# Affine Torsion à la Cartan

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ABSTRACT. After reviewing Cartan's first presentation of affine torsion, we use differential forms and moving frames to approach in flat spaces the concept of torsion and equations of structure. These constitute the integrability conditions for the connection equations. The integration obviously yields the affine elementary (or Klein) geometry from which we start, namely the pair of affine group and its largest linear subgroup. Generalized (i.e. non-holonomic) affine spaces are then defined through the breaking of those conditions, which we consider in the bundle of frames, rather than in sections thereof. We also show Y. H. Clifton's definitions of affine connection and equations of structure, which allow for the development of differential geometry without the modern distortion of Cartan's original concepts. New avenues for torsion research and applications are then considered.

## 1 Introduction

In this paper, starting immediately, we immerse ourselves in the misunderstood world of torsion. Affine curvature has to do with transporting a vector along a closed curve and returning in principle with a different vector. Similarly, affine torsion has to do with representing closed curves of a "generalized affine space" on a flat space of equal dimension, representation which generally fails to close. Suppose that in 1492 we were to sail westward from Europe into the opposite shores of the Atlantic. Not to get lost, we would follow a parallel, viewed as a line of constant direction [1] (p. 9). We would thus have virtually defined the "Columbus affine connection", where rhumb lines play the role of straight lines. Intersecting meridians and parallels form curvilinear rectangles, whose sides are represented on the plane by mutually perpendicular straight segments. The representation on the plane of the curvilinear rectangles are "quasi rectangles", as they fail to close because the segments on the parallels are of different length. That failure to close is a manifestation of the torsion of the Columbus connection [2] (p. 709).

Consider next the surface of the earth endowed with the Levi-Civita (LC) connection, which has zero torsion. As per this connection or rule to navigate a manifold, only the maximum circles are lines of constant direction. According to the foregoing paragraph, however, it would appear that the representation on the plane of a curvilinear triangle obtained from intersecting two meridians with the equator should close. It does not, since there are at least two right angles in the triangle's representation on the plane. What is wrong? The curves used for the foregoing geometric interpretation of torsion have to be "infinitesimally small" [2] (p.708), and it is difficult to see graphically whether a curve fails to close by a small quantity of first (failure to close) or higher order. Because the Columbus connection has the property of teleparallelism (TP, simply put: path-independent comparison of vectors at a distance), it is mathematically legitimate to attribute the non-closing of the representation of closed curves to the torsion of the Columbus connection. The same conclusion cannot be drawn for the LC connection, however, because there is not TP. These subtleties result from issues of integrability of the connection equations.

## 2 Cartan's Approach to Affine Torsion

Dieudonné explains Cartan's extending of Klein ideas in geometry as resulting from replacing the group $G$ of flat space geometry with an object called the "principal fiber space", represented as a family of isomorphic subgroups $G_0$, parametrized by the different points of the flat space (see Gardner [3]). Modernly, the pair $(G, G_o)$ is called an elementary [4] or Klein geometry [5], characterizing flat spaces: affine, Euclidean, projective, conformal, etc. $G$ is built from $G_0$ and translations. In generalized spaces, the translations act only "infinitesimally", meaning that they appear in differential equations which, when integrable, give $G$ through integration. Affine space is Euclidean space minus the dot product, like the surface of a table extended to infinity but minus the concept of distance. It "becomes" a vector space when we choose a point and assign to it the zero of a vector space.

Cartan introduced torsion as the "complementary" term $\Omega_i$ in

$$d\omega_i = \sum_k \omega_k \wedge \omega_{ki} + \Omega_i. \tag{1}$$

further stating that $\Omega_i$ is the "*element of double integral*"

$$\Omega_i = \sum A_i^{rs} \omega_r \wedge \omega_s, \tag{2}$$

yielding infinitesimal translations associated with arbitrary elements in two dimensions [6]. He said "complementary" because $\Omega_i$ is zero in Euclidean spaces. The integration of $A_i^{rs}\omega_r \wedge \omega_s$ over the aforementioned curvilinear rectangle yields the vector with which to close the open one.

Cartan explained that, in Euclidean 3-space, $E^3$, the $\omega_k$ and $\omega_{ij}$ are linear in the differentials of the three coordinates $(x)$, with coefficients which depend on $(x)$ plus three additional ones $(u)$ that label all the orthonormal bases [6]. These constitute a bundle because of how they are organized. For the moment, let us think of frame bundles as neat sheaves of wheat (warning: the term sheaf is used in mathematics differently!), which is a faithful picture in $E^2$. Each stem represents a fiber, meaning all bases at one point of $E^2$, different points of the stem representing different frames, labelled by $u$. Cut the sheaf, perpendicularly for simplicity. The surface of the cut is a cross section of the bundle, meaning one and only one basis at each point of $E^2$. $\omega_k$ depends on $x$, $u$ and $dx$; $d\omega_i$ and $\omega_{ki}$ depend on $x$, $u$, $dx$ and $du$, but $\Omega_i$ does not depend on $du$. A cross section is a particular field of bases. On them, everything depends only on the coordinates $x$ because one has "chosen" the values for the $u$'s. In two-dimensional affine space, the fiber is a four-dimensional manifold (four-dimensional group of linear transformations).

Equations 1-2 are born in affine space, in the form

$$d\omega^i = \sum_k \omega^k \wedge \omega_k^i + \Omega^i , \qquad \Omega^i = \sum_{r,s} A^i_{rs} \omega^r \wedge \omega^s. \tag{3}$$

In the Euclidean specialization of affine space, one raises and lowers indices with impunity (up to minus signs in the pseudo-Euclidean case).

What are the $\omega^k$ and the $\omega_i^j$? Take the 2-dimensional sphere and, say, $d\mathbf{r} = d\rho\hat{\mathbf{e}}_\rho + \rho d\phi \hat{\mathbf{e}}_\phi$. Circumflex marks are used to denote that the $\hat{\mathbf{e}}$'s are orthonormal. The basis $(\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\phi)$ varies from point to point. It clearly is a basis field. $d\mathbf{r}$ is a vector-valued element of a line integral, like $\Omega^i\hat{\mathbf{e}}_i$ is a vector-valued element of a surface integral. We seek the expression for $d\mathbf{r}$ that belongs to the whole bundle of Euclidean frames. Since $ds^2 \equiv d\mathbf{r}\cdot d\mathbf{r} = d\rho^2 + \rho^2 d\phi^2$, we first think of $d\rho$ as $\omega^1$ and of $\rho d\phi$ as $\omega^2$. However, the most general $\omega^i$'s which satisfy $ds^2 = (\omega^1)^2 + (\omega^2)^2$ are $\omega^1 = \cos\alpha\ d\rho + \sin\ \alpha\ \rho d\phi$ and $\omega^2 = -\sin\alpha\ d\rho + \cos\alpha\,\rho d\phi$, which result from applying the most general rotation matrix to the column matrix $\begin{Bmatrix} d\rho \\ \rho d\phi \end{Bmatrix}$. Although $dx$ and $dy$ are not equal to $d\rho$ and $\rho d\phi$, one however has that $\cos\alpha\ d\rho + \sin\ \alpha\ \rho d\phi$ equals $\cos\beta\ dx + \sin\ \beta\ dy$; both of them are the same differential $\omega^1$under a change of coordinates $(\rho, \phi, \alpha)$ to $(x, y, \beta)$. Technically, each of these two forms of $\omega^1$ is a "pull back" $\sigma$ of the other under a coordinate transformation. The right way of writing $x = \rho\cos\phi$ is $\sigma x = \rho\cos\phi$, where the pull-back $\sigma$ is in this case a coordinate transformation. One usually ignores $\sigma$.

Consider now the $\omega^i_k$, again in Euclidean space. In the same way as the set of the $\omega^i$'s depends on the parameters $u$ of the rotation, so does the general orthonormal basis $\{\mathbf{e}_i\}$, in order for $d\mathbf{r}$ $(=\omega^i\mathbf{e}_i)$ not to depend on $u$. Suppose we refer the $d\mathbf{e}_i$ to the basis $(\mathbf{i},\mathbf{j})$, differentiate and, after differentiating, we replace the $(\mathbf{i},\mathbf{j})$ in terms of $\{\mathbf{e}_i\}$ itself. The $\omega^k_i$, which are nothing but the coefficients in the equations $d\mathbf{e}_i = \omega^k_i\mathbf{e}_k$, depend on the $x'$s, the $u$'s and, linearly, on the $dx'$s and the $du$'s.

$\wedge$ is the symbol for exterior product, which is hidden in the vector product $\times$ of $E^3$. For $n \neq 3$, the direction perpendicular to the plane of two vectors is not defined. In three dimensions, computing with $\wedge$ replaces term for term computing with $\times$, except that the different terms are no longer the components of a vector. The same product $\wedge$ is at work in the algebra of the integrands. We shall refer to an element of line integral, of double integral, ... of $r-$integral as a differential $1-$form, $2-$form, ... $r-$form. Because one unfortunately writes double integrands as $dxdy$, $d\rho d\phi$, etc., instead of $dx \wedge dy$, $d\rho \wedge d\phi$, etc., one cannot just substitute the differentials of the coordinates in $dxdy$, $d\rho d\phi$, etc., in order to change coordinates in integrands. One can, however, do so in $dx \wedge dy$, $d\rho \wedge d\phi$, etc. Jacobians then emerge spontaneously. The same considerations apply to $r$-forms.

The action of the operator $d$ on differential forms ($\omega^i$, $A^i_{rs}\omega^r \wedge \omega^s$, etc.) is such that

$$\int_{\partial R} \alpha = \int_R d\alpha, \tag{4}$$

where $\partial R$ is the boundary of the integration domain $R$. This generalized theorem comprises, in particular, the usual theorems of Stokes and Gauss. The integrands are scalar-valued, not withstanding the vector calculus. There are, however, other types of differential forms, like $\omega^i \mathbf{e}_i$, which is vector-valued.Sabela

## 3 Torsion in Affine and Euclidean Spaces

We now formally define affine space and provide the concept of differential form and exterior differentiation in Cartan's work. The torsion is Cartan's exterior differential of the translation 1-form.

An affine space $AFF(n)$ is a set of elements, called points, such that pairs (A,B) of them can be put in correspondence with the vectors, denoted $\mathbf{AB}$, of a vector space of dimension $n$, in such a way that (a) $\mathbf{AB} = -\mathbf{BA}$, (b) $\mathbf{AB} = \mathbf{AC} + \mathbf{CB}$, and (c) if O is an arbitrary point of $AFF(n)$ and $\mathbf{a}$ is an arbitrary point of the vector space, there is a unique point A such that $\mathbf{OA} = \mathbf{a}$. If the vector space is Euclidean (i.e. endowed with a dot product), the affine space is said to be a Euclidean space. An affine (respectively Euclidean) frame is the pair constituted by a point and a vector basis (respectively orthonormal basis). The affine (respectively Euclidean) geometry is the pair constituted by the group $G$ of all transformations between affine (respectively Euclidean) frames and the subgroup $G_0$ of those transformations that leave a point fixed. The same group $G_0$ acts on each of all fibers in one-to-one correspondence between members of $G_0$ and bases. The frame bundle is thus already present in the concept $(G, G_0)$ of Klein geometry. It is trivial to adapt these considerations to pseudo-Euclidean cases.

The group $G$ for the surface of the table is made of translations, linear transformations and their products. A concept of length on the table restricts the linear transformations to rotations (Using non-orthonormal bases in Euclidean geometry is extending Euclidean geometry into affine geometry; in differential geometry, one speaks of the extension of an affine connection into a Euclidean connection).

Without attempting to be self contained, we now discuss how the terms tangent vector, differential form and differentiation are to be understood in Cartan's writings. Cartan does not define *tangent vectors* in his papers on connections. A modern definition of tangent vector at a point compatible with his approach to geometry would be the well known "equivalence class of curves which go through the point and such that they give the same $dx^i/d\lambda$ at the point, $\lambda$ being the parameter on each curve". Unlike the definition of tangent vectors as differential operators acting on functions, this definition allows for a pictorial representation of tangent vectors as tiny arrows associated with arbitrarily close pairs of points on the manifold.

*Differential forms* are usually defined as antisymmetric covariant tensors. But, in Cartan's work on differential geometry, they are functions of hypersurfaces, whose evaluation is their integration, as reported in section 2. Consistently with this, $\omega^i \mathbf{e}_i$ (also written with the Kronecker delta as $\delta_i^j \omega^i \mathbf{e}_j$) is an element of vector-valued integral whose integration on a curve gives the radius vector associated with its end points. Needless to say that, in order to perform the integration, the vector basis at all points of the curve has to be the same. Hence, in affine space, one has to express $\delta_i^j \omega^i \mathbf{e}_j$ in terms of rectilinear bases

In Cartan's work, the action of $d$ on non-scalar-valued forms is connection-dependent, and $d$ no longer satisfies $dd = 0$ unless the connection is TP. Thus $d$ is not the operator known modernly as *exterior differentiation* but a generalization thereof. When Cartan differentiates $\omega^i \mathbf{e}_i$ he says that he is exterior differentiating. Kähler, whose work is highly relevant in connection with the use of differential forms in quantum mechanics [7], also uses only the term exterior differentiation for both, exterior and exterior-covariant differentiation. One can write $d\mathbf{r} = \omega^i \mathbf{e}_i$ for Euclidean spaces. In non-holonomic generalizations, one may wish to write, however, $\tilde{d}\mathbf{r}$ instead of $d\mathbf{r}$ when operating on it with $d$.

For lack of space, we shall assume that readers know exterior differentiation. We recall that, given an $r$-form $\alpha_r$ and an $s-$form $\beta_s$, the following is satisfied: $d(\alpha_r \wedge \beta_s) = d\alpha_r \wedge \beta_s + (-1)^r \alpha_r \wedge d\beta_s$. In the case of $d(\omega^i \mathbf{e}_i)$, $r$ is 1 and $s$ is zero ($\mathbf{e}_i$ is a scalar-valued $0-$form). Hence:

$$d(\omega^i \mathbf{e}_i) = d\omega^i \mathbf{e}_i - \omega^i \wedge d\mathbf{e}_i = d\omega^i \mathbf{e}_i - \omega^j \wedge \omega_j^i \mathbf{e}_i = (d\omega^i - \omega^j \wedge \omega_j^i)\mathbf{e}_i. \quad (5)$$

In affine space, $d(\omega^i \mathbf{e}_i) = d(dx^i \mathbf{a}_i) = ddx^i \mathbf{a}_i = 0$ and, therefore, $d\omega^i - \omega^j \wedge \omega_j^i = 0$. This still holds in $E^n$, but, whereas the $\omega_{ij}$ are independent in the affine case, they are not independent in $E^n$ since $0 = d(\delta_{ij}) = d(\mathbf{e}_i.\mathbf{e}_j) = \omega_{ij} + \omega_{ji}$. In the extension of Euclidean to affine geometry, we use arbitrary bases. Defining then $g_{ij} \equiv \mathbf{e}_i.\mathbf{e}_j$, we get $dg_{ij} = \omega_{ij} + \omega_{ji}$, which still exhibits that the Euclidean $\omega_{ij}$ are not independent.

## 4 Integrability and Formal Definition of Torsion

The affine group $G(aff, n)$ (for space dimension $n$) is a $(n + n^2)$-differentiable manifold with a frame bundle structure. Its action is given by

$$\mathbf{P} = \mathbf{Q} + A^{\mu}\mathbf{a}_{\mu}, \qquad \mathbf{e}_{\mu} = A^{\nu}_{\mu}\mathbf{a}_{\nu}, \qquad \det A^{\nu}_{\mu} \neq 0, \tag{6}$$

where $(\mathbf{Q}, \mathbf{a}_{\mu})$ is a fixed frame (We use Greek indices for general dimension). In block matrix form, we have

$$\begin{Bmatrix} \mathbf{P} \\ \mathbf{e}_{\mu} \end{Bmatrix} = g \begin{Bmatrix} \mathbf{Q} \\ \mathbf{a}_{\nu} \end{Bmatrix} = \begin{bmatrix} 1 & A^{\nu} \\ 0 & A^{\nu}_{\mu} \end{bmatrix} \begin{Bmatrix} \mathbf{Q} \\ \mathbf{a}_{\nu} \end{Bmatrix} \tag{7}$$

where $\{\}$ designates column matrices with $1+n$ rows, and where $g$ is an $(n+1)\times(n+1)$ matrix member of $G(aff, n)$. This group of matrices thus is an $(n^2+n)$-dimensional hypersurface in the $(n+1)^2$-dimensional manifold of all $(n+1)$ $(n+1)$ matrices. For the bundle of Euclidean frames, the group $G$ is a $[n+(1/2)n(n+1)]$-dimensional hypersurface in the same $(n+1)^2$ -dimensional manifold. One readily gets

$$\begin{Bmatrix} d\mathbf{P} \\ d\mathbf{e}_{\mu} \end{Bmatrix} = \begin{bmatrix} 0 & dA^{\nu} \\ 0 & dA^{\nu}_{\mu} \end{bmatrix} \begin{Bmatrix} \mathbf{Q} \\ \mathbf{a}_{\nu} \end{Bmatrix} = dg \begin{Bmatrix} \mathbf{Q} \\ \mathbf{a}_{\nu} \end{Bmatrix} = dg \cdot g^{-1} \begin{Bmatrix} \mathbf{P} \\ \mathbf{e}_{\mu} \end{Bmatrix}. \tag{8}$$

Equations (8) are also written as

$$d\mathbf{P} = \omega^{\mu}\mathbf{e}_{\mu}, \qquad d\mathbf{e}_{\mu} = \omega^{\nu}_{\mu}\mathbf{e}_{\nu}. \tag{9}$$

With the $\omega^{\mu}$ and $\omega^{\nu}_{\mu}$ implicit in $dg \cdot g^{-1}$ of Eqs. (8), this is an obviously integrable differentiable because it was obtained by actual differentiation of the system (6), i.e. the group $G$.

The $\omega^{\mu}$ and $\omega^{\nu}_{\mu}$ might be chosen to be more general than those above. $d\mathbf{P}$ then does not mean the differential of some vector-valued function $\mathbf{P}$, but a vector-valued integrand to be integrated on curves. On the other hand, $d\mathbf{e}_{\mu} = \omega^{\nu}_{\mu}\mathbf{e}_{\nu}$ means the difference between $\{\mathbf{e}_{\mu}(x+dx, u+du)\}$ and $\{\mathbf{e}_{\mu}(x,u)\}$ . We can pull this to a cross section. Using $\{\mathbf{e}_{\mu}(x+dx\} = \{\mathbf{e}_{\mu}(x)\} + \omega^{\nu}_{\mu}\mathbf{e}_{\nu}$, we can express what vector at $x+dx$ equals a given vector at $x$. We integrate $\omega^{\nu}_{\mu}\mathbf{e}_{\nu}$ on curves and add the result to $\{\mathbf{e}_{\mu}(x)\}$. We have thus parallel transported to $x'$ the basis at $x$.

Those integrations are curve-dependent in general, which is to say that the system (9) is not integrable. The Frobenius integrability conditions applied to $d\mathbf{P}-\omega^{\mu}\mathbf{e}_{\mu} = 0$ and $d\mathbf{e}_{\mu} - \omega^{\nu}_{\mu}\mathbf{e}_{\nu} = 0$ yield [8]

$$d\omega^{\nu} - \omega^{\lambda} \wedge \omega^{\nu}_{\lambda} = 0, \qquad d\omega^{\nu}_{\mu} - \omega^{\lambda}_{\mu} \wedge \omega^{\nu}_{\lambda} = 0, \tag{10}$$

i.e. the vanishing of affine torsion and curvature. Equations (6)-(7) are equations in the bundle, where the $A^{\nu}$and $A^{\nu}_{\lambda}$ are coordinates. Equations (8)-(9) may be written with the $\omega^{\mu}$ and $\omega^{\nu}_{\mu}$ of the bundle or (as is almost overwhelmingly the case in applications) on a section thereof.

There is an important difference between the two equations (10) from the perspective of integrability. The equation $d\omega^{\nu}_{\mu} - \omega^{\lambda}_{\mu} \wedge \omega^{\nu}_{\lambda} = 0$ (which contains only the $\omega^{\nu}_{\lambda}$'s) is the integrability condition for the subsystem $d\mathbf{e}_{\mu} = \omega^{\nu}_{\mu}\mathbf{e}_{\nu}$. But we cannot say that $d\omega^{\nu} - \omega^{\lambda} \wedge \omega^{\nu}_{\lambda} = 0$ is the integrability condition for the subsystem $d\mathbf{P} = \omega^{\mu}\mathbf{e}_{\mu}$. For this subsystem to be considered on its own, $\{\mathbf{e}_{\mu}\}$ has to be the same for all $x$. In other words, the other integrability condition has to be satisfied at the same time. Hence, either we consider the full system (9) or the subsystem $d\mathbf{e}_{\mu} = \omega^{\nu}_{\mu}\mathbf{e}_{\nu}$ for purposes of integrability. Thus, if the torsion is zero, we have not gained anything since we would also need zero affine curvature in order to have integrability of the subsystem $d\mathbf{P} = \omega^{\mu}\mathbf{e}_{\mu}$. That is the reason why, when we constructed a curvilinear triangle on the sphere with LC connection, its representation on the plane did not close; the $\mathbf{e}_{\mu}$ do not constitute vector-valued functions on the sphere (whether punctured or not) if endowed with the LC connection.

We now give two more non-trivial examples of manifolds endowed with TP. On the surface of a table drill a hole, just a point $O$ for simplicity. Define the circles centered at $O$ and the radial lines from it as lines of constant direction (which would make sense to inhabitants in a flat world with $O$ as the source of energy). Put the usual metric on the plane. The metric-compatible affine connection so defined is a TP connection, since the "transport" of the vectors $\hat{\mathbf{e}}_{\rho}$ and $\hat{\mathbf{e}}_{\phi}$ (and therefore any vector whatsoever) then is path independent. In fact, there is no need to speak of transport; there is geometric equality at initio. A curvilinear rectangle from the intersection of radial lines and circles goes in the plane into a rectangle which fails to close. Assuming we computed the $\omega^{\mu}$ and integrated $d\mathbf{P} = \omega^{\mu}\mathbf{e}_{\mu}$ in terms of the now constant basis field ($\hat{\mathbf{e}}_{\rho}$, $\hat{\mathbf{e}}_{\phi}$), we would obtain the vector that allows us to close the rectangle.

For another example, define "parallels" on the torus by cutting it with planes perpendicular to the axis of symmetry, and define "meridians" by cutting with planes containing that axis. Consider those lines as being of constant direction. We are thus defining a TP connection. The representation in the plane of curvilinear rectangles formed with two parallels and two meridians does not close. That connection has torsion.

We now proceed to reproduce Clifton's formal definition of affine connection, which includes the definition of torsion [9]. *An affine connection is a* 1*-form* $(\omega^\mu, \omega^\nu_\lambda)$ *on the frame bundle* $B(M)$ *taking values in the Lie algebra of the affine group and satisfying the conditions:*

*(1) The* $n^2 + n$ *real-valued* 1*-forms are linearly independent.*

*(2) The forms* $\omega^\mu$ *are the solder forms* (i.e. those introduced in the paragraph after Eq. (3)).

*(3) The pull-back of* $\omega^\nu_\mu$ *to the fibers are the left invariant forms of the affine group* (i.e. the $dg \cdot g^{-1}$ given implicitly in Eq. (8)).

*(4) The forms* $\Omega^\nu = d\omega^\nu - \omega^\lambda \wedge \omega^\nu_\lambda$*, called torsion, and* $\Omega^\nu_\mu = d\omega^\nu_\mu - \omega^\lambda_\mu \wedge \omega^\nu_\lambda$*, called affine curvature, are quadratic exterior polynomials in the* $n$ *forms* $\omega^\mu$*:*

$$\Omega^\nu = R^\nu{}_{\lambda\mu}\omega^\lambda \wedge \omega^\mu, \qquad \Omega^\nu_\pi = R^\nu_\pi{}_{\lambda\mu}\omega^\lambda \wedge \omega^\mu. \tag{11}$$

Lack of space impedes us to explain some of the concepts involved in this definition.

## 5 Finslerian, Kaluza-Klein and Quantum Torsions

(**a**) *Finslerian torsion.* Let S(M) be the bundle of directions of a differentiable manifold M. If M is of dimension $n$, S(M) is of dimension $2n - 1$. If we consider the total space of this bundle simply as a topological space, one can construct a frame bundle over it with the same set of frames B(M) of $n$-dimensional vector spaces that we have considered so far. The fibers now are of smaller dimension than before, since S(M) plays the role of "base space" that M played before. The sum of the dimensions of the new base space and fiber has to be the same as for the old bundle, namely the dimension of B(M). Let the signature be Lorentzian. Let us use the index zero for the "odd" directions (i.e. time-like). The role of the $\omega^\mu$ is now played by the $\omega^\mu$ and the $\omega^i_0$. We define Finslerian torsion, within the definition of affine-Finsler connection, as follows [9]:

*An affine-Finsler connection is a* 1*-form* $(\omega^\mu, \omega^\nu_\lambda)$ *on a* $(n^2 + n)$*-dimensional manifold* $B(M)$ *taking values in the Lie algebra of the affine group and satisfying the conditions:*

*(1) The* $n^2 + n$ *real-valued* 1*-forms are linearly independent.*

*(2) The forms* $\omega^\mu$ *are the soldering forms.*

*(3) The forms* $\omega^i_0$ *vanish on the fibers of* $B(M)$ *over* $S(M)$*.*

*(4) The pullbacks of $\omega_0^0, \omega_i^0, \omega_j^i$ into the fibers of $B(M)$ over $S(M)$ are the left invariant forms of the linear (sub)group that leaves the direction of a vector unchanged.*

*(5) The forms $\Omega^\nu = d\omega^\nu - \omega^\lambda \wedge \omega_\lambda^\nu$, called torsion, and $\Omega_\mu^\nu = d\omega_\mu^\nu - \omega_\mu^\lambda \wedge \omega_\lambda^\nu$, called affine curvature, are quadratic exterior polynomials in the $2n-1$ forms $\omega^\mu$, $\omega_0^i$:*

$$\Omega^\nu = R^\nu{}_{\lambda\mu}\omega^\lambda \wedge \omega^\mu + S^\nu{}_{\lambda i}\omega^\lambda \wedge \omega_0^i \tag{12}$$

$$\Omega_\pi^\nu = R_\pi^\nu{}_{\lambda\mu}\omega^\lambda \wedge \omega^\mu + S_\pi{}^\nu{}_{\lambda i}\omega^\lambda \wedge \omega_0^i + T_\pi{}^\nu{}_{ij}\omega_0^i \wedge \omega_0^j, \tag{13}$$

*where $R^\nu{}_{\lambda\mu}$, $R_\pi{}^\nu{}_{\lambda\mu}$ and $T_\pi{}^\nu{}_{ij}$ are antisymmetric in the last 2 subscripts.*

Theory similar to that of the pre-Finslerian case can thus be developed in Finsler geometry

(**b**) *Kaluza-Klein torsion.* By Kaluza-Klein (KK) torsion we mean standard torsion of the (KK type of) space that is motivated by an observation Cartan made in the course of the simple computation that follows. He did not solve the issue that he implicitly raised. Finsler bundles provide a natural way of removing the critique present in that observation, namely that standard differential geometry is simply a theory of moving frames. It is not the method of the moving frame that is under question but the limited use that is made of it. The critique applies equally to those presentations of differential geometry that do not use moving frames but which yield equivalent results.

Consider an orthonormal basis and a point not coincident with the origin of the basis (in 3-dimensional Euclidean space, to help the imagination and as in the original work [10]). We perform an "infinitesimal" Euclidean motion of the basis, while leaving the point fixed. Let the translation be given by $\omega^i$, and let the rotation be given $\omega_r^i$. The coordinates $x^i$ of the point with respect to that basis will change as a result. We have:

$$dx^i + x^r\omega_r^i + \omega^i = 0. \tag{14}$$

Exterior differentiating, we obtain

$$0 + dx^r \wedge \omega_r^i + x^r d\omega_r^i + d\omega^i = 0. \tag{15}$$

Substitution of $dx^r$ from the original equation yields

$$d\omega^i - \omega^j \wedge \omega_j^i + x^k(d\omega_k^i - \omega_k^j \wedge \omega_j^i) = 0. \tag{16}$$

Since this has to be valid for all $x$, the equations of structure of affine space follow. This emphasizes that these equations are about a geometry where points remain fixed and frames move. Curves with deep geometric significance like autoparallels (geometric equality) are not the result of the motion of a point independently of frames, but a succession of origins of frames. This becomes clear by pushing standard affine connections to their Finsler bundles. The equation $d\mathbf{u} = 0$ defining lines of constant direction in that push-forward become

$$d\mathbf{e}_0 = 0. \tag{17}$$

In order to deal with the motion of the point, one is tempted to use equivalence of active and passive transformations. However, moving the basis may not be equivalent to moving the point. In other words, that equivalence may not always hold. There is the following elegant way of creating new geometry. In the Finsler bundle, the $\omega_0^i$ are the coefficients in $d\mathbf{u} = d\mathbf{e}_0 = \omega_0^i \mathbf{e}_i$. We view $\mathbf{u}$ as independent of the frames themselves, as reflected in the translation element

$$\tilde{d}\wp = d\mathbf{P} + \mathbf{u} d\tau = \omega^\mu \mathbf{e}_\mu + \mathbf{u} d\tau \tag{18}$$

on a manifold $M^4 \oplus M^1$ [11]. $d\mathbf{P}$ is the translation element on $M^4$. $\mathbf{u}$ is the unit vector on $M^1$ manifolds, not contained in the spacetime manifold itself. The torsion is similarly defined, $\check{\mathbf{\Omega}}.\equiv d(\tilde{d}\wp)$,where we use the hat in $\check{\mathbf{\Omega}}$ to emphasize that we are in KK geometry, rather than in Finsler geometry.

In the Finsler bundle for spacetime, the torsion

$$\Omega^0 = CF, \qquad \Omega^i = 0, \tag{19}$$

or $\mathbf{\Omega} = \Omega^0 \mathbf{e}_0$ is an invariant, since the group in the fibers now is the rotation group in three dimensions. More on this in the next section. In the KK space, that torsion is written as

$$\Omega^4 = CF, \qquad \Omega^\mu = 0. \tag{20}$$

The fact that the fifth dimension differs from the other four even more than time differs from space carries to the components of the KK torsion. Integrations containing $d\tau$ and $dx^\mu$ as factors in the integrand would require specialized treatment.

(**c**) *Quantum torsion.* The structure of spacetime is determined by the $\omega^\mu$'s and $\omega_\nu^\lambda$'s. One may try to obtain these differential forms via integration of the equations of structure. In that case, each specification of the affine torsion and curvature must satisfy the Bianchi identities. If we require geometric equality, we have TP and, therefore, zero affine curvature. Suppose that we then wish to specify the torsion. It has to be such that its exterior (covariant) derivative is zero, as required by the first Bianchi identity. Or we could simply specify the interior covariant derivative, as it complements the specification of the exterior covariant derivative. But that is precisely what the Kähler replacement for the Dirac equation does [7]; it specifies both. It is then natural to postulate that the torsion may be given by a Kähler equation, which is a quantum mechanical equation. It takes the form

$$\partial u = a \vee u, \tag{21}$$

where $\partial$ represents the sum of the exterior and interior derivatives and $a$ is an input differential form (like, for example, $m + eA$, where $m$ is mass, $e$ is charge and $a$ is electromagnetic 1-form). If $a$ is vector-valued, we have $\partial u = a(\vee, \otimes)u$, where $\vee$ is the product for the differential forms and $\otimes$ is the product for the valuedness tensors [12], though Kähler does not make explicit the sign for tensor product. Similarly, we must have $\partial u = a(\vee, \vee)u$ if the differential forms are Clifford valued on tangent Clifford spaces. We would tentatively have

$$\partial \check{\mathbf{\Omega}} = \mathbf{a}(\vee, \vee)\check{\mathbf{\Omega}}. \tag{22}$$

where $\mathbf{a}$ is a Clifford-valued differential form and $\check{\mathbf{\Omega}}$ is the KK torsion. Furthermore, we shall see in the next section that the torsion appears to be intimately connected with the electromagnetic field. Thus, $m + eA$ is equally connected with the potential for the torsion. We postulate the equation

$$\partial \check{\mathbf{\Omega}} = \mathbf{u} \vee \tilde{d}\wp(\vee, \vee)\check{\mathbf{\Omega}}. \tag{23}$$

where the scalar-plus-bivector-valued differential form $\mathbf{u} \vee \tilde{d}\wp$ is insinuated by the good working of the Kähler equation with $a$ given by $m + eA$ (up to universal constants and the imaginary unit) [7]. Notice that only the differential invariants $\omega^\mu$'s and $\omega_0^i$ enter this equation, directly and through their derivatives. They are present in $\partial$ since this operator depends on the connection. They are present in $\tilde{d}\wp$, and they are present in $\check{\mathbf{\Omega}}$ since this is $d(\tilde{d}\wp)$. Thus, in the system of two equations of structure, the curvature equation retains its geometric flavor. The second one is of the KD type, i.e. the central equation of the Kähler calculus of differential forms, and of quantum mechanics. In that sense, spacetime is quantized. The differential invariants that determine the space are the solution of a system that combines classical (gravitation) and quantum mechanical equations (the other interactions).

## 6 Applications of Affine Torsion

One may define affine-Finsler connections and metric-compatible affine-Finsler connections (or "metric-Finsler connections"). Affine-Finsler bundles exist even if a length of curves is not defined. An affine-Finsler bundle is a refibration over $S(M)$ of the bundle of tangent bases to $M$. Metric-Finsler bundles exist even if the metric is (pseudo-)Riemannian. They are refibrations over $S(M)$ of the bundle of pseudo-orthonormal tangent bases to $M$. In addition, the positive and Lorentzian signatures are specially adapted to the standard Riemann-cum-torsion and Finslerian connections. In this view, the spacetime of special relativity is the flat metric-Finsler space [13].

Suppose we ask ourselves what the autoparallels look like in TP. If we make the S terms of the torsion equal to zero, the autoparallels look like the equations of the geodesics (i.e. as in general relativity) accompanied by a term which looks exactly like the Lorentz force, up to physical constants and upon labelling with $E$'s and $B$'s the coefficients of the $R^o$ piece of the Finslerian torsion. The $R^i$ components of the torsion do not contribute to the equations of the autoparallels. This suggests ab initio geometric unification of gravitation and electromagnetism at the level of equations of the motion

$$0 = -l_{,i}\, dt + dl_{\cdot i} + C[E_i + (B_k u^j - B_j u^k)]dt$$

[14]. One must not ask what torsion will get us the equations of motion of relativity, but rather how does one avoid obtaining them, and what does it mean doing so. One can, of course, single out the electromagnetic interaction by choosing the torsion to be $R^o \mathbf{e}_0$, where $R^o$ is the negative of the electromagnetic form $F$ (up to physical constants).

Assume further TP in order to have equality of vectors at a distance and to be able to perform integrations of vector-valued differential forms, like the so called energy-momentum "tensors". The first Bianchi identity then states that the exterior covariant derivative of the torsion is zero. If one linearizes and chooses torsions with $R^i = 0$, one obtains the homogeneous pair of Maxwell's equations, always up to multiplicative constants [15].

The affine connection, $\omega_\mu^\nu$, can be written as the sum of the LC connection, $\alpha_\mu^\nu$, and the contorsion, $\beta_\mu^\nu$, whose components are linear combinations of the components of the torsion. The equation $d\omega_\mu^\nu - \omega_\mu^\lambda \wedge \omega_\lambda^\nu = 0$ expressing that the affine curvature is zero can be written as

$$d(\alpha_\mu^\nu + \beta_\mu^\nu) - (\alpha_\mu^\lambda + \beta_\mu^\lambda) \wedge (\alpha_\lambda^\nu + \beta_\lambda^\nu) = 0. \tag{24}$$

It is clear that by leaving $d\alpha_\mu^\nu - \alpha_\mu^\lambda \wedge \alpha_\lambda^\nu$ on the left and everything else on the right and contracting with the Einstein contraction one obtains a geometric version of Einstein's equations, but with a richer right hand side. One can specialize this to the pure electromagnetic torsion, $-F\mathbf{e}_0$, if one so wishes. But this can also be carried out in the KK space, where the pure electromagnetic torsion is given as $-F\mathbf{u}$ and Sabelawhere $\mathbf{u}$ spans the fifth dimension.

It is clear that the $O(3)$ symmetry (or $SU(2)$ in quantum mechanical equations) is implicit in the $R^i$ piece of the torsion that accompanies $F$ but does not appear in the equations of motion. One can hardly think of a better candidate for a classical representation of the weak interaction.

Viewing the specification of the torsion from the quantum perspective of the previous section leads one to replace with advantage the Dirac equation with the Kähler equation, where fermions are represented by inhomogeneous differential forms. Even in the case that the form is scalar-valued, it has in principle 32 real components, which decomposes into four eight component solutions in the presence of rotational and time translation symmetries. In the absence of one or two of those symmetries, the solutions must, however, have 16 or 32 components. We have shown elsewhere the possibility for viewing strongly interacting particles as composites of virtual solutions (quarks) that depend on only eight real components, like Dirac fermions [16].

There is finally the issue of Maxwell's second pair. It is our view that the second pair of Maxwell's equations (and the first one also, except that the absence of source makes the need less apparent) is to be obtained from a degeneration of the information in a collective quantum system. Clearly this is emerging as a formidable undertahing where torsion plays the central role.

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