Torsion tensor and its geometric interpretation

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ABSTRACT. Generally, spin is considered to be the source of torsion, but there are several other possibilities in which torsion emerges in different contexts. In some cases a phenomenological counterpart is absent, in some other cases torsion arises from sources without spin as a gradient of a scalar field. Accordingly, we propose two classification schemes. The *first* one is based on the possibility to construct torsion tensors from the product of a covariant bivector and a vector and their respective space-time properties. The *second* one is obtained by starting from the decomposition of torsion into three irreducible pieces. Their space-time properties again lead to a complete classification. The classifications found are given in a $\mathbf{U}_4$, a four dimensional space-time where the torsion tensors have some peculiar properties. The irreducible decomposition is useful since most of the phenomenological work done for torsion concerns four dimensional cosmological models. In the second part of the paper two applications of these classification schemes are given. The modifications of energy-momentum tensors are considered that arise due to different sources of torsion. Furthermore, we analyze the contributions of torsion to shear, vorticity, expansion and acceleration. Finally the generalized Raychaudhuri equation is discussed.

1   Introduction

The issue to enlarge the classical scheme of General Relativity is felt strongly today since several questions strictly depend on the fact if the spacetime connection is symmetric or not. General Relativity is essentially a classical theory which does not take into account quantum effects. These ones must be considered in any theory which deals with gravity

on a fundamental level. Passing from $\mathbf{V}_4$ to $\mathbf{U}_4$ manifolds, is the first straightforward generalization which tries to include the spin fields of matter into the same geometrical scheme of General Relativity. The Einstein–Cartan–Sciama–Kibble (ECSK) theory is one of the most serious attempts in this direction [1]. However, this mere inclusion of spin matter fields does not exhaust the role of torsion which seems to give important contributions in any fundamental theory.

For example, a torsion field appears in (super)string theory if we consider the fundamental string modes; we need, at least, a scalar mode and two tensor modes: a symmetric and an antisymmetric one. The latter one, in the low energy limit for the effective string action, gives the effects of a torsion field [2].

Furthermore, several attempts of unification between gravity and electromagnetism have to take into account torsion in four and in higher–dimensional theories such as Kaluza-Klein ones [3].

Any theory of gravity considering twistors needs the inclusion of torsion [4] while supergravity is the natural arena in which torsion, curvature and matter fields are treated in an analogous way [5].

Besides, several people agree with the line of thinking that torsion could have played some specific role in the dynamics of the early universe and, by the way, it could have yielded macroscopically observable effects today. In fact, the presence of torsion naturally gives repulsive contributions to the energy-momentum tensor so that cosmological models become singularity-free [6]. This feature, essentially, depends on spin alignments of primordial particles which can be considered as the source of torsion [7]. If the universe undergoes one or several phase transitions, torsion could give rise to topological defects (*e.g.* torsion walls [8]) which today can act as intrinsic angular momenta for cosmic structures as galaxies. Furthermore, the presence of torsion in an effective energy-momentum tensor alters the spectrum of cosmological perturbations giving characteristic lengths for large scale structures [31].

All these arguments, and several more, do not allow to neglect torsion in any comprehensive theory of gravity which takes into account the non gravitational counterpart of the fundamental interactions.

However, in most articles, a clear distinction is not made among the different kinds of torsion. Usually torsion is related with the spin density of matter, but very often there are examples where it cannot be derived from it and assumes meanings quite different from the models

with spinning fluids and particles. It can be shown that there are many independent torsion tensors with different properties.

In order to clarify these points, we have found two classification schemes of torsion tensors based on the geometrical properties of vectors and bivectors that can be used to decompose them. The first classification found is based on the possibility to construct the torsion tensors as the tensor product of a simple covariant bivector and a contravariant vector. Such objects are well understood in General Relativity and they can be easily classified [12]. Then classifying all the possible combinations, we find 24 independent tensors. We call these tensors *elementary torsions*.

The second classification follows from the decomposition at one point of a $\mathbf{U}_4$ space-time of the torsion tensors into three irreducible tensors with respect to the Lorentz group. Again we could use vectors and bivectors to identify their geometrical properties. It follows that the elements of the second classification are generally expressed as a combination of "elementary torsion tensors", while the "elementary torsion tensors" are generally non-irreducible.

One of the main results of our classifications is that in many theories, such as the Einstein-Cartan-Sciama-Kibble theory, torsion is related to its sources by an algebraic equation; it follows that these two classifications clarify the nature of the sources too. This feature leads to recognize which tensors can be generated by the spin and which not and which do not even have a physical origin.

To our knowledge, two other classifications have been already proposed. The first one was given in [9] and it is based on the properties of the Riemann and Ricci tensors as defined in a $\mathbf{U}_4$ spacetime compared with the Weyl and Ricci tensors as defined in a $\mathbf{V}_4$ spacetime. The second, given in [10], deals with the algebraic classification of spacetimes with torsion following from the application of the BRST operator.

The classifications of the torsion tensors show of how the different sources of torsion can influence the physical phenomena. It is well known[1] that the ECSK theory can be recast in a nonminimal coupling version of ordinary General Relativity, where the energy-momentum tensor is modified by the torsion sources. This contributions are calculated and discussed for all types found in the classification.

From a phenomenological point of view, the torsion theories may be relevant in cosmology. This because the kinematical quantities, shear,

vorticity, acceleration, expansion and their evolution equations are modified by the presence of torsion.

The paper is organized as follows. In Sec.2, we give general notations and definitions following [1] and references therein.

In Sec.3, we give the first classification through the elementary torsion tensors and in 4 we specify the elements of the second classification. In Sec.5, without any pretence of completeness, we review some relevant torsion models that have been used in literature. As a result it is shown that just three of them can be related to the usual spin sources generally treated in literature [1]; while the other kinds have sources of a physically different origin or even, to our knowledge, their possible sources have not a clear physical interpretation.

In Secs.6 and 7, some applications are given. In the former, the contributions to the energy-momentum tensor in the ECSK theory are calculated. In the latter, the contributions of torsion to the different components of the gradient of the 4-velocity U_a are obtained. This discussion is completed finding the most general expression of the Raychaudhuri equation. General discussion and conclusions are given in Sec.8.

2 General Definitions

In this section, we give general definitions of torsion and associated quantities which, below, will be specified in the particular $\mathbf{U}_4$ spacetimes. We shall use, essentially, the notation in [1].

The torsion tensor $S_{ab}{}^c$ it the antisymmetric part of the affine connection coefficients Γ^c_{ab}, that is

$$S_{ab}{}^c = \frac{1}{2}\left(\Gamma^c_{ab} - \Gamma^c_{ba}\right) \equiv \Gamma^c_{[ab]}, \tag{1}$$

where $a, b, c = 0, \ldots 3$.

In General Relativity it is postulated that $S_{ab}{}^c = 0$. It is a general convention to call $\mathbf{U}_4$ a 4-dimensional space-time manifold endowed with metric and torsion. The manifolds with metric and without torsion are labeled as $\mathbf{V}_4$ (see [11]).

Often in the calculations, torsion occurs in linear combinations as in the *contortion tensor*, defined as

$$K_{ab}{}^c = -S_{ab}{}^c - S^c{}_{ab} + S_b{}^c{}_a = -K_a{}^c{}_b, \tag{2}$$

and in the *modified torsion tensor*

$$T_{ab}{}^c = S_{ab}{}^c + 2\delta_{[a}{}^c S_{b]} \tag{3}$$

where $S_a \equiv S_{ab}{}^b$.

According to these definitions, it follows that the affine connection can be written as

$$\Gamma^c_{ab} = \{^c_{ab}\} - K_{ab}{}^c, \tag{4}$$

where $\{^c_{ab}\}$ are the usual Christoffel symbols of the symmetric connection. The presence of torsion in the affine connection implies that the covariant derivatives of a scalar field ϕ do not commute, that is

$$\tilde{\nabla}_{[a}\tilde{\nabla}_{b]}\phi = -S_{ab}{}^c \tilde{\nabla}_c \phi; \tag{5}$$

and for a vector v^a and a covector w_a, one has the following relations

$$(\tilde{\nabla}_a \tilde{\nabla}_b - \tilde{\nabla}_b \tilde{\nabla}_a)v^c = R_{abd}{}^c v^d - 2S_{ab}{}^d \tilde{\nabla}_d v^c, \tag{6}$$

and

$$(\tilde{\nabla}_a \tilde{\nabla}_b - \tilde{\nabla}_b \tilde{\nabla}_a)w_d = R_{abc}{}^d w_d - 2S_{ab}{}^d \tilde{\nabla}_d w_c \tag{7}$$

where the Riemann tensor is defined as

$$R_{abc}{}^d = \partial_a \Gamma^d_{bc} - \partial_b \Gamma^d_{ac} + \Gamma^d_{ae}\Gamma^e_{bc} - \Gamma^d_{be}\Gamma^e_{ac}. \tag{8}$$

The contribution to the Riemann tensor of torsion can be explicitly given by

$$R_{abc}{}^d = R_{abc}{}^d(\{\}) - \nabla_a K_{bc}{}^d + \nabla_b K_{ac}{}^d + K_{ae}{}^d K_{bc}{}^e - K_{be}{}^d K_{ac}{}^e \tag{9}$$

where $R_{abc}{}^d(\{\})$ is the tensor generated by the Christoffel symbols. The symbols $\tilde{\nabla}$ and ∇ have been used to indicate the covariant derivative with and without torsion respectively.

From Eq.(9), the expressions for the Ricci tensor and the curvature scalar are

$$R_{ab} = R_{ab}(\{\}) - 2\nabla_a S_c + \nabla_b K_{ac}{}^b + K_{ae}{}^b K_{bc}{}^e - 2S_e K_{ac}{}^e \tag{10}$$

and

$$R = R(\{\}) - 4\nabla_a S^a + K_{ceb}K^{bce} - 4S_a S^a. \tag{11}$$

2.1 Bivectors and Tetrads Fields

In the following, we will use the tetrad fields. They are defined at each point of the manifold as a base of orthonormal vectors e_A^a, where $A, B, C \cdots = 0, 1, 2, 3$ label the tetrad vectors and $a, b, c, \cdots = 0, 1, 2, 3$ are the component indices; e_0^a is a time-like vector and e_I^a is space-like.

Correspondingly, a cotetrad e_a^A is defined such that

$$e_A^a e_b^A = \delta_b^a \,, \tag{12}$$

$$e_A^a e_a^B = \delta_A^B \,. \tag{13}$$

The tetrad metric is

$$\eta_{AB} = \eta^{AB} = diag(-1, 1, 1, 1), \tag{14}$$

then the space-time metric can be reconstructed in the following way

$$g_{ab} = \eta_{AB} e_a^A e_b^B. \tag{15}$$

In the construction of the torsion tensors, it will be useful to consider expression of the simple bivectors. These are given by the skew-symmetric tensor product of two vectors. Generally a bivector B^{ab} is simple, if and only if it satisfies the equation

$$B^{[ab} B^{c]d} = 0. \tag{16}$$

By the tetrad vectors in a N-dimensional manifold, one can construct the $N(N-1)/2$ simple bivectors

$$F_{AB}^{ab} = e_A^{[a} e_B^{b]} \tag{17}$$

and any bivector B^{ab} is expressed as

$$B^{ab} = B^{AB} e_A^a e_B^b \tag{18}$$

with $B^{AB} = -B^{BA}$.

2.2 The decomposition of torsion tensor

An important property of torsion is that it can be decomposed with respect to the Lorentz group into three irreducible tensors, i.e. it can be written as

$$S_{ab}{}^c = {}^T S_{ab}{}^c + {}^A S_{ab}{}^c + {}^V S_{ab}{}^c. \tag{19}$$

Torsion has 24 components, of which ${}^T S_{ab}$ has 16 components, ${}^A S_{ab}$ has 4 and ${}^V S_{ab}$ has the remaining 4 components [13],[14],[15].

One has

$$^V S_{ab}{}^c = \frac{1}{3}(S_a \delta_b^c - S_b \delta_a^c), \tag{20}$$

where $S_a = S_{ab}{}^b$,

$$^A S_{ab}{}^c = g^{cd} S_{[abd]} \tag{21}$$

which is called the axial (or totally anti-symmetric) torsion, and

$$^T S_{ab}{}^c = S_{ab}{}^c - {}^A S_{ab}{}^c - {}^V S_{ab}{}^c \tag{22}$$

which is the traceless non totally anti-symmetric part of torsion.

For the sake of brevity, in the following, we will refer respectively to the tensor (20) as a V-torsion, to the tensor (21) as an A-torsion and to the tensor (22) as a T-torsion.

The dual operation (see [13],[15]) defined as

$$^+ S_{ab}{}^c = \frac{1}{2}\epsilon^{de}{}_{ab} S_{de}{}^c \tag{23}$$

has the relevant property, that it associates an A-torsion tensor to a V-torsion tensor and vice versa. Then it associates a T-torsion to a T-torsion.

2.3 The Einstein-Cartan field equations

The introduction of torsion as an extension of the gravitational field theories has some relevant consequences.

The closest theory to General Relativity is the Einstein-Cartan-Sciama-Kibble (ECSK) theory. It is described by

$$L = \sqrt{-g}\left(\frac{R}{2k} + \mathcal{L}_m\right), \tag{24}$$

which is the lagrangian density of General Relativity depending on the metric tensor g_{ab} and on the connection Γ^c_{ab}, where R is the curvature scalar (11) and $\mathcal{L}_m$ the Lagrangian function of matter fields, which yields

$$\mathbf{t}^{ab} = \frac{\delta \mathcal{L}_m}{\delta g_{ab}}, \tag{25}$$

which is the symmetric stress–energy tensor while

$$\tau_c{}^{ba} = \frac{\delta \mathcal{L}_m}{\delta K_{ab}{}^c}, \tag{26}$$

is the source of torsion. In many instances, it can be identified with a spin density. But, as will be clear from the following sections, there are many cases in which the source of the torsion field (26) is not spin.

From the variation of (24) and introducing the canonical energy-momentum tensor

$$\mathbf{\Sigma^{ab}} = \mathbf{t^{ab}} + {}^*\tilde{\nabla}_c(\tau^{abc} - \tau^{bca} + \tau^{cab}), \tag{27}$$

where we used the abridged notation ${}^*\tilde{\nabla}_c := \tilde{\nabla}_c + 2S_{cd}{}^d$, the following field equations are derived [1]

$$G^{ab} = k\mathbf{\Sigma}^{ab}, \tag{28}$$

and

$$T_{ab}{}^c = k\tau_{ab}{}^c, \tag{29}$$

where $k = 8\pi G$, $c = 1$.

Equation (28) is the generalizes the Einstein equations in a U_4.

Unlike Eq.(28), Eq.(29) is algebraic so that it is always possible to cast Eq.(28) in a pure Einstein one, by substituting the torsion terms with their sources. It results in defining an effective energy–momentum tensor as the source of the Riemannian part of the Einstein tensor [1]. In doing so, one obtains

$$G^{ab}(\{\}) = k\tilde{\mathbf{t}}^{ab}, \tag{30}$$

where $G^{ab}(\{\})$ is the Riemannian part of the Einstein tensor. The effective energy–momentum tensor is

$$\tilde{\mathbf{t}}^{ab} = \mathbf{t}^{ab} + k \left[-4\tau^{ac}{}_{[d}\tau^{bd}{}_{c]} - 2\tau^{acd}\tau^{b}{}_{cd} + \tau^{cda}\tau_{cd}{}^{b} \right.$$
$$\left. + \frac{1}{2}g^{ab}(4\tau_{e}{}^{c}{}_{[d}\tau^{ed}{}_{c]} + \tau^{ecd}\tau_{ecd}) \right]. \tag{31}$$

The tensor $\mathbf{t}^{ab}$ can be of the form

$$\mathbf{t}^{ab} = (p + \rho)u^{a}u^{b} - pg^{ab}, \tag{32}$$

if standard perfect–fluid matter is considered. But when spin fluids are considered, one has to define a different stress–energy tensor in which the spin contributions are taken into account as in [19],[20],[21], [22].

3 The classification of elementary torsion tensors

It can be observed that a tensor with all properties of torsion can be constructed as the tensor product of a bivector F_{ab} with a vector Σ^{c}.

It is well known that any generic bivector in a four dimensional manifold can be always reduced into the sum of two simple bivectors with a particular choice of the coordinates (see e.g. [12]). In analogy with the electromagnetic case, we can call the bivector with the timelike vector, the *electric* term and the one with two spacelike vectors, the *magnetic* term and label them respectively with E_{ab} and B_{ab}. Then we can introduce the concept of elementary torsion tensor given as the tensor product of a *simple* bivector with a vector.

We say that a bivector A_{ab} and a vector V^{c} are orthogonal if $V^{a}A_{ab} = 0$.

We consider only the cases where any four-vector Σ^{c} is either orthogonal to a simple bivector or is one of its components. All the other possible cases are combinations of these two cases.

Then the 24 elementary torsion tensors can be classified according to the space-time properties of their bivectors and the corresponding vectors.

At this point an important remark is necessary. Any generic torsion tensor can be decomposed in terms of these elementary parts. Let us practically construct the elementary torsion tensors by the vectors of a tetrad. In general, we have

$$S^{(el)c}_{ABCab} = e^{A}_{[a}e^{B}_{b]}e^{c}_{C}, \tag{33}$$

and then any torsion tensor can be expressed as

$$S_{ab}{}^c = S_{AB}{}^C e^A_{[a} e^B_{b]} e^c_C \,, \tag{34}$$

where the coefficients have to be

$$S_{AB}{}^C = S_{ab}{}^c e^{[a}_A e^{b]}_B e^C_c \,. \tag{35}$$

The classification of elementary torsion tensors in which Σ^a does not lie on the plane defined by the bivector is then the following:

a) if E_{ab} is a bivector obtained from the antisymmetric tensor product of a timelike covector and a spacelike covector, Σ^a must be any spacelike vector orthogonal to E_{ab}. The pure electric case is represented just by one family of tensors. It will be labeled with the symbol Es;

b) in the pure magnetic case, one has that Σ^a can be either a spacelike vector, a timelike vector or a null vector, leading to three family of tensors. These three families will be labeled respectively as Bs, Bt and Bn;

c) In the null case, it turns out that there are two possibilities for Σ^a, *i.e.* it can be either a null vector or a spacelike vector. The labels will be Nn and Ns respectively.

Regarding the case in which the vector Σ^a lies on the plane described by the bivector, it can be noted that, if $B \equiv C$ in (33), we have V-torsions.

Finally, let us note that the previous discussion changes if a null tetrad, defined by $l^a = e^a_0 - e^a_1$, $n^a = e^a_0 + e^a_1$, $m^a = e^a_2 - ie^a_3$ and $m^{*a} = e^a_2 + ie^a_3$, is considered.

In this case, it follows that the elementary torsions like

$$S_{ab}{}^c = m_{[a} l_{b]} l^c \tag{36}$$

bear all properties of a T-torsion.

4   Irreducible tensors in four dimensions

To classify the torsion tensors, according to their irreducible properties, let us first consider the V-torsion. It follows, from Eq.(20), that the V-torsion is characterized by a covector

$$S_a = S_{ab}{}^b. \tag{37}$$

S_a can be either time-like, space-like or light-like. So we have three different possible types of V-torsion, which can be labeled respectively by the symbols Vt, Vs and $V\ell$.

It can be noted that the V-torsion is expressed as a combination of elementary torsion tensors.

From Eqs.(20) and (12), it follows that

$$^{V}S_{ab}{}^{c} = \frac{2}{3}S_{[a}e^{A}_{b]}e^{c}_{A}$$ (38)

The A-torsion can be expressed by the equation

$$^{A}S_{abc} = \epsilon_{abcd}\sigma^{d}.$$ (39)

Its properties can be characterized by the space-time properties of the vector σ^{d}. As for the V-torsion, we label the A-torsion with At, As or $A\ell$ depending on whether the vector σ^{a} is time-like, space-like or light-like.

The statement given in §2.2 can be proved here by direct calculation. In fact, we have

$$\epsilon^{de}{}_{ab}S_{[d}\delta^{c}_{e]} = \epsilon^{dc}{}_{ab}S_{d},$$ (40)

which is an A-torsion, on the other hand

$$\epsilon^{ef}{}_{ab}\epsilon_{def}{}^{c}S^{d} = S_{[a}\delta^{c}_{b]},$$ (41)

which is a V-torsion.

Finally, the T-torsion tensors can be constructed through a combination of elementary torsion tensors of the forms

$$^{T}S_{ab}{}^{c} = V_{[a}e^{A}_{b]}C_{A}{}^{B}e^{c}_{B},$$ (42)

and

$$^{T}S_{ab}{}^{c} = \epsilon^{ef}{}_{ab}V_{[e}e^{A}_{f]}C_{A}{}^{B}e^{c}_{B},$$ (43)

where $C_{A}{}^{B}$ is an arbitrary matrix. By the null–trace conditions

$$V_{[a}e^{A}_{b]}C_{A}{}^{B}e^{b}_{B} = 0,$$ (44)

and

$$\epsilon^{ef}{}_{ab}V_{[e}e^{A}_{f]}C_{A}{}^{B}e^{b}_{B} = 0,$$ (45)

on (42) and (43), we obtain 7 constraints on the matrix $C_{A}{}^{B}$, by fixing V_a. As a consequence, $C_{A}{}^{B}$ has 9 independent components. In order to

get the 16 components of the T-torsion from the expressions (42) and (43), we have to impose a further condition. If $V^2 \equiv V^a V_a \neq 0$, we can impose that

$$C_A{}^B e_a^A e_B^b V^a V_b = 0 \,. \tag{46}$$

which reduces one of the constraints following from (44) to

$$C_A{}^A = 0 \,, \tag{47}$$

If V is a null vector, the constraint (46) follows from (44) and the equation (47) has to be imposed as a supplementary constraint on the matrix $C_A{}^B$.

From the previous discussion, it follows that the T-torsion tensors can be classified according to the nature of the vector V_a which can be time-like, space-like, or null. We label the T-torsion with Tt, Ts or $T\ell$ depending on whether the 4-vector V^a is time-like, space-like or light-like.

5   Some examples

In a first group of examples, we show how some torsion tensors frequently found in literature, can be classified according to the irreducible tensors classification given above.

1. Scalar fields ϕ produce torsion only in nonminimally coupled theories with a $\xi \phi^2 R$ term in the Lagrangian density or in a R^2 theory in a $\mathbf{U}_4$ (where the Ricci scalar is R coupled to itself). As a result, the torsion is related to the gradient of the field. For example, in homogeneous cosmologies, one obtains a Vt tensor. In a Schwarzchild solution one deals with a Vs tensor. See for example [26],[27],[28],[24].

2. According to [14] and [16], it turns out that the only torsion tensors compatible with a Friedmann-Lemaitre-Robertson-Walker universe are of class Vt and At. A cosmological solution with a torsion tensor of class At is discussed also in [25].

3. Examples of torsion of class $V\ell$ and $A\ell$ are found in [9] to describe null electromagnetic plane waves.

4. As and Vs tensors introduce anisotropies in a spacetime, since the spacelike vector yields a privileged direction.

5. The spin of classical Dirac particles is the source of an *As* torsion for massive particles and of an *An* torsion for a massless neutrino [1]. The *At* torsion is generated by tachyon Dirac particles.

6. An example of T-torsion tensors can be found in simple supergravity where torsion is given in terms of the Rarita–Schwinger spinors (see, for example, [30]). They contribute also to torsion in the Weyssenhoff spin fluids (see below and discussion in [29]).

7. The Lanczos tensor was considered in [9] as a candidate of torsion in a non-ECSK theory. It is a sort of Weyl tensor potential and it bears all the characteristics of a traceless torsion tensor. Then its properties depend on the symmetries of spacetime.

8. The influence of an *At* torsion on cosmological perturbations is discussed in [31].

9. The helicity flip of fermions can be induced by a *Al* torsion [32].

10. The same kind of torsion can induce a geometrical contribution to the Berry phase of Dirac particles [33].

    The next group of examples is related to elementary torsion tensors found in literature.

11. The torsion tensors related to spin, usually found in the literature, are generated by the Weyssenhoff spinning particle and the classical Dirac particle. In the first case, the torsion tensor is a *Bs* tensor, in the second case, one has a *As* tensor. Spin fluids *à la* Weyssenhoff can be found in [1], [19],[21],[22], [18],[34], and have been discussed by many other authors.

12. Cosmological models with a *Bs* torsion have been studied in [34].

6   The role of the energy-momentum tensor

After straightforward calculation, one obtains that the contribution of the antisymmetric and vector parts of torsion to the energy-momentum tensor are respectively proportional to the following expressions

$$^A\mathbf{t}_a^b = 2\sigma^b\sigma_a + \delta_a^b\sigma^c\sigma_c\,, \tag{48}$$

and

$$^V\mathbf{t}_a^b = \frac{8}{3}S^bS_a - \frac{4}{3}\delta_a^bS^cS_c\,. \tag{49}$$

The contribution of the T-torsion, when expressed from (42) is

$$^{T_1}\mathbf{t}^{ab} = -C^{cd}C_{(cd)}V^aV^b - V^cC_{cd}V^{(a}C^{b)d} + \frac{1}{2}V^cV_c(C_f{}^aC^{fb} - C^a{}_fC^{bf})$$

$$-\frac{1}{2}V^cV^dC_c{}^aC_d{}^b + \frac{1}{2}g^{ab}(C^{cd}C_{(cd)}V^fV_f - \frac{1}{2}V^cC_{cd}V^fC_f{}^d)\,, \qquad (50)$$

otherwise when the T-torsion is expressed by (43)

$$^{T_2}\mathbf{t}^{ab} = C^{cd}C_{cd}V^aV^b + V^dV_d(C_f{}^aC^{fb} + C^a{}_fC^{bf}) - V^cV^dC_c{}^aC_d{}^b$$

$$-V^fC_{fd}(V^aC^{bd} + V^bC^{ad}) + \frac{1}{2}(V^cV^dC_{cf}C^f{}_d - V^fV_fC^{cd}C_{cd})\,. \quad (51)$$

In (50) and in (51) we have used the expression $C_a{}^b = C_A{}^B e^A{}_a e_B{}^b$. The presence of contributions of distinct irreducible tensors does not lead to interaction terms except when the two classes of T-torsion are present.

An elementary torsion tensor, $S_{ij}{}^k = F_{ij}\Sigma^k$, contributes to the energy-momentum tensor with a symmetric tensor proportional to

$$^e\mathbf{t}_a{}^b = -2\Sigma^2 F^{bc}F_{ac} + F^2\Sigma^b\Sigma_a - \frac{1}{2}F^2\Sigma^2\delta_a^b\,, \qquad (52)$$

where $\Sigma^2 = \Sigma_a\Sigma^a$ and $F^2 = F^{ab}F_{ab}$.

Expression (31), through Eqs.(3) and (29), is the final result involving also ordinary perfect–fluid matter.

7   Torsion vs. shear, expansion, vorticity and acceleration

It has often been pointed out in the literature how torsion can modify the behaviour of fluids. In [20] it was shown that the presence of a torsion generated by a Weyssenhoff fluid generalizes the Bernoulli theorem, through an extension of the definition of the vorticity. In the same way such a modification of the vorticity has led some authors to argue about the possibility of having cosmological models with torsion which could avert the initial singularity [18]. An extended analysis of this problem

has been made by restating the Raychaudhuri equation in the presence of torsion for a Weyssenhoff fluid [35][38].

In [39] it was considered an inflationary Bianchi I universe in the ECSK theory. In this paper it was shown how torsion could contribute to an isotropic expansion universe even in anisotropic universes, if the energy density of spin was sufficiently large to counterbalance the anisotropic terms. As a result it followed that this model supplies a rapid isotropization mechanism of the universe during the inflationary stage.

The previous considerations suggest considering how the kinematical quantities are modified by each of the irreducible components of torsion.

In [40] a gauge invariant and covariant formalism for cosmological perturbations was formulated. In this derivation an important rôle is attributed to the Raychaudhuri equation. Such formulation has been extended recently in ([41]) for the ECSK theory. It follows that important tests for torsion in the primordial universe through its effects on the spectrum of perturbations. A complete study of perturbations for all the irreducible torsion tensors can be useful to extend this program.

7.1 Kinematics

One of the consequences of introducing torsion in a space-time is that the definition of some quantities can be modified. This is the case of the kinematical quantities, defined from the following decomposition of the covariant derivative of the four velocity U_a [23]

$$\tilde{\nabla}_a U_b = \tilde{\sigma}_{ab} + \frac{1}{3} h_{ab} \tilde{\theta} + \tilde{\omega}_{ab} - U_a \tilde{a}_b \tag{53}$$

where $h_{ab} = g_{ab} + U_a U_b$ and

$$\tilde{\theta} = \tilde{\nabla}_a U^a = \theta - 2 S^c U_c, \tag{54}$$

$$\tilde{\sigma}_{ab} = h_a^c h_b^d \tilde{\nabla}_{(c} U_{d)} = \sigma_{ab} + 2 h_a^c h_b^d K_{(cd)}{}^e U_e, \tag{55}$$

$$\tilde{\omega}_{ab} = h_a^c h_b^d \tilde{\nabla}_{[c} U_{d]} = \omega_{ab} + 2 h_a^c h_b^d K_{[cd]}{}^e U_e, \tag{56}$$

and the acceleration

$$\tilde{a}_c = U^a \tilde{\nabla}_a U_c = a_c + U^a K_{ac}{}^d U_d. \tag{57}$$

The quantities without the tilde are those usually defined in General Relativity.

We will summarize in Tables I and II the contributions to this objects given by the irreducible torsion tensors.

	$^{V}S_{ab}{}^{c}$	$^{A}S_{ab}{}^{c}$
$\tilde{\theta} =$	$\theta - 2S^cU_c$	θ
$\tilde{a}_b =$	$a_b - S_b - S_aU^aU_b$	a_b
$\tilde{\omega}_{ab} =$	ω_{ab}	$\omega_{ab} - \epsilon_{abcd}\sigma^dU^c$
$\tilde{\sigma}_{ab} =$	σ_{ab}	σ_{ab}

Table I: *Contributions of V-torsion and A-torsion to the kinematical quantities.*

	$^{T_1}S_{ab}{}^{c}$	$^{T_2}S_{ab}{}^{c}$
$\tilde{\theta} =$	θ	θ
$\tilde{a}_b =$	$a_b + 2V_{[b}C_{a]c}U^cU^a$	$a_b - 2\epsilon_{efhb}V^eC_{fc}U^hU^a$
$\tilde{\omega}_{ab} =$	$\omega_{ab} + h_a^ch_b^dV_{[c}C_{d]}{}^{e}U_e$	$\omega_{ab} + h_a^ch_b^d\epsilon_{efcd}V^eC^{fg}U_g$
$\tilde{\sigma}_{ab} =$	$\sigma_{ab} - 2h_a^ch_b^d(V_eC_{(cd)} - C_{e(c}V_{d)})U^e$	$\sigma_{ab} - 2h_a^ch_b^d\epsilon_{efg(a}V^eC^f{}_{d)}U^g$

Table II: *Contributions of the two T-torsions to the kinematical quantities.*

7.2   The derivation of the Raychaudhuri equation

Given the four-velocity U_a $(U_aU^a = -1)$, bearing in mind the identity

$$U^b\tilde{\nabla}_c\tilde{\nabla}_bU_a = \tilde{\nabla}_c(U^b\tilde{\nabla}_bU_a) - \tilde{\nabla}_cU^b\tilde{\nabla}_bU_a \qquad (58)$$

and from Eq.(8)

$$U^b\tilde{\nabla}_c\tilde{\nabla}_bU_a = U^b\tilde{\nabla}_b\tilde{\nabla}_cU_a + R_{cba}{}^{d}U_dU^b - 2U^bS_{ab}{}^{c}\tilde{\nabla}_dU_c \qquad (59)$$

we find the equation

$$\frac{1}{3}h_{ca}\tilde{\theta} + \tilde{\sigma}_{ca} + \tilde{\omega}_{ca} - U_c\tilde{a}_a = \tilde{\nabla}_c\tilde{a}_a$$

$$-\left(\frac{1}{9}h_{ca}\tilde{\theta} + \frac{2}{3}\tilde{\theta}\tilde{\sigma}_{ca} + \frac{2}{3}\tilde{\theta}\tilde{\omega}_{ca} + 2\tilde{\sigma}_c^b\tilde{\omega}_{ba}\right.$$

$$\left.\tilde{\sigma}_c^b\tilde{\sigma}_{ba} + \tilde{\omega}_c^b\tilde{\omega}_{ba} - \frac{1}{3}U_c\tilde{\theta}\tilde{a}_a - U_c\tilde{a}^b\tilde{\sigma}_{ba} - U_c\tilde{a}^b\tilde{\omega}_{ba}\right)$$

$$-R_{cba}{}^d U_d U^d - 2U^b S_{ab}{}^c\left(\frac{1}{3}h_{dc}\tilde{\theta} + \tilde{\sigma}_{dc} + \tilde{\omega}_{dc} - U_d\tilde{a}_c\right). \qquad (60)$$

Contracting the indices in Eq.(60), one obtains immediately the most general expression for the Raychaudhuri equation, i.e.

$$\dot{\tilde{\theta}} = \tilde{\nabla}_c\tilde{a}^c - \frac{1}{3}\theta^2 - \tilde{\sigma}^{ab}\tilde{\sigma}_{ab} + \tilde{\omega}^{ab}\tilde{\omega}_{ab} - R_{ab}U^aU^b - 2U^bS_{ab}{}^d\left(\frac{1}{3}h_d^a\tilde{\theta} + \tilde{\sigma}_d^a + \tilde{\omega}_d^a - U_d\tilde{a}^a\right).$$
$$(61)$$

This is the Raychaudhuri equation in its most general form in presence of torsion. Simpler versions of this equation have already been discussed in [35], [36], [38] and in [37].

8   Discussion and Conclusions

As we discussed in Sec.5, there are several ways to build a torsion tensor. In this paper, we deal with the problem of finding a geometrical classification of torsion tensors. A decomposition of torsion into irreducible tensors has been already given (see e.g. [13],[14], and, for a systematic account, [15]). Essentially, one has three classes of tensors: traceless, vector and totally antisymmetric ones. However, we propose to add a classification scheme to this decomposition. Our proposal is based on the space-time properties of 4-vectors and bivectors which can be used to construct these torsion tensors. According to this classification, we have shown that it is possible to construct two tensors of the same irreducible class, with distinct properties, due to the fact that one can use space-like, time-like, or null 4-vectors.

As a byproduct, we found also a second decomposition and classification scheme based on elementary torsions. These elementary tensors are given by the tensor product of simple bivectors and vectors. As a

consequence, the classification of these tensors is based on the space-time properties of the simple bivectors (which we distinguished in *electric* and *magnetic* bivectors), and on those of the vectors.

These two classifications, in our opinion, can help to distinguish the physical situations associated to different torsion tensors.

As an application, we have provided a general scheme for the modification induced by torsion tensors on kinematical quantities (such as shear, vorticity, expansion and acceleration). Moreover, we discussed a general form of the Raychaudhuri equation which can be physically relevant for the study of many issues such as the cosmological perturbations (e.g. see [41]).

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