Linearized Torsion Waves in a Tensor-Tensor Theory of Gravity

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We investigate a linearized tensor-tensor theory of gravity with torsion and a perturbed torsion wave solution is discovered in background Minkowski spacetime with zero torsion. Furthermore, gauge transformations of any perturbed tensor field are derived in general background non-Riemannian geometries. By calculating autoparallel deviations, both longitudinal and transverse polarizations of the torsion wave are discovered.

1 Introduction

It is well known that gravitational energy cannot be localized in general relativity (GR) according to the equivalence principle. Moreover, the conventional expressions for gravitational energy-momentum density are non-covariant and vanish locally in Riemann normal coordinates [10] [9]. By exploring the mathematical analogy of gravitational and electromagnetic fields, the Bel tensor $\bar{\mathrm{B}}$ is defined by

$$\bar{\mathrm{B}} = \star\,\bar{\tau}_{bcd} \otimes e^b \otimes e^c \otimes e^d \tag{1}$$

with associated 3-forms

$$\bar{\tau}_{bcd} = \frac{1}{2}\left((\mathrm{i}_{X_b}\bar{\mathrm{R}}_{cq} \wedge \star\,\bar{\mathrm{R}}_d{}^q - \bar{\mathrm{R}}_{cq} \wedge \mathrm{i}_{X_b} \star\,\bar{\mathrm{R}}_d{}^q) + (\mathrm{i}_{X_b}\bar{\mathrm{R}}_{dq} \wedge \star\,\bar{\mathrm{R}}_c{}^q - \bar{\mathrm{R}}_{dq} \wedge \mathrm{i}_{X_b} \star\,\bar{\mathrm{R}}_c{}^q)\right).$$

$\{e^a\}$ is an orthonormal co-frame with its dual $\{X_a\}$ and $\bar{\mathrm{R}}_{ab}$ are Riemann curvature 2-forms.[1] $\star$ denotes Hodge map and i_X the interior derivative. $\bar{\mathrm{B}}$ has a number of properties in common with the stress-energy tensor of electromagnetic fields [2] [12] [3] [8]. By using differential

[1]Greek indices $a, b, \cdots$ run over 0 to 3 and Latin indices $\alpha, \beta, \cdots$ over 1 to 3.

forms, it can easily be proved that $\bar{\mathrm{B}}$ is totally symmetric, traceless, and $\bar{\nabla}\cdot\bar{\mathrm{B}} = 0$ in Ricci-flat spacetime [15]. $\bar{\nabla}$ is Levi-Civita connection. Since Bel tensor holds these interesting properties, it has long been considered as gravitational energy density. Dereli and Tucker (DT) [6] proposed a tensor-tensor theory of gravity by introducing a symmetric tensor field Φ in non-Riemannian spacetime with metric-compatible connection. In DT theory, generalized Bel 3-forms τ_{abc}[2] are naturally appeared in field equations given by

$$-\frac{1}{2\kappa}\,\mathrm{R}_{ab}\wedge\star e^{ab}{}_{c}=\Phi^{ab}\tau_{cab}+\lambda\tau^{\Phi}_{c}\,, \tag{2}$$

$$\frac{1}{2\kappa}\,T^{c}\wedge\star\, e_{abc}=\lambda(\,\Phi_{bc}\star\mathrm{D}\Phi^{c}{}_{a}-\Phi_{ac}\star\mathrm{D}\Phi^{c}{}_{b}\,) + \frac{1}{2}\{\,\mathrm{D}(\Phi_{ac}\star\mathrm{R}^{c}{}_{b})-\mathrm{D}(\Phi_{bc}\star\mathrm{R}^{c}{}_{a})\}, \tag{3}$$

$$\lambda\mathrm{D}\star\mathrm{D}\Phi_{ab}=\frac{1}{2}\,\mathrm{R}_{ac}\wedge\star\mathrm{R}_{b}{}^{c} \tag{4}$$

where

$$\tau^{\Phi}_{c}=\frac{1}{2}(\mathrm{i}_{X_c}\mathrm{D}\Phi^{ab}\,\star\mathrm{D}\Phi_{ab}+\mathrm{D}\Phi_{ab}\wedge\mathrm{i}_{X_c}\star\mathrm{D}\Phi^{ab}),$$

and $e^{a\ldots b}{}_{c\ldots d}$ denotes $e^a\wedge\ldots\wedge e^b\wedge e_c\wedge\ldots\wedge e_d$. κ and λ are constants, Φ_{ab} components of Φ with respect to $\{e^a\}$, T^a torsion 2-forms and D covariant exterior derivative [4].

In GR, gravitational waves were first predicted from a linearized theory. Their polarizations have been fully investigated [10]. By applying perturbation analysis in DT theory, we also obtain a wave-like solution, i.e. torsion waves and Φ field waves. The polarizations of torsion waves turn out to be quite different from the gravitational waves in GR. In section 2, we derive linearized field equations in background curvature and torsion vanishing. These linearized equations are expressed in terms of coordinate-free language. In section 3, we apply Minkowski coordinates on these equations and then obtain a torsion wave and a Φ wave solution. In section 4, we derive gauge transformations of perturbed field variables in non-Riemannian geometries. Using gauge transformations, some components of torsion waves can be eliminated. In section 5, we calculating the autoparallel deviation in the torsion wave spacetime and discuss polarizations of the torsion wave.

[2] τ_{abc} is defined in terms of full curvature 2-forms R_{ab}. In the torsion-free metric-compatible connection, $\tau_{abc}=\bar{\tau}_{abc}$

2 Linearized Field Equations

In our perturbation scheme, we expand field variables e^a, $\omega^a{}_b$ and Φ_{ab} with respect to a dimensionless small parameter ε. When $\varepsilon = 0$, they return to the following background field

$$\tilde{\mathrm{R}}_{ab} = \mathrm{d}\tilde{\omega}_{ab} + \tilde{\omega}_{ac} \wedge \tilde{\omega}^c{}_b = 0, \quad (5)$$

$$\tilde{T}^a = \mathrm{d}\tilde{e}^a + \tilde{\omega}^a{}_c \wedge \tilde{e}^c = 0 \quad (6)$$

with a solution $\tilde{\Phi}_{ab}$ satisfying

$$\tilde{\mathrm{D}}\, \tilde{\Phi}_{ab} = 0 \quad (7)$$

where

$$\tilde{\Phi}_{ab} = \tilde{\Phi}(\tilde{X}_a, \tilde{X}_b).$$

$\tilde{e}^a$ and $\tilde{\omega}^a{}_b$ denote the background $\tilde{g}$-orthonormal co-frame and connection 1-forms, and $\tilde{X}_a$ is a dual of $\tilde{e}^a$. $\tilde{\mathrm{D}}$ and $\tilde{\star}$ are covariant exterior derivative and the Hodge map associated to $\tilde{\omega}^a{}_b$ and $\tilde{e}^a$, respectively. This background field which represent a flat and torsion-free spacetime with covariant constant $\tilde{\Phi}$ is an exact solution of field equations. We substitute field variables

$$e^a = \tilde{e}^a + \varepsilon\, \dot{e}^a + \ldots,$$
$$\omega^a{}_b = \tilde{\omega}^a{}_b + \varepsilon\, \dot{\omega}^a{}_b + \ldots,$$
$$\Phi_{ab} = \tilde{\Phi}_{ab} + \varepsilon\, \dot{\Phi}_{ab} + \ldots$$

into field equations (2)-(4) to obtain linearized field equations

$$\tilde{\mathrm{D}}\, \dot{\omega}^{ab} \wedge \tilde{\star}\, \tilde{e}_{abc} = -\tilde{\mathrm{D}}\, \dot{K}^{ab} \wedge \tilde{\star}\, \tilde{e}_{abc}, \quad (8)$$

$$\frac{1}{2\kappa}\, \dot{T}^c \wedge \tilde{\star}\, \tilde{e}_{abc} = \lambda(\, \tilde{\Phi}_{bc}\, \tilde{\star}\, [\dot{\mathrm{D}\Phi^c}{}_a] - \tilde{\Phi}_{ac}\, \tilde{\star}\, [\dot{\mathrm{D}\Phi^c}{}_b]\,),$$
$$+ \frac{1}{2}\, (\, \tilde{\Phi}_{ac}\, \tilde{\mathrm{D}}\, \tilde{\star}\, \tilde{\mathrm{D}}\, \dot{\omega}^c{}_b - \tilde{\Phi}_{bc}\, \tilde{\mathrm{D}}\, \tilde{\star}\, \tilde{\mathrm{D}}\, \dot{\omega}^c{}_a) \quad (9)$$

$$\tilde{\mathrm{D}}\, \tilde{\star}\, \tilde{\mathrm{D}}\, \dot{\Phi}_{ab} = \tilde{\Phi}_{cb}\, \tilde{\mathrm{D}}\, \tilde{\star}\, \dot{\omega}^c{}_a + \tilde{\Phi}_{ca}\, \tilde{\mathrm{D}}\, \tilde{\star}\, \dot{\omega}^c{}_b \quad (10)$$

where

$$[\dot{\mathrm{D}\Phi}_{ab}] = \tilde{\mathrm{D}}\dot{\Phi}_{ab} - \dot{\omega}^c{}_a\, \tilde{\Phi}_{cb} - \dot{\omega}^c{}_b\, \tilde{\Phi}_{ac}. \quad (11)$$

$\dot{\omega}^{ab}$ are perturbed torsion-free connection 1-forms and $\dot{K}^{ab}$ perturbed contorison 1-forms. (8) is considered as linearized Einstein equations with source terms $\tilde{\mathrm{D}}\,\dot{K}^{ab}\wedge\tilde{\star}\,\tilde{e}_{abc}$. It is interesting to notice that $\tilde{\mathrm{D}}\,\tilde{\star}\,\tilde{\mathrm{D}}\,\dot{\Phi}_{ab}$ are associated with the covariant d'Alembertian operator $\tilde{\nabla}\cdot\tilde{\nabla}$ defined by

$$(\tilde{\nabla}\cdot\tilde{\nabla}\,\mathrm{T})(X_1,\ldots,X_r,\,e^1,\ldots,e^s)\equiv\tilde{\nabla}_{X_a}(\tilde{\nabla}\,\mathrm{T})\,(X^a,X_1,\ldots,X_r,\,e^1,\ldots,e^s)$$

where $\mathrm{T}\in\Gamma T^s_r M$ is a tensor field. Using the definition of $\nabla\,\mathrm{T}$

$$(\nabla\,\mathrm{T})\,(X,\,X_1,\ldots,X_r,\,e^1,\ldots,e^s)=(\nabla_X\mathrm{T})\,(X_1,\ldots,X_r,\,e^1,\ldots,e^s)$$

and

$$\mathrm{D}S_{a\ldots b}{}^{c\ldots d}=\{(\nabla_{X_j}S)\,(X_a,\ldots,X_b,e^c,\ldots,e^d)\}\,e^j \tag{12}$$

for arbitrary 0-forms $S_{a\ldots b}{}^{c\ldots d}$, we obtain

$$\tilde{\mathrm{D}}\,\dot{\Phi}_{ab}=\{(\tilde{\nabla}\Phi_P)\,(\tilde{X}_j,\,\tilde{X}_a,\tilde{X}_b)\}\,e^j \tag{13}$$

where

$$\Phi_P=\dot{\Phi}_{ab}\,\tilde{e}^a\otimes\,\tilde{e}^b \tag{14}$$

and $\tilde{\nabla}$ denotes the background connection. By performing $\tilde{\mathrm{D}}\,\tilde{\star}$ on (13) gives

$$\begin{aligned}\tilde{\mathrm{D}}\,\tilde{\star}\,\tilde{\mathrm{D}}\,\dot{\Phi}_{ab}=&\tilde{\mathrm{D}}\left((\tilde{\nabla}\Phi_P)\,(\tilde{X}_j,\,\tilde{X}_a,\tilde{X}_b)\right)\wedge\tilde{\star}\,\tilde{e}^j\\ =&\tilde{\nabla}_{\tilde{X}_c}(\tilde{\nabla}\Phi_P)\,(\tilde{X}^c,\,\tilde{X}_a,\tilde{X}_b)\,\tilde{\star}\,1\\ =&(\tilde{\nabla}\cdot\tilde{\nabla}\Phi_P)\,(\tilde{X}_a,\tilde{X}_b).\end{aligned} \tag{15}$$

So (10) is a non-vacuum covariant wave equation. In the following section, we find a torsion wave solution using the ansatz $\dot{\omega}_{ab}=0$.

3 A Torsion-Wave Solution

In previous section, we chose the background field $\tilde{\mathrm{R}}_{ab}=\tilde{T}^a=0$ with the background tensor field $\tilde{\Phi}$ satisfying $\tilde{\mathrm{D}}\tilde{\Phi}_{ab}=0$. Because $\tilde{\Phi}_{ab}$ depends on $\{\tilde{e}^a\}$, it simplifies our calculations to choose the background orthonormal co-frame as $\tilde{e}^a=\mathrm{d}x^a$, i.e. Minkowski metric in Minkowski coordinates $\{x^a=t,x^\alpha\}$. So $\tilde{\omega}_{ab}=0$ and $\tilde{\mathrm{D}}$ becomes the exterior derivative d. According to the equation $\mathrm{d}\tilde{\Phi}_{ab}=0$, we obtain that components $\tilde{\Phi}_{ab}$

should be constant in this background orthonormal co-frame $\{dx^a\}$. We then have the following background fields

$$\begin{aligned}\tilde{\mathrm{R}}_{ab}&=0,\\ \tilde{T}^a&=0,\\ \tilde{e}^a&=dx^a\end{aligned} \tag{16}$$

and choose

$$\tilde{\Phi}_{ab}=\mathrm{diag}(\Phi_0,\,\Phi_1,\,\Phi_1,\,\Phi_1) \tag{17}$$

where Φ_0 and Φ_1 are constants. Furthermore, Using the ansatz

$$\dot{T}^a = d\dot{e}^a, \tag{18}$$

i.e. $\dot{\omega}^a{}_b = 0$, (8)-(10) are decoupled and then reduce to

$$\frac{1}{2\kappa}\, d\dot{e}^c \wedge \tilde{\star}\, \tilde{e}_{abc}=\lambda(\, \tilde{\Phi}_{bc}\, \tilde{\star}\, d\dot{\Phi}^c{}_a - \,\tilde{\Phi}_{ac}\, \tilde{\star}\, d\dot{\Phi}^c{}_b), \tag{19}$$

$$d\, \tilde{\star}\, d\dot{\Phi}_{ab}=0. \tag{20}$$

We can first solve (20) and then substitute the solution $\dot{\Phi}_{ab}$ into (19) to obtain a solution of $\dot{e}^a$ and $\dot{T}^a$. By solving the vacuum symmetric tensor wave equation (20), a particular solution, monochromatic plane-wave solution, are obtained

$$\dot{\Phi}_{ab} = \Re[A_{ab} \exp(i\, k_p x^p)] \tag{21}$$

with the amplitude $\mathrm{A} = A_{ab} dx^a \otimes dx^b$ and the wave vector $\mathrm{k} = k^a \partial_a$ satisfying

$$A_{ab}=A_{ba}, \qquad \text{(A is symmetric)} \tag{22}$$

$$k_p k^p=0, \qquad \text{(k is a null vector)} \tag{23}$$

where $\Re$ denotes the real part of the expression in brackets and A_{ab}, k_p are constants. $\{\partial_a\}$ is the dual of $\{dx^a\}$.

In order to solve (19), we write down these equations explicitly

$$\begin{aligned}-(\mathrm{i}_{\partial_0}\mathrm{i}_{\partial_1} d\dot{e}^c)\, \tilde{\star}\, dx_c + (\mathrm{i}_{\partial_0}\mathrm{i}_{\partial_c} d\dot{e}^c)\, \tilde{\star}\, dx_1 - (\mathrm{i}_{\partial_1}\mathrm{i}_{\partial_c} d\dot{e}^c)\, \tilde{\star}\, dx_0&=\alpha\, \tilde{\star}\, d\dot{\Phi}_{01},\\ -(\mathrm{i}_{\partial_0}\mathrm{i}_{\partial_2} d\dot{e}^c)\, \tilde{\star}\, dx_c + (\mathrm{i}_{\partial_0}\mathrm{i}_{\partial_c} d\dot{e}^c)\, \tilde{\star}\, dx_2 - (\mathrm{i}_{\partial_2}\mathrm{i}_{\partial_c} d\dot{e}^c)\, \tilde{\star}\, dx_0&=\alpha\, \tilde{\star}\, d\dot{\Phi}_{02},\\ -(\mathrm{i}_{\partial_0}\mathrm{i}_{\partial_3} d\dot{e}^c)\, \tilde{\star}\, dx_c + (\mathrm{i}_{\partial_0}\mathrm{i}_{\partial_c} d\dot{e}^c)\, \tilde{\star}\, dx_3 - (\mathrm{i}_{\partial_3}\mathrm{i}_{\partial_c} d\dot{e}^c)\, \tilde{\star}\, dx_0&=\alpha\, \tilde{\star}\, d\dot{\Phi}_{03},\end{aligned} \tag{24}$$

$$
\begin{aligned}
&-(\mathrm{i}_{\partial_1}\mathrm{i}_{\partial_2}\mathrm{d}\dot{e}^c)\,\tilde{\star}\,\mathrm{d}x_c + (\mathrm{i}_{\partial_1}\mathrm{i}_{\partial_c}\mathrm{d}\dot{e}^c)\,\tilde{\star}\,\mathrm{d}x_2 - (\mathrm{i}_{\partial_2}\mathrm{i}_{\partial_c}\mathrm{d}\dot{e}^c)\,\tilde{\star}\,\mathrm{d}x_1 = 0,\\
&-(\mathrm{i}_{\partial_1}\mathrm{i}_{\partial_3}\mathrm{d}\dot{e}^c)\,\tilde{\star}\,\mathrm{d}x_c + (\mathrm{i}_{\partial_1}\mathrm{i}_{\partial_c}\mathrm{d}\dot{e}^c)\,\tilde{\star}\,\mathrm{d}x_3 - (\mathrm{i}_{\partial_3}\mathrm{i}_{\partial_c}\mathrm{d}\dot{e}^c)\,\tilde{\star}\,\mathrm{d}x_1 = 0,\\
&-(\mathrm{i}_{\partial_2}\mathrm{i}_{\partial_3}\mathrm{d}\dot{e}^c)\,\tilde{\star}\,\mathrm{d}x_c + (\mathrm{i}_{\partial_2}\mathrm{i}_{\partial_c}\mathrm{d}\dot{e}^c)\,\tilde{\star}\,\mathrm{d}x_3 - (\mathrm{i}_{\partial_3}\mathrm{i}_{\partial_c}\mathrm{d}\dot{e}^c)\,\tilde{\star}\,\mathrm{d}x_2 = 0
\end{aligned}
\tag{25}
$$

where

$$\alpha = 2\,\kappa\lambda(\Phi_0 + \Phi_1),$$

and $\mathrm{d}x_a = \eta_{ab}\,\mathrm{d}x^b$. The equations (25) can be considered as constraint equations for $\mathrm{d}\dot{e}^a$. In general,

$$
\begin{aligned}
\mathrm{d}\dot{e}^0 &= \mathcal{A}_{ab}\,\mathrm{d}x^a \wedge \mathrm{d}x^b,\\
\mathrm{d}\dot{e}^1 &= \mathcal{B}_{ab}\,\mathrm{d}x^a \wedge \mathrm{d}x^b,\\
\mathrm{d}\dot{e}^2 &= \mathcal{C}_{ab}\,\mathrm{d}x^a \wedge \mathrm{d}x^b,\\
\mathrm{d}\dot{e}^3 &= \mathcal{D}_{ab}\,\mathrm{d}x^a \wedge \mathrm{d}x^b
\end{aligned}
\tag{26}
$$

where $\mathcal{A}_{ab}$, $\mathcal{B}_{ab}$, $\mathcal{C}_{ab}$ and $\mathcal{D}_{ab}$ are functions of x^a. Substituting (26) into (25) gives the following conditions

$$
\begin{gathered}
\mathcal{A}_{12} = \mathcal{A}_{13} = \mathcal{A}_{23} = \mathcal{B}_{23} = \mathcal{C}_{13} = \mathcal{D}_{12} = 0,\\
\mathcal{A}_{01} = \mathcal{C}_{12} = \mathcal{D}_{13},\;\; \mathcal{A}_{02} = -\mathcal{B}_{12} = \mathcal{D}_{23},\;\; \mathcal{A}_{03} = -\mathcal{B}_{13} = -\mathcal{C}_{23}
\end{gathered}
\tag{27}
$$

and the 24 unknown functions reduce to 12. For solving (24), we consider the plane-wave of Φ propagating along the x^1 direction and then (21) becomes

$$\dot{\Phi}_{ab} = \Re[A_{ab}\exp\{-i\,k(x^0 - x^1)\}]. \tag{28}$$

Substituting (28) into (24) gives a particular solution

$$
\begin{aligned}
\dot{e}^0 &= -\frac{1}{2}\,\alpha\,\dot{\Phi}_{01}\,\mathrm{d}x^0,\\
\dot{e}^1 &= \Big(\mathcal{P}(x^0) - \alpha\,\dot{\Phi}_{01}\Big)\,\mathrm{d}x^0 + \frac{1}{2}\,\alpha\,\dot{\Phi}_{01}\,\mathrm{d}x^1,\\
\dot{e}^2 &= \mathcal{C}(x^0)\,\mathrm{d}x^0 + \frac{1}{2}\,\alpha\,\dot{\Phi}_{01}\,\mathrm{d}x^2,\\
\dot{e}^3 &= \mathcal{D}(x^0)\,\mathrm{d}x^0 + \frac{1}{2}\,\alpha\,\dot{\Phi}_{01}\,\mathrm{d}x^3
\end{aligned}
\tag{29}
$$

with an extra condition on $\dot{\Phi}_{ab}$

$$A_{02} = A_{03} = 0 \tag{30}$$

where $\mathcal{P}(x^0)$, $\mathcal{C}(x^0)$, and $\mathcal{D}(x^0)$ are still unknown functions. We can also obtain the perturbed torsion wave solution

$$
\begin{aligned}
\dot{T}^0 &= \mathrm{d}\dot{e}^0 = \frac{1}{2}\,\alpha\,\Re[A_{01}\,f_1]\,\mathrm{d}x^0 \wedge \mathrm{d}x^1, \\
\dot{T}^1 &= \mathrm{d}\dot{e}^1 = \frac{1}{2}\,\alpha\,\Re[A_{01}\,f_1]\,\mathrm{d}x^0 \wedge \mathrm{d}x^1, \\
\dot{T}^2 &= \mathrm{d}\dot{e}^2 = \frac{1}{2}\,\alpha\,\Re[A_{01}\,f_1]\,(-\mathrm{d}x^0 \wedge \mathrm{d}x^2 \,+\, \mathrm{d}x^1 \wedge \mathrm{d}x^2), \\
\dot{T}^3 &= \mathrm{d}\dot{e}^3 = \frac{1}{2}\,\alpha\,\Re[A_{01}\,f_1]\,(-\mathrm{d}x^0 \wedge \mathrm{d}x^3 \,+\, \mathrm{d}x^1 \wedge \mathrm{d}x^3)
\end{aligned}
\tag{31}
$$

where $f_1 = i\,k\exp\{-i\,k(x^0 - x^1)\}$

From (19), we may expect that perturbed orthonormal co-frames $\dot{e}^a$ allow gauge transformations

$$\dot{e}^a \to \dot{e}^a + \mathrm{d}\,V^a \tag{32}$$

where V^a are 0-forms. These unknown functions $\mathcal{P}(x^0)$, $\mathcal{C}(x^0)$, and $\mathcal{D}(x^0)$ can be transformed away by performing the gauge transformations (32). In addition to gauge transformations of $\dot{e}^a$, we have to know the gauge transformations of $\dot{\Phi}_{ab}$. In the following section, we explore the gauge transformations of perturbed field variables $\dot{e^a}$, $\dot{\omega}_{ab}$, $\dot{T}^a$, $\dot{R}^a{}_b$, $\dot{\Phi}_{ab}$ in general background fields.

4 Gauge Transformations

The formulation of perturbations of spacetimes is developed in [14] which is based on the work in [7]. We adopt their perturbation formulation to discover the gauge transformations of $\dot{e^a}$, $\dot{\omega}_{ab}$, $\dot{T}^a$, $\dot{R}^a{}_b$, $\dot{\Phi}_{ab}$ in non-Riemannian spacetime. In [14] , a gauge transformation of any perturbed tensor field $\dot{\mathrm{T}}$ can be defined in terms of Lie derivative $\mathcal{L}_X$

$$\dot{\mathrm{T}}' = \dot{\mathrm{T}} + \mathcal{L}_X \tilde{\mathrm{T}}, \tag{33}$$

where X is any vector field on spacetime manifold M and $\tilde{\mathrm{T}}$ denotes the background tensor field. If $\mathcal{L}_X\tilde{\mathrm{T}}$ vanishes for all $X \in M$, we say that $\dot{\mathrm{T}}$ is gauge invariant. We first calculate the gauge transformations of $\dot{e}^a$. By expanding metric g with respect to ε gives

$$g = \tilde{g} \,+ \varepsilon\,\dot{g} \,+\, O(\varepsilon^2) \tag{34}$$

with

$$\tilde{g}{=}\tilde{e}^a \otimes \tilde{e}_a,$$
$$\dot{g}{=}\dot{e}^a \otimes \tilde{e}_a + \tilde{e}_a \otimes \dot{e}^a$$

and using

$$\mathcal{L}_V \tilde{e}^a {=} \tilde{\mathrm{D}} V^a + \mathrm{i}_V \tilde{T}^a + \tilde{\nabla}_V \tilde{e}^a \tag{35}$$

where $\tilde{\nabla}_V$ denotes the background connection, we obtain [3]

$$\mathcal{L}_V \tilde{g} {=} (\tilde{\mathrm{D}} V^a + \mathrm{i}_V \tilde{T}^a) \otimes \tilde{e}_a + \tilde{e}_a \otimes (\tilde{\mathrm{D}} V^a + \mathrm{i}_V \tilde{T}^a) \tag{36}$$

where V is a vector field on M. So gauge transformations of $\dot{e}^a$ are

$$\dot{e}^{a\prime} = \dot{e}^a + \tilde{\mathrm{D}} V^a + \mathrm{i}_V \tilde{T}^a. \tag{37}$$

Expand $(3,0)$ torsion tensor T with respect to ε

$$T = 2\,\tilde{T}_a \otimes \tilde{e}^a \; + \; 2\,\varepsilon(\dot{T}_a \otimes \tilde{e}^a + \tilde{T}_a \otimes \dot{e}^a) + O(\varepsilon^2) \tag{38}$$

and then calculate $\mathcal{L}_V \tilde{T}$ to obtain

$$\mathcal{L}_V \tilde{T} {=} 2\,\{(\tilde{\mathrm{D}} \mathrm{i}_V \tilde{T}^a + \mathrm{i}_V \tilde{\mathrm{R}}^a{}_c \wedge \tilde{e}^c + V^c \tilde{\mathrm{R}}^a{}_c) \otimes \tilde{e}_a + \tilde{T}_a \otimes (\tilde{\mathrm{D}} V^a + \mathrm{i}_V \tilde{T}^a)\}.$$

So gauge transformations of $\dot{T}^a$ are

$$\dot{T}^{a\prime} = \dot{T}^a + \tilde{\mathrm{D}} \mathrm{i}_V \tilde{T}^a + \mathrm{i}_V \tilde{\mathrm{R}}^a{}_c \wedge \tilde{e}^c + V^c \tilde{\mathrm{R}}^a{}_c. \tag{39}$$

It is easily to show that perturbed connection 1-forms are gauge invariant, i.e. $\dot{\omega}'_{ab} = \dot{\omega}_{ab}$. We next expand $(4,0)$ curvature tensor R with respect to ε which gives

$$\mathrm{R} = 2\,\tilde{\mathrm{R}}_{ab} \otimes \tilde{e}^{ab} + 2\,\varepsilon(\,\dot{\mathrm{R}}_{ab} \otimes \tilde{e}^{ab} + \tilde{\mathrm{R}}_{ab} \otimes \dot{e}^a \wedge \tilde{e}^b + \tilde{\mathrm{R}}_{ab} \otimes \tilde{e}^a \wedge \dot{e}^b) + O(\varepsilon^2)$$

and using $\mathcal{L}_V \tilde{\mathrm{R}}$ we obtain

$$\dot{\mathrm{R}}'_{ab} = \dot{\mathrm{R}}_{ab} + \tilde{\mathrm{D}} \mathrm{i}_V \tilde{\mathrm{R}}_{ab}. \tag{40}$$

Finally, gauge transformations of Φ_{ab} are

$$\dot{\Phi}'_{ab} = \dot{\Phi}_{ab} + \mathrm{i}_V \tilde{\mathrm{D}} \tilde{\Phi}_{ab}. \tag{41}$$

[3] $\tilde{\nabla}_V \tilde{g}$ vanishes because of metric-compatible connection $\tilde{\nabla}$

We have shown gauge transformations of $\dot{e}^a$, $\dot{T}^a$, $\dot{\omega}_{ab}$, $\dot{\mathrm{R}}^a{}_b$, and $\dot{\Phi}_{ab}$ in general background fields. By considering the background fields (16) and (17), we can see that $\dot{T}^a$, $\dot{\omega}_{ab}$, $\dot{\mathrm{R}}^a{}_b$, and $\dot{\Phi}_{ab}$ are gauge invariant. $\dot{e}^a$ is still a gauge dependent variable and its gauge transformations are

$$\dot{e}^{a\prime} = \dot{e}^a + \mathrm{d}V^a \tag{42}$$

which are the same as (32).

We perform the following gauge transformations

$$\begin{aligned}
V^0 &= \frac{1}{2}\,\alpha\,\Re[\,\frac{i}{k}\,A_{01}\exp\{-i\,k(x^0-x^1)\}],\\
V^1 &= \frac{1}{2}\,\alpha\,\Re[\,\frac{i}{k}\,A_{01}\exp\{-i\,k(x^0-x^1)\}] + \int_{a_0}^{x^0}(\,\mathcal{P}(x^0)\mathrm{d}x^0\,) + \mathcal{P}(a_0)\,x^0,\\
V^2 &= \int_{b_0}^{x^0}\big(\,\mathcal{C}(x^0)\mathrm{d}x^0\,\big) + \mathcal{C}(b_0)\,x^0,\\
V^3 &= \int_{d_0}^{x^0}\big(\,\mathcal{D}(x^0)\mathrm{d}x^0\,\big) + \mathcal{D}(d_0)\,x^0
\end{aligned} \tag{43}$$

where a_0, b_0, d_0 are constants, so (29) becomes

$$\begin{aligned}
\dot{e}^{0\prime} &= -\frac{1}{2}\,\alpha\,\Re[A_{01}\exp\{-i\,k(x^0-x^1)\}]\mathrm{d}x^1 = -\frac{1}{2}\,\alpha\,\dot{\Phi}_{01}\mathrm{d}x^1,\\
\dot{e}^{1\prime} &= -\frac{1}{2}\,\alpha\,\Re[A_{01}\exp\{-i\,k(x^0-x^1)\}]\mathrm{d}x^0 = -\frac{1}{2}\,\alpha\,\dot{\Phi}_{01}\mathrm{d}x^0,\\
\dot{e}^{2\prime} &= \frac{1}{2}\,\alpha\,\Re[A_{01}\exp\{-i\,k(x^0-x^1)\}]\,\mathrm{d}x^2 = \frac{1}{2}\,\alpha\,\dot{\Phi}_{01}\mathrm{d}x^2,\\
\dot{e}^{3\prime} &= \frac{1}{2}\,\alpha\,\Re[A_{01}\exp\{-i\,k(x^0-x^1)\}]\,\mathrm{d}x^3 = \frac{1}{2}\,\alpha\,\dot{\Phi}_{01}\mathrm{d}x^3
\end{aligned} \tag{44}$$

where $\mathcal{C}(x^0)$, $\mathcal{D}(x^0)$ and $\mathcal{P}(x^0)$ have been transformed away.

We compare (44) to the linearized gravitational wave in GR. In the linearized GR, two different polarizations of the plane-wave solutions in transverse and traceless gauges have been found [10] and by considering the plane-wave propagating along x^1, the solution gives

$$\begin{aligned}
\dot{g}_{GR} &= h^1_{GR}\,\mathrm{d}x^2\otimes\mathrm{d}x^2 - h^1_{GR}\,\mathrm{d}x^3\otimes\mathrm{d}x^3,\\
\dot{g}_{GR} &= h^2_{GR}\,\mathrm{d}x^2\otimes\mathrm{d}x^3 + h^2_{GR}\,\mathrm{d}x^3\otimes\mathrm{d}x^2
\end{aligned} \tag{45}$$

with

$$h_{GR}^1 = \Re[A_+ \exp\{-i\,k(x^0 - x^1)\}]$$
$$h_{GR}^2 = \Re[A_\times \exp\{-i\,k(x^0 - x^1)\}] \quad (46)$$

where A_+ and $A_\times$ are amplitudes. The torsion wave solution is

$$\dot{g}' = h\,\mathrm{d}x^2 \otimes \mathrm{d}x^2 + h\,\mathrm{d}x^3 \otimes \mathrm{d}x^3, \quad (47)$$

where

$$h = \alpha\,\Re[A_{01} \exp\{-i\,k(x^0 - x^1)\}]. \quad (48)$$

If we consider particles following geodesics, one may expect that the polarizations of torsion waves are transverse but not traceless. So we obtain another possible polarization mode of gravitational waves. On the other hands, if we assume particles following autoparallels instead of geodesics, autoparallel deviation should give us different polarizations of torsion waves. In the next section, we will investigate this possibility.

5 Autoparallel Deviation in a Torsion-Wave Spacetime

Consider a family of autoparallels $\gamma : [0,1] \times [0,1] \to M,\ \tau,\, s \mapsto p = \gamma(\tau, s)$ in a spacetime manifold M and tangent vectors $\gamma_* \partial_\tau$ at every point of $\gamma(\tau, s)$ satisfying

$$\nabla_{\gamma_* \partial_\tau} \gamma_* \partial_\tau = 0 \quad (49)$$

with affine parametrization

$$g(\gamma_* \partial_\tau, \gamma_* \partial_\tau) = -1. \quad (50)$$

We may consider $\gamma_* \partial_\tau|_{\gamma_0} \equiv \dot{C}$ as particle's 4-velocity along $\gamma_0 \equiv \gamma(\tau, 0)$ and $\gamma_* \partial_s|_{\gamma_0} \equiv S$ the separation vector from the particle to its neighbor particles. The definition of the $(3,1)$ curvature tensor R gives

$$\nabla_{\dot{C}} \nabla_S \dot{C} - \nabla_S \nabla_{\dot{C}} \dot{C} - \nabla_{[\dot{C}, S]} \dot{C} = \mathrm{R}(\dot{C}, S, \dot{C}, -)|_{\gamma_0}$$
$$\equiv \hat{\mathrm{R}}(\dot{C}, S, \dot{C}, -) \quad (51)$$

where hats indicate evaluation over the image of γ_0. Using (49),

$$\mathcal{L}_{\gamma_* \partial_\tau} \gamma_* \partial_s = \gamma_* [\partial_\tau,\, \partial_s] = 0 \quad (52)$$

and the definition of $(2,1)$ torsion tensor

$$\nabla_{\dot{C}} S - \nabla_S \dot{C} - [\dot{C},\, S] = \hat{\mathrm{T}}(\dot{C}, S, -), \tag{53}$$

(51) becomes

$$\nabla_{\dot{C}} \nabla_{\dot{C}} S - \nabla_{\dot{C}} \left(\hat{\mathrm{T}}(\dot{C}, S, -) \right) = \hat{\mathrm{R}}(\dot{C}, S, \dot{C}, -) \tag{54}$$

which is the equation of autoparallel deviation. S indicates how neighboring particles behave when a torsion wave passes through them.

In torsion-wave spacetime, we construct an orthonormal frame $\{X_a\}$

$$\begin{aligned} X_0 &= \partial_0 + \varepsilon\, f \partial_1 + O(\varepsilon^2), \\ X_1 &= \partial_1 + \varepsilon\, f \partial_0 + O(\varepsilon^2), \\ X_A &= \partial_A - \varepsilon\, f \partial_2 + O(\varepsilon^2) \end{aligned} \tag{55}$$

where $A = 2, 3$ and

$$f = \frac{1}{2}\, h = \frac{1}{2}\, \alpha\, \Re[A_{01} \exp\{-i\, k(x^0 - x^1)\}]. \tag{56}$$

Since

$$\nabla_{\hat{X}_0} \hat{X}_0 = \hat{\omega}^c{}_0(\hat{X}_0) \hat{X}_c = O(\varepsilon^2), \tag{57}$$

it can be identified with the particle's 4-velocity $\dot{C}$ up to first-order in ε, i.e.

$$\hat{X}_0 = \hat{\partial}_0 + \varepsilon\, \hat{f} \hat{\partial}_1 + O(\varepsilon^2) = \dot{C} + O(\varepsilon^2). \tag{58}$$

Calculating (54) with respect to $\{\hat{X}_a\}$ to first-order gives

$$\frac{\mathrm{d}^2 S^{\bar{a}}}{\mathrm{d}t^2} - \varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \left(\hat{\dot{T}}{}^{\bar{a}}{}_{\bar{0}\bar{c}}\, S^{\bar{c}} \right) = 0 \tag{59}$$

where $\hat{\dot{T}}{}^{\bar{a}}{}_{\bar{0}\bar{c}}$ are components of $\dot{T}^a$ with respect to $\{X_a\}$ and t is the coordinate time parameter.[4] $^-$ on indices denotes the components of

[4] $t = \tau + O(\varepsilon^2)$.

tensor fields with respect to $\{X_a\}$. So substituting the torsion wave solution into (59), we obtain

$$\begin{aligned}
\frac{\mathrm{d}^2 S^{\bar{0}}}{\mathrm{d}t^2} &= \varepsilon\,\alpha\,\Re[A_{01}\,\hat{f}_1]\frac{\mathrm{d}S^{\bar{1}}}{\mathrm{d}t} + 2\,\varepsilon\,k^2\,\hat{f}S^{\bar{1}} + O(\varepsilon^2),\\
\frac{\mathrm{d}^2 S^{\bar{1}}}{\mathrm{d}t^2} &= \varepsilon\,\alpha\,\Re[A_{01}\,\hat{f}_1]\frac{\mathrm{d}S^{\bar{1}}}{\mathrm{d}t} + 2\,\varepsilon\,k^2\,\hat{f}S^{\bar{1}} + O(\varepsilon^2),\\
\frac{\mathrm{d}^2 S^{\bar{A}}}{\mathrm{d}t^2} &= -\varepsilon\,\alpha\,\Re[A_{01}\,\hat{f}_1]\frac{\mathrm{d}S^{\bar{A}}}{\mathrm{d}t} - 2\,\varepsilon\,k^2\,\hat{f}S^{\bar{A}} + O(\varepsilon^2).
\end{aligned} \tag{60}$$

For solving (60), we expand $S^{\bar{a}}$ with respect to ε

$$S^{\bar{a}} = \tilde{S}^{\bar{a}} + \varepsilon\,\dot{S}^{\bar{a}} + O(\varepsilon^2) \tag{61}$$

and consider the following initial conditions

$$\begin{aligned}
S^{\bar{a}}(0) &= \tilde{S}^{\bar{a}}\\
\frac{\mathrm{d}S^{\bar{a}}}{\mathrm{d}t}(0) &= 0.
\end{aligned} \tag{62}$$

The solution for zeroth-order and first-order are

$$\begin{aligned}
\tilde{S}^{\bar{0}} &= 0,\\
\tilde{S}^{\bar{\alpha}} &= \text{constant}
\end{aligned} \tag{63}$$

and

$$\begin{aligned}
\dot{S}^{\bar{0}} &= -2\,\hat{f}\,\tilde{S}^{\bar{1}},\\
\dot{S}^{\bar{1}} &= -2\,\hat{f}\,\tilde{S}^{\bar{1}},\\
\dot{S}^{\bar{A}} &= 2\,\hat{f}\,\tilde{S}^{\bar{A}}
\end{aligned} \tag{64}$$

where

$$\begin{aligned}
\hat{f}\,(0) &= 0,\\
\hat{f}_1\,(0) &= 0.
\end{aligned} \tag{65}$$

So the solutions of (60) to first-order are

$$\begin{aligned}
S^{\bar{0}}(t) &= -2\,\varepsilon\,\hat{f}\,\tilde{S}^{\bar{1}},\\
S^{\bar{1}}(t) &= \tilde{S}^{\bar{1}} - 2\,\varepsilon\,\hat{f}\,\tilde{S}^{\bar{1}},\\
S^{\bar{A}}(t) &= \tilde{S}^{\bar{A}} + 2\,\varepsilon\,\hat{f}\,\tilde{S}^{\bar{A}}.
\end{aligned} \tag{66}$$

In the linearized GR, it is well-known that a gravitational wave is transverse. The solution (66) shows that the torsion wave has not only transverse parts but also longitudinal parts. From this solution, we can clarify the polarization of the torsion wave. Consider a particle **Q** sitting at the centre of a sphere and suppose the particle **Q** and every particle on the sphere carries a clock. All the clocks have been synchronized with respect to background spacetime (i.e. Minkowski spacetime with coordiantes $\{x^a\}$ and torsion field vanishing) before the torsion wave comes. Moreover, all the particle are at rest in the background spacetime. Using the separation vector S, we define the distances of **Q** and neighboring particles sitting in three different spatial directions $\hat{X}_\alpha$ by $\sqrt{g(S_L, S_L)}$, $\sqrt{g(S_{T_2}, S_{T_2})}$ and $\sqrt{g(S_{T_3}, S_{T_3})}$, where

$$\begin{aligned} S_L &= S^{\bar{1}} \hat{X}_1, \\ S_{T_2} &= S^{\bar{2}} \hat{X}_2, \\ S_{T_3} &= S^{\bar{3}} \hat{X}_3. \end{aligned}$$

When a torsion wave propagates along x^1 passing through these particles, the distance between these neighboring particles will change. The changes of the distance can be obtained by calculating $g(S, S)$ to first-order which give

$$\sqrt{g(S_L, S_L)} = |\tilde{S}^{\bar{1}}|\,(\,1 - 2\,\varepsilon\,\hat{f}\,) + O(\varepsilon^2), \tag{67}$$

$$\sqrt{g(S_{T_2}, S_{T_2})} = |\tilde{S}^{\bar{2}}|\,(\,1 + 2\,\varepsilon\,\hat{f}\,) + O(\varepsilon^2), \tag{68}$$

$$\sqrt{g(S_{T_3}, S_{T_3})} = |\tilde{S}^{\bar{3}}|\,(\,1 + 2\,\varepsilon\,\hat{f}\,) + O(\varepsilon^2). \tag{69}$$

If the distances (to first-order) between the particle **Q** and those neighboring particles sitting on the transverse plane are expanded, the longitudinal distances will be contracted. Attempts to detect gravitational waves are well underway [13] [5] [11]. It would also be interesting to detect longitudinal modes of gravitational waves.

6 Conclusion

We first discover a plane wave solution in linearized DT theory. The plane wave solution consists of torsion waves and tensor waves. We use gauge transformations to simplified our solution. To investigate the polarizations of torsion waves, we calculate the equations of motion of particles. We assume particles following time-like autoparallels instead of geodesics and then calculating autoparallel deviation. We then obtain both longitudinal and transverse polarization.

Since gravitational wave observations are underway, and if Laser Interferometer Gravitational Wave Observatory (LIGO) [1] and Laser Interferometer Space Antenna (LISA) [5] [11] detect longitudinal modes of gravitational waves, these calculations may be used to explain the data.

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