0) \Phi &= m E , \\
 -i\tau^l(x) (\nabla_l + b_l - ieA_l) \Phi &= m H , \quad (44)
 \end{aligned}$$

where

$$\tau^l(x) = e^l_{(k)}(x) \tau^k, \quad w^l(x) = e^l_{(k)}(x) w^k, \quad \tilde{w}^l(x) = e^l_{(k)}(x) \tilde{w}^k .$$

After excluding non-dynamical variables  $\Phi_0(x)$  and  $H(x)$ , we get

$$\begin{aligned}
 & -(\nabla_0 + b_0 - ieA_0) E \\
 +i\tau^l(x)(\nabla_l + b_l - ieA_l) & \left(-\frac{i}{m}\right) \tau^k(x) (\nabla_k + b_k - ieA_k) \Phi = m \Phi , \\
 & (\nabla_0 + b_0 - ieA_0) \Phi \\
 +i\tilde{w}^l(x)(\nabla_l - ieA_l) & \frac{i}{m} w^k(x) (\nabla_k + b_k - ieA_k) E = m E . \quad (45)
 \end{aligned}$$

Allowing for commutative relations

$$\begin{aligned}
 \tau^k(x) (\nabla_l + b_l - ieA_l) &= (\nabla_l + b_l - ieA_l) \tau^k(x) , \\
 (\nabla_l - ieA_l) w^k(x) &= w^k(x) (\nabla_l + b_l - ieA_l) ,
 \end{aligned}$$

eqs. (45) reduce to ( $\bullet$  represents diad multiplication of vectors)

$$\begin{aligned}
 & -(\nabla_0 + b_0 - ieA_0) E \\
 +\frac{1}{m} \tau^l(x) \tau^k(x) & (\nabla_l + b_l - ieA_l) (\nabla_k + b_k - ieA_k) \Phi = m \Phi , \\
 & +(\nabla_0 + b_0 - ieA_0) \Phi \\
 -\frac{1}{m} \tilde{w}^l(x) \bullet w^k(x) & (\nabla_l + b_l - ieA_l) (\nabla_k + b_k - ieA_k) E = m E ,
 \end{aligned}$$

or

$$\begin{aligned}
 D_0 E + \frac{1}{m} \tau^l(x) \tau^k(x) D_l D_k \Phi &= m \Phi , \\
 D_0 \Phi - \frac{1}{m} \tilde{w}^l(x) \bullet w^k(x) D_l D_k E &= m E . \quad (46)
 \end{aligned}$$

Instead of  $\Phi(x)$  and  $E(x)$  let us introduce  $\psi(x)$  and  $\varphi(x)$ :

$$\frac{1}{2} (\Phi + iE) = \psi, \quad \frac{1}{2} (\Phi - iE) = \varphi. \quad (47)$$

Eqs. (46) will look

$$\begin{aligned} 2m \psi &= +2iD_0 \psi + \frac{1}{m} \tau^l(x) \tau^k(x) D_l D_k (\psi + \varphi) \\ &\quad - \frac{1}{m} \tilde{w}^l(x) \bullet w^k(x) D_l D_k (\psi - \varphi), \\ 2m \varphi &= -2iD_0 \varphi + \frac{1}{m} \tau^l(x) \tau^k(x) D_l D_k (\psi + \varphi) \\ &\quad + \frac{1}{m} \tilde{w}^l(x) \bullet w^k(x) D_l D_k (\psi - \varphi). \end{aligned} \quad (48)$$

Making the formal change  $iD_0 \implies (iD_0 + m)$ , we get to

$$\begin{aligned} 0 &= +2iD_0 \psi + \frac{1}{m} \tau^l(x) \tau^k(x) D_l D_k (\psi + \varphi) \\ &\quad - \frac{1}{m} \tilde{w}^l(x) \bullet w^k(x) D_l D_k (\psi - \varphi), \\ 4m \varphi &= -2iD_0 \varphi + \frac{1}{m} \tau^l(x) \tau^k(x) D_l D_k (\psi + \varphi) \\ &\quad + \frac{1}{m} \tilde{w}^l(x) \bullet w^k(x) D_l D_k (\psi - \varphi). \end{aligned} \quad (49)$$

From eq. (49), taking  $\psi(x)$  as a big component and  $\varphi(x)$  as small we arrive at

$$\begin{aligned} iD_0 \psi(x) &= \frac{1}{2m} [ \tilde{w}^l(x) \bullet w^k(x) - \tau^l(x) \tau^k(x) ] D_l D_k \psi(x), \\ 4m^2 \varphi(x) &= + [ \tau^l(x) \tau^k(x) + \tilde{w}^l(x) \bullet w^k(x) ] D_l D_k \psi(x). \end{aligned} \quad (50)$$

From (50), allowing for identity  $\tau^l(x) \tau^k(x) = -g^{lk}(x) + \tilde{w}^k(x) \bullet w^l(x)$  and using notation

$$\begin{aligned} \tilde{w}^l(x) \bullet w^k(x) + \tilde{w}^k(x) \bullet w^l(x) &= w^{(lk)}(x), \\ \tilde{w}^l(x) \bullet w^k(x) - \tilde{w}^k(x) \bullet w^l(x) &= j^{lk}(x), \\ j^{ps} &= -i\epsilon_{psj} \tau_j, \quad j^{lk}(x) = e_{(p)}^l e_{(s)}^k j^{ps}, \end{aligned}$$

we will obtain

$$\begin{aligned}
 4m^2 \varphi(x) &= (-D^l D_l + w^{(lk)}(x) D_l D_k) \psi, \\
 iD_0 \psi(x) &= \frac{1}{2m} (D^l D_l + \frac{1}{2} j^{lk}(x) [D_l, D_k]_-) \psi(x). \quad (51)
 \end{aligned}$$

Thus, the Pauli equation for 3-vector wave function  $\psi$  in Riemannian space is (compare with (29))

$$iD_t \psi = \frac{1}{2m} (D^l D_l + \frac{1}{2} j^{lk}(x) [D_l, D_k]_-) \psi. \quad (52)$$

One additional point should be stressed. Take notice that the non-relativistic wave function is constructed in terms of relativistic ones as follows:

$$\psi(x) = \frac{1}{2} [ \Phi_i(x) + i E_i(x) ], \quad \text{where} \quad E_i(x) = \Phi_{0i}(x); \quad (53)$$

this function  $\psi$  is complex even if we start with real-valued relativistic components.

Let us add some detail about the term  $\frac{1}{2} j^{lk}(x) [D_l, D_k]_-$  in (52):

$$\frac{1}{2} j^{lk}(x) [D_l, D_k]_- \psi = \frac{1}{2} j^{lk}(x) (-ieF_{lk} + \nabla_l b_k - \nabla_k b_l + b_l b_k - b_k b_l) \psi.$$

Taking into account relations

$$\begin{aligned}
 \nabla_l b_k - \nabla_k b_l &= + j^{dc} (\nabla_l e_{(d)m}) (\nabla_k e_{(c)}^m) \\
 &+ \frac{1}{2} j^{dc} e_{(d)m} \{ \nabla_l \nabla_k - \nabla_k \nabla_l \} e_{(c)m}
 \end{aligned}$$

and

$$\begin{aligned}
 b_l b_k - b_k b_l &= \frac{1}{4} (j^{ps} j^{cd} - j^{cd} j^{ps}) e_{(p)}^n (\nabla_l e_{(s)n}) e_{(c)}^m (\nabla_k e_{(d)m}) \\
 &= - j^{dc} (\nabla_l e_{(d)m}) (\nabla_k e_{(c)}^m),
 \end{aligned}$$

we find

$$\begin{aligned}
 &b_l b_k - b_k b_l + \nabla_l b_k - \nabla_k b_l \\
 &= \frac{1}{2} j^{dc} e_{(d)}^m R_{klmn} e_{(c)}^n = \frac{1}{2} j^{mn}(x) R_{lkmn}; \quad (54)
 \end{aligned}$$

$R_{lkmn}$  is a Riemann curvature tensor for 3-space, and eq. (52) can be written as

$$iD_t\psi = \frac{1}{2m} [D^l D_l - ie\frac{1}{2} j^{lk}(x)F_{lk} + \frac{1}{4}j^{lk}(x)j^{mn}(x)R_{lkmn}] \psi. \quad (55)$$

In turn, one can readily verify

$$\begin{aligned} \frac{1}{4}j^{lk}(x)j^{mn}(x)R_{lkmn} &= \frac{1}{4} (-i\epsilon_{prc}\tau_c)e_{(p)}^l e_{(r)}^k (-i\epsilon_{std}\tau_d)e_{(s)}^m e_{(t)}^n R_{lkmn} = \\ &= -\frac{1}{4} \vec{\tau}(\vec{e}^l \times \vec{e}^k)R_{lkmn}\vec{\tau}(\vec{e}^m \times \vec{e}^n). \end{aligned} \quad (56)$$

Thus, the Pauli equation for meson in a curved space looks as follows (in ordinary units)

$$\begin{aligned} i\hbar D_t \psi &= -\frac{\hbar^2}{2m} [ -D^l D_l + i \frac{e}{\hbar c} \frac{1}{2} j^{lk}(x) F_{lk} \\ &+ \frac{1}{4} \vec{\tau} ( \vec{e}^l \times \vec{e}^k ) R_{lkmn} \vec{\tau} ( \vec{e}^m \times \vec{e}^n ) ] \psi. \end{aligned} \quad (57)$$

## 5 The wave equation for a scalar particle in Riemannian space, non-relativistic approximation

Now let us turn the case of a scalar particle. The Klein-Fock-Gordon equation in a curved space is

$$[ (i\hbar\nabla_\alpha + \frac{e}{c}A_\alpha)g^{\alpha\beta}(x)(i\hbar\nabla_\beta + \frac{e}{c}A_\beta) - \frac{\hbar^2}{6} R - m^2c^2 ]\Psi = 0. \quad (58)$$

Take notice on additional interaction term through scalar curvature  $R(x)$  (F. Gürsey [90]). This equation may be changed to the form more convenient in application. With the use of the known relations [91]

$$\begin{aligned} \nabla_\alpha g^{\alpha\beta}(x)\nabla_\beta \Phi &= \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g} g^{\alpha\beta} \frac{\partial}{\partial x^\beta} \Psi, \\ \nabla_\alpha g^{\alpha\beta} A_\beta &= \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g} g^{\alpha\beta} A_\beta, \quad g = \det (g_{\alpha\beta}) \end{aligned}$$

eq. (58) is changed to

$$\left[ \frac{1}{\sqrt{-g}} (i\hbar \frac{\partial}{\partial x^\alpha} + \frac{e}{c} A_\alpha) \sqrt{-g} g^{\alpha\beta} (i\hbar \frac{\partial}{\partial x^\beta} + \frac{e}{c} A_\beta) - \frac{\hbar^2}{6} R - m^2 c^2 \right] \Psi = 0. \quad (59)$$



What is the Schrödinger's non-relativistic equation in the curved space-time? Let us begin with a generally covariant first order equations for a scalar particle (take notice to the additional interaction term through the Ricci scalar

$$\begin{aligned} (i \nabla_\alpha + \frac{e}{c\hbar} A_\alpha) \Phi &= \frac{mc}{\hbar} \Phi_\alpha, \\ (i \nabla_\alpha + \frac{e}{c\hbar} A_\alpha) \Phi &= \frac{mc}{\hbar} (1 + \sigma \frac{R(x)}{m^2 c^2 / \hbar^2}) \Phi. \end{aligned} \quad (60)$$

With the notation

$$1 + \sigma \frac{R(x)}{m^2 c^2 / \hbar^2} = \Gamma(x)$$

eqs. (60) read

$$\begin{aligned} (i \partial_\alpha + \frac{e}{c\hbar} A_\alpha) \Phi(x) &= \frac{mc}{\hbar} \Phi_\alpha, \\ (\frac{i}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g} + \frac{e}{c\hbar} A_\alpha) g^{\alpha\beta} \Phi_\beta &= \frac{mc}{\hbar} \Gamma \Phi. \end{aligned} \quad (61)$$

In the space-time models of the type (16), one can easily separate time- and space- variables in eq. (61):

$$\begin{aligned} (i\partial_0 + \frac{e}{c\hbar} A_0) \Phi &= \frac{mc}{\hbar} \Phi_0, & (i\partial_l + \frac{e}{c\hbar} A_l) \Phi &= \frac{mc}{\hbar} \Phi_l, \\ (i \frac{\partial}{\partial x^0} + \frac{i}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial x^0} + \frac{e}{c\hbar} A_0) \Phi_0 & & & \\ + (\frac{i}{\sqrt{-g}} \frac{\partial}{\partial x^k} \sqrt{-g} + \frac{e}{c\hbar} A_k) g^{kl} \Phi_l &= \frac{mc}{\hbar} \Gamma \Phi. \end{aligned} \quad (62)$$

Now one should separate the rest energy term by means of the substitutions:

$$\Phi \implies \exp(-i \frac{mc^2 t}{\hbar}) \Phi, \quad \Phi_0 \implies \exp(-i \frac{mc^2 t}{\hbar}) \Phi_0, \quad \Phi_l \implies \exp(-i \frac{mc^2 t}{\hbar}) \Phi_l.$$

As a result, eq. (62) will give

$$(i\hbar \partial_t + mc^2 + eA_0) \Phi(x) = mc^2 \Phi_0(x), \quad (63)$$

$$(i\hbar \partial_t + mc^2 + i\hbar \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial t} + e A_0) \Phi_0 + c(\frac{i\hbar}{\sqrt{-g}} \frac{\partial}{\partial x^k} \sqrt{-g} + \frac{e}{c} A_k) g^{kl} \Phi_l = mc^2 \Gamma \Phi(x), \quad (64)$$

$$(i \hbar \partial_l + \frac{e}{c} A_l) \Phi(x) = mc \Phi_l(x). \quad (65)$$

With the help of (65), the non-dynamical variable  $\Phi_l$  can be readily excluded:

$$(i\hbar \partial_t + mc^2 + eA_0) \Phi(x) = mc^2 \Phi_0(x), \quad (66)$$

$$(i\hbar \partial_t + mc^2 + i\hbar \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial t} + e A_0) \Phi_0 + \frac{1}{m} [(\frac{i\hbar}{\sqrt{-g}} \partial_k \sqrt{-g} + \frac{e}{c} A_k) g^{kl} (i \hbar \partial_l + \frac{e}{c} A_l)] \Phi(x) = mc^2 \Gamma \Phi(x). \quad (67)$$

Now we are to introduce a small  $\varphi$  and big  $\Psi$  components:

$$\begin{aligned} \Phi - \Phi_0 &= \varphi, & \Phi + \Phi_0 &= \Psi, \\ \Phi &= \frac{\Psi + \varphi}{2}, & \Phi_0 &= \frac{\Psi - \varphi}{2}. \end{aligned} \quad (68)$$

Substituting eq. (68) into (66) and (67) one gets

$$\begin{aligned} (i\hbar \partial_t + mc^2 + eA_0) \frac{\Psi + \varphi}{2} &= mc^2 \frac{\Psi - \varphi}{2}, & (69) \\ (i\hbar \partial_t + mc^2 + i\hbar \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial t} + e A_0) \frac{\Psi - \varphi}{2} \\ + \frac{1}{m} [(\frac{i\hbar}{\sqrt{-g}} \partial_k \sqrt{-g} + \frac{e}{c} A_k) g^{kl} (i\hbar \partial_l + \frac{e}{c} A_l)] \frac{\Psi + \varphi}{2} &= mc^2 \Gamma \frac{\Psi + \varphi}{2}, \end{aligned} \quad (70)$$

or after simple calculation we arrive at

$$(i\hbar \partial_t + eA_0) \frac{+\varphi + \Psi}{2} = -mc^2 \varphi, \quad (71)$$

$$\begin{aligned} & \left( i\hbar \partial_t + i\hbar \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial t} + e A_0 \right) \frac{\Psi - \varphi}{2} \\ & + \frac{1}{m} \left[ \left( \frac{i\hbar}{\sqrt{-g}} \partial_k \sqrt{-g} + \frac{e}{c} A_k \right) g^{kl} \left( i\hbar \partial_l + \frac{e}{c} A_l \right) \right] \frac{\Psi + \varphi}{2} \\ & = mc^2 (\Gamma + 1) \frac{\varphi}{2} + mc^2 (\Gamma - 1) \frac{\Psi}{2}. \end{aligned} \quad (72)$$

In this point, it is better to consider two different cases. The first possibility is when one poses an additional requirement  $\Gamma = 1$ , which means the absence of the non-minimal interaction term through  $R$ -scalar. Then at  $\Gamma = 1$ , from the previous equations – ignoring small component compared with big one – it follows

$$(i\hbar \partial_t + eA_0) \frac{\Psi}{2} = -mc^2 \varphi, \quad (73)$$

$$\begin{aligned} & \left( i\hbar \partial_t + i\hbar \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial t} + e A_0 \right) \frac{\Psi}{2} \\ & + \frac{1}{m} \left[ \left( \frac{i\hbar}{\sqrt{-g}} \partial_k \sqrt{-g} + \frac{e}{c} A_k \right) g^{kl} \left( i\hbar \partial_l + \frac{e}{c} A_l \right) \right] \frac{\Psi}{2} = mc^2 \varphi. \end{aligned} \quad (74)$$

Excluding the small constituent we arrive at

$$\begin{aligned} & \left[ i\hbar \left( \partial_t + \frac{1}{2\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial t} \right) + e A_0 \right] \Psi \\ & = \frac{1}{2m} \left[ \left( \frac{i\hbar}{\sqrt{-g}} \partial_k \sqrt{-g} + \frac{e}{c} A_k \right) (-g^{kl}) \left( i\hbar \partial_l + \frac{e}{c} A_l \right) \right] \Psi \end{aligned} \quad (75)$$

With the help of substitution  $\Psi \implies (-g)^{-1/4} \Psi$  the obtained equation can be simplified:

$$\begin{aligned} & (i\hbar \partial_t + e A_0) \Psi = \\ & = \frac{1}{2m} \left[ \left( \frac{i\hbar}{\sqrt{-g}} \partial_k \sqrt{-g} + \frac{e}{c} A_k \right) (-g^{kl}) \left( i\hbar \partial_l + \frac{e}{c} A_l \right) \right] \Psi, \end{aligned} \quad (76)$$

which is the the Schrödinger equation in curved space.

The second possibility is when  $\Gamma \neq 1$ , then from (72)

$$\begin{aligned}
 (i\hbar \partial_t + eA_0) \frac{\Psi}{2} &= -mc^2 \varphi, & (77) \\
 (i\hbar \partial_t + i\hbar \frac{1}{\sqrt{-g}} \frac{\partial\sqrt{-g}}{\partial t} + eA_0) \frac{\Psi}{2} \\
 + \frac{1}{m} [ (\frac{i\hbar}{\sqrt{-g}} \partial_k \sqrt{-g} + \frac{e}{c} A_k) g^{kl} (i\hbar \partial_l + \frac{e}{c} A_l) ] \frac{\Psi}{2} \\
 &= mc^2 (\Gamma + 1) \frac{\varphi}{2} + mc^2 (\Gamma - 1) \frac{\Psi}{2}. & (78)
 \end{aligned}$$

With the use of (77) we can derive the following equation for the big component  $\Psi$ :

$$\begin{aligned}
 (i\hbar \partial_t + i\hbar \frac{1}{\sqrt{-g}} \frac{\partial\sqrt{-g}}{\partial t} + eA_0) \frac{\Psi}{2} \\
 + \frac{(\Gamma + 1)}{2} (i\hbar \partial_t + eA_0) \frac{\Psi}{2} - mc^2 (\Gamma - 1) \frac{\Psi}{2} \\
 - \frac{1}{2m} [ (\frac{i\hbar}{\sqrt{-g}} \partial_k \sqrt{-g} + \frac{e}{c} A_k) g^{kl} (i\hbar \partial_l + \frac{e}{c} A_l) ] \Psi. & (79)
 \end{aligned}$$

This equation can be rewritten as follows:

$$\begin{aligned}
 [ (\frac{1}{2} + \frac{1}{2} \frac{\Gamma(x) + 1}{2}) (i\hbar \partial_t + eA_0) + \frac{i\hbar}{2\sqrt{-g}} \frac{\partial\sqrt{-g}}{\partial t} ] \Psi \\
 = \frac{1}{2m} [ (\frac{i\hbar}{\sqrt{-g}} \partial_k \sqrt{-g} + \frac{e}{c} A_k) (-g^{kl}) (i\hbar \partial_l + \frac{e}{c} A_l) ] \Psi + mc^2 \frac{(\Gamma(x) - 1)}{2} \Psi. & (80)
 \end{aligned}$$

It remains to recall that

$$\Gamma(x) = 1 + \frac{1}{6} \frac{\hbar^2 R(x)}{m^2 c^2},$$

so the previous equation will take the form

$$\begin{aligned}
 [ (1 + \frac{1}{24} \frac{\hbar^2 R(x)}{m^2 c^2}) (i\hbar \partial_t + eA_0) + \frac{i\hbar}{2\sqrt{-g}} \frac{\partial\sqrt{-g}}{\partial t} ] \Psi = \\
 = \frac{1}{2m} [ (\frac{i\hbar}{\sqrt{-g}} \partial_k \sqrt{-g} + \frac{e}{c} A_k) (-g^{kl}) (i\hbar \partial_l + \frac{e}{c} A_l) ] \Psi + mc^2 \frac{\hbar^2 R}{12m^2 c^2} \Psi & (81)
 \end{aligned}$$

and finally

$$\begin{aligned} & \left[ \left( 1 + \frac{1}{24} \frac{\hbar^2 R(x)}{m^2 c^2} \right) (i\hbar \partial_t + e A_0) + \frac{i\hbar}{2\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial t} \right] \Psi \\ &= \frac{1}{2m} \left[ \left( \frac{i\hbar}{\sqrt{-g}} \partial_k \sqrt{-g} + \frac{e}{c} A_k \right) (-g^{kl}) \left( i\hbar \partial_l + \frac{e}{c} A_l \right) \right] + \hbar^2 \frac{R}{6} \Psi \quad (82) \end{aligned}$$

which should be considered as a Schrödinger equation in a space-time with non-vanishing scalar curvature  $R(x) \neq 0$  when allowing for a non-minimal interaction term through scalar curvature  $R(x)$ .

In addition, several general comments may be given. The wave function of Schrödinger equation  $\Psi$  does not coincide with the initial scalar Klei-Fock wave function  $\Phi$ . Instead we have the following

$$\Psi = \Phi + \Phi_0, \quad \Phi_0 \text{ belongs to } \{ \Phi_0, \Phi_1, \Phi_2, \Phi_3 \}. \quad (83)$$

One may have looked at this fact as a non-occasional and even necessary one (in this context see recent discussion of the problem in [92,93]). Indeed, let one start with a neutral scalar particle theory. Such a particle cannot interact with electromagnetic field and its wave function is real. However, by general consideration, certain non-relativistic limit in this theory must exist. It is the case in fact: the added term in (83)

$$\Phi_0 = i \frac{\hbar}{mc} \partial_0 \Phi \quad (84)$$

is imaginary even if  $\Phi^* = +\Phi$ . All the more, that situation is in accordance with the the mathematical structure of the Schrödinger equation itself, it cannot be written for a real-valued wave function at all.

The same property was seen in the theory of a vector particle: even if the wave function of the relativistic particle of spin 1 is taken real, the corresponding wave function in the non-relativistic approximation turns to be complex-valued. By general consideration, one may expect an analogous result in the theory of a spin 1/2 particle: if the non relativistic approximation is done in the theory of Majorana neutral particle [94] with the real 4-spinor wave function then the corresponding Pauli spinor wave function must be complex-valued.

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