Twisted space, chiral gauge and magnetism

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RÉSUMÉ. L'article commence par une brève revue d'ensemble des espaces affines avec torsion avec des applications mathématiques aux équations d'onde spinorielles. On montre ensuite comment cette théorie s'accorde avec la théorie de la jauge chirale et d'un monopôle leptonique magnétique indépendamment développée par l'auteur.

ABSTRACT. The paper begins with a brief survey on twisted spaces with mathematical applications to spinorial wave equations. Then, it is shown how this theory is in accordance with the theory of chiral gauge and of a leptonic magnetic monopole independently developed by the author.

I. A SHORT SURVEY ON TWISTED SPACES

1. Definitions and first results.

a) In a space with rectilinear coordinates : $x^i = a^i_j x^{j'}$, the gradients $T^i_{;k} = \partial_k T^i$ and $T_{i;k} = \partial_k T_i$, of a vector T^i or a covector T_i, are both tensors.

b) In the case of curved coordinates $x^j = x^j\left(x^{1'}, x^{2'}, \dots, x^{n'}\right)$, $\partial_k T^i$ and $\partial_k T_i$ are not tensors. The corresponding tensors are the *covariant derivatives* :

$$\nabla_k T^i = \partial_k T^i + \Gamma^i_{rk} T^i \; ; \; \nabla_k T_i = \partial_k T_i - \Gamma^r_{ik} T_r \qquad (1)$$

Γ^i_{rk} are the *connection* coefficients. Their transformation formula is :

$$\Gamma^{i'}_{k'j'} = \Gamma^{i}_{kj}\frac{\partial x^{i'}}{\partial x^{i}}\frac{\partial x^{k}}{\partial x^{k'}}\frac{\partial x^{j}}{\partial x^{j'}} + \frac{\partial x^{i'}}{\partial x^{i}}\frac{\partial^2 x^{i}}{\partial x^{k'}\partial x^{j'}} \quad (2)$$

Definition : A space with an *affine connection* is a manifold $\mathcal{M}_n$ where ∇_k and Γ^{i}_{kj} are defined.

Not any metric g_{ij} is supposed to exist, and even if there is a metric, it is not – a priori – linked to Γ^{i}_{kj}. A connection coefficient Γ^{i}_{kj} is not a tensor by itself, but it defines two tensors :

A curvature **:** $-R^{i}_{qkl} = \dfrac{\partial\Gamma^{i}_{ql}}{\partial x^{k}} - \dfrac{\partial\Gamma^{i}_{qk}}{\partial x^{l}} + \Gamma^{i}_{pk}\Gamma^{p}_{ql} - \Gamma^{i}_{pl}\Gamma^{p}_{qk}$ (3)

A torsion : $S^{i}_{[kj]} = \Gamma^{i}_{kj} - \Gamma^{i}_{jk}$ (4)

They are related by the equality :

$$\left(\nabla_k\nabla_l - \nabla_l\nabla_k\right)T^{i} = -R^{i}_{qkl}T^{q} + S^{p}_{[kl]}\frac{\partial T^{i}}{\partial x^{p}} \quad (5)$$

If $S^{p}_{[kj]} \neq 0$, we have a *twisted space.***.** If $R^{i}_{qkl} = S^{p}_{[kj]} = 0$, *the space is euclidean and (5) entails the commutation of covariant derivatives.*

2. Geometrical interpretation.

a) The *parallel transport* of a tensor T along a direction $\xi^{k} = \dfrac{dx^{k}}{dt}$ is defined as :

$$\nabla_{\xi}T = \xi^{k}\nabla_{k}T \quad (6)$$

Therefore, the transported vector *depends on the curve* along which it has been transported.

b) **Definition :** A geodesic line is a line enveloped by the parallel transport of its tangent.

c) It follows, from a) and b), that *generally a geodesic rectangle is not closed* (Fig.1) because in an infinitesimal rectangle, one finds a gap, the principal part of which is due to the torsion and it is of the order of the area of the rectangle :

$$x^{k}_{P} - x^{k}_{T} = S^{k}_{ij}\left(\frac{dx^{i}}{dt}\right)_{P}\left(\frac{dx^{j}}{dt}\right)_{T} dt^{2} + 0\left(dt^{3}\right) \quad (7)$$

The terms $0\left(dt^{3}\right)$ are due to the curvature.

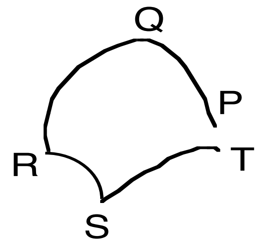

Fig. 1

Note : Now, we enter into the frame of Quantum Mechanics and we shall make use of the space-time coordinates with an imaginary time and the usual von Neumann γ matrices.

3. The case of a flat twisted space.

Definition : A flat twisted space is a space with torsion $\left(\Gamma^{\lambda}_{[\mu\nu]} \neq 0\right)$ and rectilinear geodesics.

Let us consider with Rodichev [2] a spinor equation with an affine connection defined by an antisymmetric tensor $\Phi_{[\lambda\mu\nu]}$:

$$\Gamma_{\lambda[\mu\nu]} = \Phi_{[\lambda\mu\nu]} \tag{8}$$

Thus we have the following covarariant derivative, where γ_λ are the Dirac matrices $\left(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 0\right)$:

$$\nabla_\mu = \partial_\mu\psi - \frac{i}{4}\Phi_{[\lambda\mu\nu]}\gamma_\mu\gamma_\nu\psi \tag{9}$$

Now, Rodichev introduces the lagrangian :

$$L = \frac{1}{2}\left\{\overline{\psi}\gamma_\mu\left(\nabla_\mu\psi\right) - \left(\nabla_\mu\overline{\psi}\right)\gamma_\mu\psi\right\} \Leftrightarrow L = \frac{1}{2}\overline{\psi}\gamma_\mu\left[\nabla_\mu\right]\psi \tag{10}$$

The form of L may be simplified by introducing the pseudovector φ_μ conjugated to $\Phi_{[\lambda\mu\nu]}$:

$$\varphi_\mu = \frac{i}{3\,!}\varepsilon_{\mu\nu\lambda\sigma}\Phi_{[\lambda\mu\nu]} \tag{11}$$

So we have :

$$L = \frac{1}{2}\left\{\overline{\psi}\gamma_\mu\left[\partial_\mu\right]\psi - \frac{1}{2}\varphi_\mu\overline{\psi}\gamma_\mu\gamma_5\psi\right\} \tag{12}$$

from which we can derive the equation :

$$\gamma_\mu \left(\partial_\mu - \frac{1}{2} \varphi_\mu \gamma_5 \right) \psi = 0 \tag{13}$$

This equation is a geometrical construction. But actually we already know it from physics because the same equation was found in a quite different way, [3], [4], We shall see it later, but let us first follow Rodichev a little farther.

4. The curvature of a flat twisted space.

Despite it is « flat », such a space has a curvature. In the particular case of Rodichev, if we introduce (11) in (3), we find the following curvature :

$$R = R^i_{1i1} = \Phi_{[\mu\nu\lambda]}\Phi_{[\mu\nu\lambda]} = -6\varphi_\mu\varphi_\mu \tag{14}$$

Now Rodichev considers the Einstein-Like Lagrangian :

$$L = \frac{1}{2}\left\{\overline{\psi}\,\gamma_\mu \left(\nabla_\mu \psi\right) - \left(\nabla_\mu \overline{\psi}\right)\gamma_\mu \psi\right\} + 6b\varphi_\mu\varphi_\mu \tag{15}$$

where b is a Lagrange parameter. The variation of φ_μ yields :

$$\varphi_\mu = \frac{1}{24b}\overline{\psi}\,\gamma_\mu\gamma_5\psi \tag{16}$$

and the variation of ψ gives the Weyl equation [8] (see also [4], [5]).

$$\gamma_\mu\partial_\mu\psi + \frac{1}{48b}\left(\overline{\psi}\,\gamma_\mu\gamma_5\psi\right)\gamma_\mu\gamma_5\psi = 0 \tag{17}$$

But owing to the algebraic properties of bilinear spinorial forms [9] :

$$\begin{aligned} &J_\mu = i\overline{\psi}\,\gamma_\mu\psi;\ \Sigma_\mu = i\overline{\psi}\,\gamma_\mu\gamma_5\psi;\ \Omega_1 = \overline{\psi}\psi;\ \Omega_2 = i\overline{\psi}\gamma_5\psi \\ &-J_\mu J_\mu = \Sigma_\mu\Sigma_\mu = \Omega_1^2 + \Omega_2^2 \end{aligned} \tag{18}$$

we can write (17) equivalently :

$$\gamma_\mu\partial_\mu\psi + \frac{1}{48b}\left(\Omega_1 - i\Omega_2\gamma_5\right)\psi = 0 \tag{19}$$

A consequence of (14) and (18) is :

$$R = \frac{-1}{2b}\left(\Omega_1^2 + \Omega_2^2\right) \tag{20}$$

This equality will find its signification later because *the total curvature* $\left(\Omega_1^2 + \Omega_2^2\right)$ *is a chiral invariant* and it plays a role in the harmony of the whole system.

II CHIRAL GAUGE AND MAGNETIC MONOPOLE

Now, we shall follow a different way, starting from the notion of *chiral gauge* (in quantum mechanics and electromagnetism). It is, in principle, independent from the preceding one but they concur to the same geometrical conclusions.

1. Chiral gauge and linear equation of a leptonic monopole.

We start from the massless Dirac equation :

$$\gamma_\mu \partial_\mu \psi = 0 \tag{21}$$

This equation has two gauge invariances :

a) *The classical phase invariance :* $\psi \rightarrow e^{i\vartheta}\psi$, which defines a co-variant derivative with the Lorentz potential A_μ, from which one can find the Dirac theory of the electron (**adding a mass term**) . The covariant de-rivative and the transformations are :

$$\nabla_\mu = \partial_\mu - i\frac{e}{\hbar c}A_\mu;$$
$$\psi \rightarrow e^{i\frac{e}{\hbar c}\phi}\psi, \; A_\mu \rightarrow A_\mu + \partial_\mu\phi \tag{22}$$

and the Dirac equation of the electron is :

$$\gamma_\mu\left(\partial_\mu - i\frac{e}{\hbar c}A_\mu\right)\psi + \frac{m_0 c}{\hbar} = 0 \tag{23}$$

b) *The second invariance law is the* ***chiral invariance*** [3], [4], [5] :

$$\psi \rightarrow e^{i\frac{e}{\hbar c}\gamma_5\vartheta}\psi, (\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4) \tag{24}$$

This second invariance can be developed only for the massless equation (21) because, contrary to the ordinary phase invariance, the second is based, not on the commutation of the unit matrix *I* with γ_μ matrices, but on the

anticommutation of γ_5 with the four matrices γ_μ. **A second covariant derivative** is so defined,with a **pseudo potential** B_μ [1] :

$$\nabla_\mu = \partial_\mu - \frac{g}{\hbar c}\gamma_5 B_\mu \,;\, \gamma_5\gamma_\mu + \gamma_\mu\gamma_5 = 0\,; \qquad (25)$$
$$\psi \to e^{i\frac{e}{\hbar c}\gamma_5 \phi}\psi \,;\, B_\mu \to B_\mu + i\partial_\mu\phi$$

Just as the first covariant derivative (22) leads to the Dirac electron, the second derivative (25) leads to the theory of a *leptonic monopole* which is in accordance with the electromagnetic theories of Maxwell and Pierre Curie. We cannot develop here all the results, but the reader can find the theory in the references ([3],…,[7]) and the main experimental results in ([7],…,[14]).

We shall only give some theoretical results, at first, the linear equation invariant with respect to (25) :

$$\gamma_\mu\left(\partial_\mu - \frac{g}{\hbar c}\gamma_5 B_\mu\right)\psi = 0 \qquad (26)$$

The main differences between (26) and (23) are : 1) The *pseudoscalar* magnetic charge *operator* $\frac{g}{\hbar c}\gamma_5$ which appears in (26). 2) The absence of mass term in (26), because it should commute with γ_μ and so, it would brake the chiral symmetry (25) which is based on the anticommutation of the exponant of $e^{i\frac{e}{\hbar c}\gamma_5\vartheta}$ and of the charge operator $\frac{g}{\hbar c}\gamma_5$ with the four matrices γ_μ.

An important point is that, if we apply to the spinor ψ the Weyl transformation (where ξ, η are two component spinors) :

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[1] Don't forget ! B_μ is not an induction but a pseudovector. These pseudopotentials are often named « Cabibbo & Ferrari potentials » (*Nuovo Cimento*, **23**, p. 1147, 1962). They were already defined by Louis de Broglie in : *Une nouvelle théorie de la lumière*, Hermann, Paris, 1942.

$$U\psi = \frac{1}{\sqrt{2}}\left(\gamma_4 + \gamma_5\right)\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{27}$$

the equation (25) splits in two separate equations with $iB_\mu = \{\mathbf{B}, iW\}$:

$$\begin{aligned} &\left(\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{s}.\nabla - i\frac{g}{\hbar c}\left(W + \mathbf{s}.\mathbf{B}\right)\right)\xi = 0 \\ &\left(\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{s}.\nabla + i\frac{g}{\hbar c}\left(W - \mathbf{s}.\mathbf{B}\right)\right)\eta = 0 \end{aligned} \tag{28}$$

They correspond respectively to the left and to the right monopole, which have respectively a positive and a negative charge, in conformity with the symmetry laws emitted one century before by Pierre Curie. Here, it is a simple consequence of the chiral gauge invariance.

1) In the domain of beta radioactivity, the influence of a monopole beam and the possibility of an emission of monopoles instead of neutrinos with β electrons were predicted by theory and then observed [12].

2) Owing to the preceding equations, several laws and effects were predicted and some were experimentally confirmed. Several problems of motion of a monopole in an electromagnetic field, already solved in different ways, were solved by our equations : it is the case, for instance, for the Poincaré problem [5], [7] of a monopole in a coulombian field (Birkeland effect), or the Dirac quantization of the product $eg = \frac{n\hbar c}{2}$ in the same problem [4].

On the contrary, some effects in nuclear physics and biology enter in the same frame but they were not predicted by theory ([10], [11], [13],[14]).

2. Chiral invariant equation of a monopole and twisted space..

The analogy between the Rodichev equation (13) and my equation (26) is evident, despite their different premises, and we deduce the equality :

$$\varphi_\mu = \frac{g}{\hbar c} B_\mu \tag{29}$$

which gives an electromagnetic sense to the geometrical tensor of Rodichev.

We can thus assert that :

When a monopole is plunged in an external electromagnetic field, he « sees » the surrounding space as a twisted space with a non symmetric affine connection defined by the pseudopotential of the field.

2. Chiral gauge and nonlinear equations.

It is imposible to speak of chirality without introducing a rotation angle. In order to find it, we shall slightly modify the chiral transformation (24) :

$$\psi' = e^{i\gamma_5(\vartheta/2)}\psi \tag{30}$$

and examine the 16 quadratic tensorial quantities obtained from the Dirac spinor, on the basis of a Clifford algebra :

$$\begin{gathered}\Omega_1 = \bar{\psi}\psi\ ,\ J_\mu = i\bar{\psi}\gamma_\mu\psi\ ,\Sigma_\mu = i\bar{\psi}\gamma_\mu\gamma_5\psi, \\ M_{\mu\nu} = \bar{\psi}\gamma_{[\mu}\gamma_{\nu]}\psi\ ,\Omega_2 = -\overline{\psi\gamma_5}\psi\end{gathered} \tag{31}$$

Denoting by primes the quantities (31) transformed by (29), we find :

$$J'_\mu = J_\mu\ ;\ \Sigma'_\mu = \Sigma_\mu \qquad (32)$$

$$\begin{pmatrix}\Omega'_1 \\ \Omega'_2\end{pmatrix} = R\begin{pmatrix}\Omega_1 \\ \Omega_2\end{pmatrix};\begin{pmatrix}\bar{M}'_{\mu\nu} \\ M'_{\mu\nu}\end{pmatrix} = R\begin{pmatrix}\bar{M}_{\mu\nu} \\ M_{\mu\nu}\end{pmatrix};R = \begin{pmatrix}\cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta\end{pmatrix} \tag{33}$$

Owing some algebra, it is easy to prove that finally, all the scalars that can be built with the tensors (30), are functions of only one of them :

$$\rho = \left(\Omega_1^{\ 2} + \Omega_2^{\ 2}\right)^{1/2} \tag{34}$$

Here appears the second sense of the Rodichev *total curvature [20]* : this curvature is at the same time the principal *chiral invariant* which gives an electromagnetic sense to the total curvature of the space.

According to the first relation (32), ρ is the radius of a circle in the plane, called « chiral plane » [4], [5], [6], with the coordinates :

$$\Omega_1 = \rho\cos A,\Omega_2 = \rho\sin A \tag{35}$$

where A is a polar angle. The chiral transformation (29) is thus a rotation of an arbitrary angle ϑ in the *chiral plane* and the transformation (24) takes the elementary form :

$$A' = A + \vartheta \tag{36}$$

Now, if we look for a nonlinear chiral invariant equation, we find only one possible Lagrangian density, up to an arbitrary function $M\left(\rho^2\right)$ which has the dimension of a mass :

$$L = \frac{1}{2}\bar{\psi}\gamma_\mu\left[\partial_\mu\right]\psi - \frac{g}{\hbar c}\bar{\psi}\gamma_\mu\gamma_5 B_\mu\psi + \frac{1}{4}\frac{M\left(\rho^2\right)c}{\hbar} \tag{37}$$

The corresponding field equation is :

$$\gamma_\mu \left(\partial_\mu \psi - \frac{g}{\hbar c} \gamma_5 B_\mu \right) \psi + \frac{m\left(\rho^2\right) c}{2\hbar} \left(\Omega_1 - i\Omega_2 \gamma_5 \right) \psi = 0 \qquad (38)$$

where $m\left(\rho^2\right) = \dfrac{\partial M}{\partial\left(\rho^2\right)}$. So we have a general nonlinear equation of a massive nonlinear monopole which includes as particular cases all the known nonlinear spinorial equations (which were all written without external field) [4]. One can show that this equation is CPT and chiral invariant [6].

3. Another kind of nonlinear chiral invariant equation.

We have found a manner to find a nonlinear chiral invariant equation by adding an invariant term to the linear lagrangian, i.e. a function of the principal chiral invariant (34), which is the radius of a vector in the chiral plane $\left\{\Omega_1, \Omega_2\right\}$. Such a term does not depend on the angle A, it does not change the chirality - see (36) - and we have a nonlinear equation, *every solution* of which is chiral invariant.

Now we look for an equation of another kind, which is not chiral invariant, but in which we define *a class of invariant solutions* : *a class of monopole-solutions.*

Such solutions could be characterized by a zero-vector of the chiral plane $\left(\Omega_1 = \Omega_2 = 0\right)$: because the polar angle A of such a vector (and the so defined chiral phase) is now arbitrary ; the corresponding solutions of the equation do not depend on it any more and so we find the invariance again, *but only for these solutions*.

We shall start from a lagrangian which is not chiral invariant : the most simple choice is the Dirac Lagrangian with its linear mass term. And shall add a term $\left(\Omega_1^2 + \Omega_2^2\right)$ with a Lagrange parameter λ. This Lagrange parameter will select the solutions $\Omega_1^2 + \Omega_2^2 = 0$. But before going further, let us make a remark. The tensorial quantities (31) obey many relations, among which :

$$-J_\mu J_\mu = \Sigma_\mu \Sigma_\mu = \rho^2 = \Omega_1^2 + \Omega_2^2 \qquad (39)$$

This means that the electric current J_μ and the magnetic current Σ_μ have opposite norms : J_μ is timelike and Σ_μ spacelike. If $\rho^2 = 0$, they are both on the light-cone This is why I have called this problem : « *The Dirac equation on the light-cone* » [15], which means that both currents are isotropic. Now, a zero length vector has a no defined polar angle A which is thus arbitrary and it may be eliminated by a choice of gauge.

So, we start from the Lagrangian with a magnetic pseudo potential term :

$$L = \frac{1}{2}\overline{\psi}\gamma_\mu\left[\partial_\mu\right]\psi - \frac{g}{\hbar c}\overline{\psi}\gamma_\mu\gamma_5 B_\mu\psi - \frac{m_0 c}{\hbar}\overline{\psi}\psi + \frac{\lambda}{2}\left(\Omega_1^{\ 2} + \Omega_2^{\ 2}\right) \qquad (40)$$

A variation with respect to $\overline{\psi}$ gives the equation :

$$\gamma_\mu\left(\partial_\mu\psi - \frac{g}{\hbar c}\gamma_5 B_\mu\right)\psi - \frac{m_0 c}{\hbar}\psi + \lambda\left(\Omega_1 - i\Omega_2\gamma_5\right)\psi = 0 \quad (41)$$

but a variation with respect to λ gives : $\Omega_1^{\ 2} + \Omega_2^{\ 2} = 0$, which means :

$$\Omega_1 = \Omega_2 = 0 \qquad (42)$$

So that the term in λ is eliminated in (40). One could think that it remains the Dirac equation, but it is not so because we have the condition (42) which is implied by [4] :

$$\psi = e^{i\vartheta}\gamma_2\psi^* = e^{i\vartheta}\psi_c \qquad (43)$$

where ψ_c is the charge conjugated spinor and ϑ an arbitrary phase. Up to this phase, (42) is the condition of Majorana [15] and we can introduce this condition in the same manner, so that (40) takes the form :

$$\gamma_\mu\left(\partial_\mu\psi - \frac{g}{\hbar c}\gamma_5 B_\mu\right)\psi - \frac{m_0 c}{\hbar}e^{i\vartheta}\gamma_2\psi^* = 0 \qquad (44)$$

This is as to say the « magnetic Majorana equation » with the intcraction of a magnetic monopole. Of course, the equation is nonlinear. Nevertheless, if we introduce in (43) the Weyl transformation (27), this condition becomes (with the arbitrary phase ϑ) :

$$\xi = e^{i\vartheta}is_2\eta^*, \ \eta = -e^{i\vartheta}is_2\xi^* \qquad (45)$$

and, just like in the linear case (28), the equation is broken in two separated equations in ξ and η [15] :

$$\begin{aligned}
&\frac{1}{c}\frac{\partial\xi}{\partial t} - \mathbf{s}.\nabla\xi - i\frac{g}{\hbar c}\left(W + \mathbf{s}.\mathbf{B}\right)\xi - e^{i\vartheta}im_0 cs_2\xi^* = 0\\
&\frac{1}{c}\frac{\partial\eta}{\partial t} + \mathbf{s}.\nabla\eta + i\frac{g}{\hbar c}\left(W - \mathbf{s}.\mathbf{B}\right)\eta + e^{i\vartheta}im_0 cs_2\eta^* = 0
\end{aligned} \qquad (46)$$

The covariant derivatives are the same as in the equations (26) and (38) so that the chirality is linked once more with the torsion of the space.

Nevertheless, there is a great difference with the equations (28) : in the present case the equations are separated but, in virtue of (45), they are not independent. It is still more evident owing to the Weyl transformation (27), because the link between these equations is simply expressed by the equalities :

$$\Omega_1^{\;2} + \Omega_2^{\;2} = 0 \quad \Rightarrow \quad \xi^{+}\eta = 0 \tag{47}$$

III A POST SCRIPTUM ON TWISTED SPACES AND SPIN FLUIDS.

We have seen in the first part of this paper that, in an affine space, a geodesic rectangle is not closed : if we start from an initial point, following a geodesic line and if we try to go back to the starting point, along an infinitesimal curve, we miss the point and the image of a loop is in fact an arc of helicoid with a certain infinitesimal « thread ».

The interesting point is that, if *the space is twisted*, the gap between the intial and the final point (the « thread » of the helicoid) is of the *second ord*er with respect to the loop : the *order of the area of the former* ; but if the space has a symmetric connection, i.e. if it is not twisted, the gap is smaller, of the order of the curvature : at least the third order.

What happens with a twisted space is analogous to what happens in a s*pin fluid*, for instance in the fluid defined by the density of a quantum wave. A spin density, like the one defined by a solution of the Dirac equation is not at all the angular momentum due to the curl of a fluid droplet. Because the spin of a droplet dv in a fluid with a spin density is of the same order as dv itself, while the angular momentum of the same droplet in an ordinary fluid is of two orders higher.

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