On the torsion of the intrinsic spacetime

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ABSTRACT. We start from the relativistic invariance of the Dirac equation. That invariance necessarily implies to use the Clifford space algebra. That algebra allows to express the wave equation, to see more tensors, particularly four spacetime vectors forming an orthogonal basis of the spacetime. We can associate to the wave, in each point, a Lorentz dilation applying the spacetime tangent to an intrinsic spacetime manifold S_w into the observer's spacetime manifold S_{obs}. We calculate the torsion of the intrinsic spacetime manifold in the case of the plane wave. Contrarily to S_{obs}, S_w is not isotropic.

Résumé : Nous partons de l'invariance relativiste de l'équation de Dirac. Cette invariance relativiste implique nécessairement l'usage de l'algèbre de Clifford de l'espace physique. Celle-ci permet aussi d'écrire l'équation d'onde, de voir plus de tenseurs, particulièrement quatre vecteurs formant une base de l'espace-temps. On peut associer à l'onde, en chaque point de l'espace-temps, une dilatation de Lorentz appliquant l'espace tangent à une variété intrinsèque d'espace-temps S_w, sur la variété d'espace-temps de l'observateur S_{obs}. On calcule la torsion de la variété d'espace-temps intrinsèque. Cette variété, contrairement à notre espace-temps usuel, n'est pas isotrope.

1 - The Dirac equation

The special relativity was the main tool used by Louis de Broglie to get the idea of the wave linked to the motion of a particle [1]. The first attempt to give a relativistic wave equation for the electron, only with first order derivatives, was made by P. A. M. Dirac [2] and gave a lot of good results. The main ones were in the case of the H atom. More, the electron's spin was implicated by the wave equation, linking spin to relativity. It has been early understood, for geometrical reasons, that spin and torsion must also be linked. [1] As we want to study the relativistic invariance, we shall use here the Weyl's representation for the Dirac equation :

$$0 = [\gamma^\mu(\partial_\mu + iqA_\mu) + im]\psi \ ; \quad q = \frac{e}{\hbar c} \ ; \quad m = \frac{m_0 c}{\hbar} \tag{1}$$

$$\gamma_0 = \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \ ; \quad \psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \ ;$$

$$\gamma_j = -\gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \ , \quad j = 1,\ 2,\ 3. \tag{2}$$

$$I = \sigma_0 = \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ ; \quad \sigma_1 = -\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3}$$

$$\sigma_2 = -\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ ; \quad \sigma_3 = -\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Dirac equation was written so as to be relativistic invariant, but that invariance is far from any classical physics. To get that invariance it is necessary to associate to each event with coordinates $(x^0,\ x^1,\ x^2,\ x^3)$, $x^0 = ct$, the matrix

$$x = x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}. \tag{4}$$

Then we consider the set $SL(2,\mathbb{C})$ of the 2×2 complex matrices

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{5}$$

verifying

$$1 = \det(M) = \alpha\delta - \beta\gamma. \tag{6}$$

To each M is associated the transformation R defined by

$$R : x \mapsto x' = MxM^\dagger \tag{7}$$

which is a Lorentz transformation because

$$\begin{aligned} \det(x') &= ({x'}^0)^2 - ({x'}^1)^2 - ({x'}^2)^2 - ({x'}^3)^2 = \det(MxM^\dagger) \\ &= \det(M)\det(x)\det(M^\dagger) = |\det(M)|^2 \det(x) = \det(x) \\ &= (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \end{aligned} \tag{8}$$

[1] See for instance the postscript by G. Lochak in the end of his contribution here [3]

But it is not trivial that, with

$$x'^{\mu} = R^{\mu}_{\nu} x^{\nu} \tag{9}$$

we also get

$$R^0_0 > 0 \; ; \quad \det(R^{\nu}_{\mu}) = 1. \tag{10}$$

So R is an element of the restricted Lorentz group $\mathcal{L}^{\uparrow}_{+}$. The application f defined by

$$f : M \mapsto R \tag{11}$$

is an homomorphism from $SL(2,\mathbb{C})$ into $\mathcal{L}^{\uparrow}_{+}$ whose kernel is $\{\pm I\}$. f is not an isomorphism : These two groups have the same Lie algebra, and are too often identified, but they are actually different, since the kernel is not $\{I\}$. Now we use

$$\begin{aligned} \partial'_{\mu} &= \frac{\partial}{\partial x'^{\mu}} \; ; \quad \partial_{\nu} = R^{\mu}_{\nu} \partial'_{\mu} \\ \widehat{M} &= \begin{pmatrix} \delta^* & -\gamma^* \\ -\beta^* & \alpha^* \end{pmatrix} \; ; \; N = \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix} \end{aligned} \tag{12}$$

And the reward is the general relation

$$R^{\mu}_{\nu} \gamma^{\nu} = N^{-1} \gamma^{\mu} N. \tag{13}$$

which is true with any M in $SL(2,\mathbb{C})$ and for $\mu = 0,\ 1,\ 2,\ 3$. The formal invariance of the Dirac equation comes from the important assumption that under the Lorentz transformation R defined by the matrix M the wave ψ transforms as

$$\psi' = N\psi. \tag{14}$$

That gives

$$\begin{aligned} 0 &= [\gamma^{\nu}(\partial_{\nu} + iqA_{\nu}) + im]\psi \\ &= [\gamma^{\nu} R^{\mu}_{\nu}(\partial'_{\mu} + iqA'_{\mu}) + im]N^{-1}\psi' \\ &= [N^{-1}\gamma^{\mu} N(\partial'_{\mu} + iqA'_{\mu}) + im]N^{-1}\psi' \\ &= N^{-1}[\gamma^{\mu}(\partial'_{\mu} + iqA'_{\mu}) + im]\psi' \end{aligned} \tag{15}$$

Because one factor M is present in (14) whilst two factors M are present in $x' = MxM^{\dagger}$, the wave turns with half angles, a fact that never occurs in classical physics, and that is verified by many experiments. It is another reason to distinguish M and R, (7) and (14).

2 - Space algebra

What is used in (4) is nothing but the Clifford algebra of the physical space, Cl_3. It is not possible to get the relativistic invariance of the Dirac theory without that mathematical tool. Moreover, anything in the Dirac theory may be written with just that algebra [4].

The general element of the space algebra Cl_3 reads

$$u = s + \vec{v} + i\vec{w} + ip \tag{16}$$

where s is a scalar (real number), $\vec{v}$ is a vector, with three real components, $i\vec{w}$ is a pseudo-vector, $\vec{w}$ is an axial vector, and ip is a pseudo-scalar. As $i^2 = -1$, Cl_3 is a generalization of the complex field. If $(\sigma_1, \sigma_2, \sigma_3)$ is an orthonormal basis of the physical space, that is

$$\sigma_j \cdot \sigma_k = 0 \ , \ j \neq k \ ; \quad \sigma_j^2 = 1 \tag{17}$$

we can write any vector $\vec{v}$ as

$$\vec{v} = v^1\sigma_1 + v^2\sigma_2 + v^3\sigma_3 \tag{18}$$

If we use the Pauli representation (3) for the σ_j, and if we identify scalars and scalar matrices, the sum and the product of two terms in the space algebra is exactly the sum and the matrix product : Cl_3 may be identified to the $M_2(\mathbb{C})$ algebra, set of the 2×2 complex matrices. With

$$u = s + \vec{v} + i\vec{w} + ip = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{19}$$

and z^* being the complex conjugate of z, we shall need

$$u^\dagger = s + \vec{v} - i\vec{w} - ip = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} \tag{20}$$

$$\widehat{u} = s - \vec{v} + i\vec{w} - ip = \begin{pmatrix} \delta^* & -\gamma^* \\ -\beta^* & \alpha^* \end{pmatrix} \tag{21}$$

$$\overline{u} = s - \vec{v} - i\vec{w} + ip = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \tag{22}$$

To get the Dirac equation in Cl_3, we use (1) and (2) and we let

$$\nabla = \sigma^\mu \partial_\mu = \begin{pmatrix} \partial_0 - \partial_3 & -\partial_1 + i\partial_2 \\ -\partial_1 - i\partial_2 & \partial_0 + \partial_3 \end{pmatrix} = \partial_0 - \vec{\partial}$$

$$\vec{\partial} = \sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3 \tag{23}$$

$$\widehat{\nabla} = \partial_0 + \vec{\partial} = \begin{pmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{pmatrix} \tag{24}$$

$$\vec{A} = A^1\sigma_1 + A^2\sigma_2 + A^3\sigma_3 \ ; \quad A = A^\mu\sigma_\mu = A^0 + \vec{A} \tag{25}$$

So we get

$$\gamma^\mu\partial_\mu = \begin{pmatrix} 0 & \nabla \\ \widehat{\nabla} & 0 \end{pmatrix} \ ; \quad \gamma^\mu A_\mu = \begin{pmatrix} 0 & A \\ \widehat{A} & 0 \end{pmatrix}. \tag{26}$$

and the Dirac equation (1) is equivalent to the system

$$0 = \nabla\eta + iqA\eta + im\xi \tag{27}$$

$$0 = \widehat{\nabla}\xi + iq\widehat{A}\xi + im\eta \tag{28}$$

If we take the complex conjugate of that last equation, and if we multiply by $-i\sigma_2$ by the left, we get

$$0 = -i\sigma_2\widehat{\nabla}^*\xi^* + iqi\sigma_2\widehat{A}^*\xi^* + imi\sigma_2\eta^* \tag{29}$$

But we have

$$i\sigma_2\widehat{\nabla}^* = \nabla i\sigma_2 \ ; \quad i\sigma_2\widehat{A}^* = Ai\sigma_2. \tag{30}$$

Therefore (29) reads

$$0 = \nabla(-i\sigma_2\xi^*) + iqAi\sigma_2\xi^* + imi\sigma_2\eta^* \tag{31}$$

Now we let

$$\phi = \sqrt{2}\begin{pmatrix} \xi & -i\sigma_2\eta^* \end{pmatrix} = \sqrt{2}\begin{pmatrix} \xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^* \end{pmatrix}. \tag{32}$$

ϕ is a function of the space-time with value into the $M_2(\mathbb{C}) = Cl_3$ algebra and we get

$$\widehat{\phi} = \sqrt{2}\begin{pmatrix} \eta & -i\sigma_2\xi^* \end{pmatrix} = \sqrt{2}\begin{pmatrix} \eta_1 & -\xi_2^* \\ \eta_2 & \xi_1^* \end{pmatrix}. \tag{33}$$

As the Dirac equation (1) is equivalent to the system (27)-(28), and as (28) is equivalent to (31), the Dirac equation (1) is equivalent to

$$\nabla\begin{pmatrix} \eta & -i\sigma_2\xi^* \end{pmatrix} + iqA\begin{pmatrix} \eta & i\sigma_2\xi^* \end{pmatrix} + im\begin{pmatrix} \xi & i\sigma_2\eta^* \end{pmatrix} = 0 \tag{34}$$

As $\phi i\sigma_3 = i\sqrt{2}(\xi \quad i\sigma_2\eta^*)$, the Dirac equation (1) is equivalent to

$$\nabla\widehat{\phi} + qA\widehat{\phi}i\sigma_3 + m\phi i\sigma_3 = 0 \tag{35}$$

or, with $\sigma_{ij} = \sigma_i\sigma_j$,

$$\nabla\widehat{\phi} + qA\widehat{\phi}\sigma_{12} + m\phi\sigma_{12} = 0. \tag{36}$$

What is the form of the relativistic invariance here? (14) gives

$$\psi' = \begin{pmatrix}\xi' \\ \eta'\end{pmatrix} = N\psi = \begin{pmatrix}M & 0 \\ 0 & \widehat{M}\end{pmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix} = \begin{pmatrix}M\xi \\ \widehat{M}\eta\end{pmatrix}. \tag{37}$$

So (14) is equivalent to

$$\xi' = M\xi \ ; \quad \eta' = \widehat{M}\eta. \tag{38}$$

and is equivalent to

$$\phi' = M\phi \ ; \quad \phi' = \sqrt{2}\left(\xi' \quad -i\sigma_2\eta'^*\right) \tag{39}$$

And with

$$\nabla' = \sigma^\mu\partial'_\mu \tag{40}$$

we get, for any M :

$$\nabla = M^{-1}\nabla'\widehat{M} \ ; \quad A = M^{-1}A'\widehat{M}. \tag{41}$$

So the Dirac equation (36) gives

$$\begin{aligned}0 &= \nabla\widehat{\phi} + qA\widehat{\phi}\sigma_{12} + m\phi\sigma_{12} \\ &= M^{-1}\nabla'\widehat{M}\widehat{\phi} + qM^{-1}A'\widehat{M}\widehat{\phi}\sigma_{12} + m\phi\sigma_{12} \\ &= M^{-1}\left(\nabla'\phi' + qA'\phi'\sigma_{12} + m\phi'\sigma_{12}\right)\end{aligned} \tag{42}$$

That assures the invariance of the Dirac equation (36) under $SL(2,\mathbb{C})$.

3 - More tensors

As the wave ψ is a non-classical object, the tensorial densities linked to that wave have been early recognized and studied, particularly by

O. Costa de Beauregard [5] and Y. Takabayasi [6]. The main tensors, without derivatives, are

$$\begin{aligned}
\Omega_1 &= \overline{\psi}\psi \ ; \quad \overline{\psi} = \psi^\dagger\gamma_0 \\
J^\mu &= \overline{\psi}\gamma^\mu\psi \\
S^{\mu\nu} &= i\overline{\psi}\gamma^\mu\gamma^\nu\psi \\
K^\mu &= \overline{\psi}\gamma^\mu\gamma_5\psi \ ; \quad \gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\
\Omega_2 &= -i\overline{\psi}\gamma_5\psi
\end{aligned} \tag{43}$$

With 1+4+6+4+1=16 densities, that list is considered as complete, because the algebra generated by the Dirac matrices is 16-dimensionnal over $\mathbb{C}$. But the preceding densities are all real, and the dimension of the algebra, over $\mathbb{R}$, is 32. So it is just a coincidence, and we will see that there are more tensors, using the space algebra where we get

$$\Omega_1 + i\Omega_2 = \det(\phi) = \phi\overline{\phi} = \overline{\phi}\phi = 2\eta^\dagger\xi \tag{44}$$

$$J = J^\mu\sigma_\mu = \phi\phi^\dagger \tag{45}$$

$$S = S^{23}\sigma_1 + S^{31}\sigma_2 + S^{12}\sigma_3 + S^{10}i\sigma_1 + S^{20}i\sigma_2 + S^{30}i\sigma_3 = \phi\sigma_3\overline{\phi} \tag{46}$$

$$K = K^\mu\sigma_\mu = \phi\sigma_3\phi^\dagger \tag{47}$$

The relativistic behaviour of those tensors are straightforward : Ω_1 and Ω_2 are invariant since

$$\Omega'_1 + i\Omega'_2 = \det(\phi') = \det(M\phi) = \det(M)\det(\phi) = \det(\phi) = \Omega_1 + i\Omega_2. \tag{48}$$

J and K are vectors transforming as x :

$$J' = \phi'\phi'^\dagger = M\phi(M\phi)^\dagger = M\phi\phi^\dagger M^\dagger = MJM^\dagger \tag{49}$$

$$K' = \phi'\sigma_3\phi'^\dagger = M\phi\sigma_3(M\phi)^\dagger = M\phi\sigma_3\phi^\dagger M^\dagger = MKM^\dagger \tag{50}$$

S is a spacetime bivector, transforming as

$$S' = \phi'\sigma_3\overline{\phi}' = M\phi\sigma_3\overline{M\phi} = M\phi\sigma_3\overline{\phi}\ \overline{M} = MS\overline{M} \tag{51}$$

But the form itself of S and K, where only σ_3 is used, implies that there are in fact three vectors D_k and three bivectors S_k, defined by

$$D_k = \phi\sigma_k\phi^\dagger \tag{52}$$

$$S_k = \phi\sigma_k\overline{\phi}, \tag{53}$$

verifying, under a Lorentz transformation $R = f(M)$:

$$D'_k = M D_k M^\dagger \tag{54}$$

$$S'_k = M S_k \overline{M}. \tag{55}$$

With Ω_1, Ω_2, J, three D_k and three S_k, we have actually $1+1+4+(3\times 4)+(3\times 6)=36$ tensorial densities without derivative : The complex formalism is, by far, incomplete.

Evidently, anything in the Dirac theory may also be read with the Dirac matrices : ψ^t being the transposed matrix, and with

$$\tilde{\psi} = \psi^t \gamma_0 \gamma_2 \ ; \quad \check{\psi} = \psi^t \gamma_1 \gamma_3 \tag{56}$$

we get

$$\tilde{\psi}\gamma^\mu \psi = D_2^\mu - i D_1^\mu \tag{57}$$

$$\check{\psi}\gamma^\mu \gamma^\nu \psi = S_2^{\mu\nu} - i S_1^{\mu\nu} \tag{58}$$

But the tensoriality is not straighforward. It results from the fact that for any N in (12) we have

$$\gamma_0\gamma_2 N^{-1} = N^t \gamma_0 \gamma_2 \ ; \quad \gamma_1\gamma_3 N^{-1} = N^t \gamma_1 \gamma_3 \tag{59}$$

$$\tilde{\psi}' = \tilde{\psi} N^{-1} \ ; \quad \check{\psi}' = \check{\psi} N^{-1} \tag{60}$$

so we get

$$\tilde{\psi}'\gamma^\mu \psi' = R^\mu_\nu \tilde{\psi}\gamma^\nu \psi \tag{61}$$

$$\check{\psi}'\gamma^\mu \gamma^\nu \psi' = R^\mu_\rho R^\nu_\tau \check{\psi}\gamma^\rho \gamma^\tau \psi \tag{62}$$

We must also remark that $K = D_3$ and $J = D_0$. So we get four space-time vectors D_μ. These vectors form an orthogonal basis of the space-time, because we get :

$$\begin{aligned} 2D_\mu \cdot D_\nu &= D_\mu \widehat{D}_\nu + D_\nu \widehat{D}_\mu \\ &= \phi\sigma_\mu\phi^\dagger \widehat{\phi\sigma_\nu\phi^\dagger} + \phi\sigma_\nu\phi^\dagger \widehat{\phi\sigma_\mu\phi^\dagger} \\ &= \phi\sigma_\mu\phi^\dagger \widehat{\phi}\widehat{\sigma}_\nu\widehat{\phi}^\dagger + \phi\sigma_\nu\phi^\dagger \widehat{\phi}\widehat{\sigma}_\mu\widehat{\phi}^\dagger \end{aligned} \tag{63}$$

But $\widehat{\phi}^\dagger = \overline{\phi}$ and $\overline{\phi}\phi = \Omega_1 + i\Omega_2$ commutes with any element in Cl_3. It is the same for $\phi^\dagger\widehat{\phi} = \Omega_1 - i\Omega_2$, and we get

$$2D_\mu \cdot D_\nu = (\Omega_1 - i\Omega_2)\phi(\sigma_\mu\widehat{\sigma}_\nu + \sigma_\nu\widehat{\sigma}_\mu)\overline{\phi} \tag{64}$$

So we get

$$D_0 \cdot D_0 = (\Omega_1 - i\Omega_2)\phi\overline{\phi} = (\Omega_1 - i\Omega_2)(\Omega_1 + i\Omega_2) = \Omega_1^2 + \Omega_2^2 \qquad (65)$$

The Dirac theory calls ρ the invariant

$$\rho = \sqrt{\Omega_1^2 + \Omega_2^2} = |\det(\phi)| \qquad (66)$$

And the argument of the determinant is the Yvon-Takabayasi angle β :

$$\det(\phi) = \Omega_1 + i\Omega_2 = \rho e^{i\beta}. \qquad (67)$$

So we get

$$D_0 \cdot D_0 = \rho^2 \qquad (68)$$

We have $\widehat{\sigma}_0 = \sigma_0 = 1$ and $\widehat{\sigma}_k = -\sigma_k$, and with $k = 1,\ 2,\ 3$ we get

$$D_0 \cdot D_k = (\Omega_1 - i\Omega_2)\phi(-\sigma_k + \sigma_k)\overline{\phi} = 0 \qquad (69)$$

and we have $\sigma_k\widehat{\sigma}_k = -\sigma_k^2 = -1$, so we get

$$D_k \cdot D_k = -\rho^2 \qquad (70)$$

With $j = 1,\ 2,\ 3,\ k = 1,\ 2,\ 3,\ j \neq k$ we have $\sigma_j\widehat{\sigma}_k + \sigma_k\widehat{\sigma}_j = 0$, and we get

$$D_j \cdot D_k = 0 \qquad (71)$$

With the Minkowski metric :

$$g_{00} = 1\ ;\quad g_{11} = g_{22} = g_{33} = -1\ ;\quad g_{\mu\nu} = 0\ ,\ \mu \neq \nu \qquad (72)$$

We get

$$D_\mu \cdot D_\nu = g_{\mu\nu}\rho^2 \qquad (73)$$

Try to find that result with the Dirac matrices and you will understand why the space algebra is much simpler.

4 - Wave's geometry

It has been seen first by G. Lochak [7], next by D. Hestenes [8], that the ψ wave has a geometrical aspect, with a Lorentz rotation. This Lorentz rotation exists only where the invariant ρ is not null. In that case we can write

$$\phi = \sqrt{\rho}e^{i\frac{\beta}{2}}M. \qquad (74)$$

We get then

$$\overline{\phi} = \sqrt{\rho}e^{i\frac{\beta}{2}}\overline{M}$$
$$\rho e^{i\beta} = \phi\overline{\phi} = \sqrt{\rho}e^{i\frac{\beta}{2}}M\sqrt{\rho}e^{i\frac{\beta}{2}}\overline{M} = \rho e^{i\beta}M\overline{M} \tag{75}$$

Therefore $\overline{M} = M^{-1}$, $\det(M) = 1$ and M is an element of $SL(2,\mathbb{C})$. Generally M is called a Lorentz rotation, but M is not a Lorentz rotation, it is an element of the covering group of $\mathcal{L}_+^\uparrow$. We must distinguish M from $R = f(M)$.

The Yvon-Takabayasi angle β is the basis of the G. Lochak's theory of the magnetic monopole [9]. But what is ρ ? Has also ρ a geometrical meaning ? Contrarily to the statistical interpretation of D. Hestenes, we think that the answer is yes, ρ is a scale parameter. To see that, we come back to (7), but now we do not restrict M with (6), we consider any matrix M in Cl_3, that is any 2×2 complex matrix. We consider always the R transformation defined by (7) and f defined by (11). Now we get

$$\det(x') = \det(M)\det(x)\det(M^\dagger) = |\det(M)|^2\det(x) \tag{76}$$

With

$$\det(M) = re^{i\theta} \tag{77}$$

we get

$$\begin{aligned}(x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2 &= x'\overline{x}' = \det(x') \\ &= r^2\det(x) = r^2 x\overline{x} = r^2[(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2]\end{aligned} \tag{78}$$

So R is a transformation multiplying each space-time length by r. We call R a Lorentz dilation and r is called the ratio of the dilation. The main results, got for any M, are [10] :

$$R_0^0 > 0 \text{ if } M \neq 0 \tag{79}$$
$$\det(R_\mu^\nu) = r^4 \tag{80}$$
$$\ker(f) = \{M/M = e^{i\frac{\theta}{2}}I\} \tag{81}$$

We note Cl_3^* the multiplicative group made of the M matrices with $r \neq 0$, and $\mathcal{D}$ the group of the Lorentz dilations, always with $r \neq 0$. [2]

[2] If $r = 0$, M and R are not invertible, and we do not get the group structure. But physically that case may not be avoided : with most of the Darwin solutions for the H atom, exist places with $\rho = 0$.

The physical signification of (79) is that, except the null case, any dilation R conserves the time's arrow. So the Dirac wave is compatible with an oriented time. The physical consequence of (80) is that $\det(R^{\nu}_{\mu}) > 0$ if $r \neq 0$, so R conserves the space-time orientation, and as R conserves the time's orientation, R conserves the space orientation : the Dirac wave is compatible with an oriented space.

Now we must examine (81). When the Dirac theory looks at the relativistic invariance, it is always with $\det(M) = 1$, that is $\theta = 0 \mod 2\pi$ and $r = 1$. In that case $\frac{\theta}{2} = 0$ or $\frac{\theta}{2} = \pi$, and $e^{i\frac{\theta}{2}} = \pm 1$. Therefore the $\{\pm I\}$ kernel, always present in quantum mechanics, is exactly what remains of the chiral gauge group used by G. Lochak for the monopole's wave equation, when we impose to use only M matrices with $\det(M) = 1$. We can also say that the chiral gauge was hidden in all the quantum theory. We can also see how significant it is to distinguish the group Cl_3^*, which includes the chiral gauge and is a 8-dimensionnal Lie group, from the $\mathcal{D}$ group, which is independent of the chiral gauge and is only a 7-dimensionnal Lie group. Cl_3^* appears as the main group, even if $\mathcal{D}$ is the geometrical group. We must not forget that it is the wave which propagates, which interferes, not the tensors or the Lorentz dilation. The wave has one more parameter, the Yvon-Takabayasi angle. It disappears from the $\mathcal{D}$ group, because the Lorentz dilation acts only on the spacetime vectors, and R. Boudet has understood that the Yvon-Takabayasi angle acts on the bivectors [11] : That can be seen with the formulas (51) or (55).

It is possible to get invariant laws under Cl_3^*, in the Dirac equation, but also in all the electromagnetic theory, with or without monopoles, with or without photons [12].

5 - Two spacetime manifolds.

There is no difference between the 2×2 complex matrix M, and the 2×2 complex matrix ϕ of the electron's wave. More precisely a Dirac wave is a function from the spacetime with value into the set Cl_3. Moreover there is no difference between the product $M'M$ which gives the product of two Lorentz dilations, and the product $M\phi$ in (39), which gives the transformation of the wave under a Lorentz dilation. Therefore we can associate to the ϕ wave, in each point of the spacetime, a Lorentz dilation. As this dilation varies with space and time, it must be seen as local, applying the local tangent spacetime into the observer's spacetime. We shall name $D = f(\phi)$ this dilation and y the general element of the tangent spacetime :

$$D : y \mapsto x = \phi y \phi^\dagger \tag{82}$$

The tangent spacetime varies from a point to another and may be seen as the tangent space to a spacetime manifold S_w, linked to the wave. So we must consider **two spacetime manifolds** and we note S_{obs} the manifold relative to the observers. D maps S_w into S_{obs}. If we use a Lorentz dilation $R = f(M)$, that is if we change from the observer of x to the observer of $x' = MxM^\dagger$, we get

$$\begin{aligned} x' &= MxM^\dagger = M(\phi y \phi^\dagger)M^\dagger \\ &= (M\phi)y(M\phi)^\dagger = \phi' y \phi'^\dagger \end{aligned} \tag{83}$$

Therefore we get $x = D(y)$ and $x' = D'(y)$ with $D' = R \circ D$, and the same y : the y term does not change, either seen by the observer of x or the observer of x'. This y is intrinsic to the wave and independent of the moving observer. We can call S_w the intrinsic manifold.

Now we let

$$x = x^\mu \sigma_\mu \ ; \quad y = y^\mu \sigma_\mu \ ; \quad D_\mu = D^\nu_\mu \sigma_\nu \tag{84}$$

and we get

$$\begin{aligned} x^\nu \sigma_\nu &= x = \phi y \phi^\dagger = \phi y^\mu \sigma_\mu \phi^\dagger \\ &= y^\mu \phi \sigma_\mu \phi^\dagger = y^\mu D_\mu = y^\mu D^\nu_\mu \sigma_\nu \end{aligned} \tag{85}$$

$$x^\nu = D^\nu_\mu y^\mu \ ; \quad \boldsymbol{\partial}_\nu = \frac{\partial}{\partial y^\nu} = D^\mu_\nu \partial_\mu. \tag{86}$$

So the components of the four vectors D_μ are the components of the matrix D^ν_μ of the Lorentz dilation D. We can apply to D and D^ν_μ all the results got with R and R^ν_μ : we have

$$D^0_0 > 0 \text{ if } \phi \neq 0 \tag{87}$$

$$\det(D^\nu_\mu) = \rho^4 \ ; \quad \det(D^\nu_\mu) > 0 \text{ if } \rho \neq 0. \tag{88}$$

So the spacetime manifold S_w has, at each point, the same time's arrow and the same space orientation.

6 - Connection of the intrinsic manifold

To compute the connection of the intrinsic manifold [11] we shall use the mobil orthogonal basis (D_0, D_1, D_2, D_3). We let

$$dx = dy^\nu D_\nu \tag{89}$$

$$dD_\mu = \Gamma^\beta_{\mu\nu} dy^\nu D_\beta. \tag{90}$$

If $\rho \neq 0$ we get

$$\begin{aligned} dx = dx^\mu \sigma_\mu = D^\mu_\nu \sigma_\mu dy^\nu = D_\nu dy^\nu \\ D_\nu = \phi\sigma_\nu\phi^\dagger = D^\mu_\nu \sigma_\mu \ ; \quad \sigma_\mu = (D^{-1})^\beta_\mu D_\beta \end{aligned} \tag{91}$$

We shall use the Lorentz dilation $\overline{D}$ verifying

$$\overline{D}(x) = \overline{\phi} x \widehat{\phi} \tag{92}$$

and we get

$$D^{-1}(x) = \rho^{-2}\overline{D}(x) \tag{93}$$

$$\begin{aligned} dD_\mu &= \boldsymbol{\partial}_\nu(D_\mu)dy^\nu = \boldsymbol{\partial}_\nu(D^\xi_\mu \sigma_\xi)dy^\nu = \boldsymbol{\partial}_\nu(D^\xi_\mu)\sigma_\xi dy^\nu \\ &= \boldsymbol{\partial}_\nu(D^\xi_\mu)(D^{-1})^\beta_\xi D_\beta dy^\nu = \Gamma^\beta_{\mu\nu} D_\beta dy^\nu. \end{aligned} \tag{94}$$

Therefore the connection verifies

$$\Gamma^\beta_{\mu\nu} = \boldsymbol{\partial}_\nu(D^\xi_\mu)(D^{-1})^\beta_\xi \ ; \quad \boldsymbol{\partial}_\nu = D^\tau_\nu \partial_\tau. \tag{95}$$

Using $\overline{D}$ we get

$$\Gamma^\beta_{\mu\nu} = \rho^{-2}\boldsymbol{\partial}_\nu(D^\xi_\mu)\overline{D}^\beta_\xi \ ; \quad \boldsymbol{\partial}_\nu = D^\tau_\nu \partial_\tau. \tag{96}$$

As $\overline{D}^0_0 = D^0_0$ and $\overline{D}^0_j = -D^j_0$ we get

$$\Gamma^0_{0\nu} = \Gamma^1_{1\nu} = \Gamma^2_{2\nu} = \Gamma^3_{3\nu} = \boldsymbol{\partial}_\nu[\ln(\rho)] = D^\mu_\nu \partial_\mu[\ln(\rho)]. \tag{97}$$

As $\overline{D}^j_0 = -D^0_j$ and $\overline{D}^k_j = D^j_k$ we get

$$\Gamma^j_{0\nu} = \Gamma^0_{j\nu} \ , \quad j = 1,\ 2,\ 3. \tag{98}$$

$$\Gamma^j_{k\nu} = -\Gamma^k_{j\nu} \ , \quad j = 1,\ 2,\ 3 \ , \quad k = 1,\ 2,\ 3, \quad k \neq j. \tag{99}$$

To compute the connection we need

$$\mathcal{S}_{(k)} + i\mathcal{S}'_{(k)} = \frac{\nabla S_k^\dagger}{\det(\phi)^\dagger} \tag{100}$$

$$\mathcal{A}_{(k)} + i\mathcal{A}'_{(k)} = \frac{A S_k^\dagger}{\det(\phi)^\dagger} \tag{101}$$

$$\tau = \frac{1}{2}[(\widehat{\nabla}\phi)\overline{\phi} - \dot{\widehat{\nabla}}\phi\dot{\overline{\phi}}] \tag{102}$$

$$\mathcal{T} + i\mathcal{T}' = \frac{\tau}{\det(\phi)} \tag{103}$$

where the points in (102) indicate on what acts the differential operator. Using the linear Dirac equation we get

$$\Gamma^0_{1\nu} = D_\nu \cdot [\mathcal{S}_{(1)} - 2q\mathcal{A}_{(2)}] + 2m\Omega_1\delta^2_\nu \tag{104}$$

$$\Gamma^2_{3\nu} = -D_\nu \cdot [\mathcal{S}'_{(1)} - 2q\mathcal{A}'_{(2)}] - 2m\Omega_2\delta^2_\nu \tag{105}$$

$$\Gamma^0_{2\nu} = D_\nu \cdot [\mathcal{S}_{(2)} + 2q\mathcal{A}_{(1)}] - 2m\Omega_1\delta^1_\nu \tag{106}$$

$$\Gamma^3_{1\nu} = -D_\nu \cdot [\mathcal{S}'_{(2)} + 2q\mathcal{A}'_{(1)}] - 2m\Omega_2\delta^1_\nu \tag{107}$$

$$\Gamma^0_{3\nu} = D_\nu \cdot \mathcal{S}_{(3)} - 2m\Omega_2\delta^0_\nu \tag{108}$$

$$\Gamma^1_{2\nu} = -D_\nu \cdot [\mathcal{S}'_{(3)} + 2qA] - 2m\Omega_1\delta^0_\nu \tag{109}$$

$$\Gamma^0_{0\nu} = D_\nu \cdot [-2\mathcal{T} + 2q\mathcal{A}'_{(3)}] - 2m\Omega_2\delta^3_\nu \tag{110}$$

As this connection is not symmetric, the intrinsic manifold has a non vanishing torsion, and this torsion includes mass terms.

Another important aspect is the fact that the third axis plays a privileged role. That can be seen with the wave equation (36) : The intrinsic manifold is not isotropic. All that we know from mechanics and astronomy indicates that the spacetime S_{obs} is isotropic, presents no privileged direction. But the intrinsic manifold S_w is not identical to the observer's manifold and may have different properties. One may be isotropic and the other not. We suppose that the existence of a privileged direction in the manifold linked to the wave of an electron is the reason of the existence of three kinds of wave equation : (36) is one of three similar equations

$$\begin{aligned}
\nabla\widehat{\phi} + qA\widehat{\phi}\sigma_{23} + m\phi\sigma_{23} &= 0 \\
\nabla\widehat{\phi} + qA\widehat{\phi}\sigma_{31} + m'\phi\sigma_{31} &= 0 \\
\nabla\widehat{\phi} + qA\widehat{\phi}\sigma_{12} + m''\phi\sigma_{12} &= 0
\end{aligned} \tag{111}$$

With the first equation, the first axis in S_w is privileged, with the second equation, the second axis in S_w is privileged, and with the third equation, which is the Dirac equation, the third axis is privileged. We can suppose that the three kinds of leptons, electrons, muons, tauons, come from that. But we lack a good theory to understand how two Dirac waves, with or without the same privileged direction, interact.

7 - Torsion in the case of a plane wave

The simpler case to solve the Dirac equation is the plane wave without electromagnetic interaction ($A = 0$). The Dirac equation reads now

$$\nabla\widehat{\phi} + m\phi\sigma_{12} = 0 \tag{112}$$

We use a plane wave verifying

$$\phi = \phi_0 e^{-\varphi\sigma_{12}} \ ; \quad \varphi = mv_\mu x^\mu \ ; \quad v = \sigma^\mu v_\mu. \tag{113}$$

where the velocity v and ϕ_0 are fixed terms. We get

$$\nabla\widehat{\phi} = \sigma^\mu\partial_\mu(\widehat{\phi}_0 e^{-\varphi\sigma_{12}}) = -mv\widehat{\phi}\sigma_{12}. \tag{114}$$

Therefore (112) is equivalent to

$$\phi_0 = v\widehat{\phi}_0 \tag{115}$$

which is equivalent to

$$\widehat{\phi}_0 = \widehat{v}\phi_0 \tag{116}$$

and implies

$$\phi_0 = v(\widehat{v}\phi_0) = v\widehat{v}\phi_0 = v\cdot v\phi_0. \tag{117}$$

Therefore if ϕ_0 is invertible we must take

$$1 = v\cdot v = v_0^2 - \vec{v}^{\,2} \tag{118}$$

$$v_0^2 = 1 + \vec{v}^{\,2} \ ; \quad v_0 = \pm\sqrt{1+\vec{v}^{\,2}}. \tag{119}$$

which is the relativistic relation for the velocity of the particle. We get also

$$D_0 = \phi\phi^\dagger = \phi_0\phi_0^\dagger \tag{120}$$

Therefore D_0 is fixed and $\partial_\mu(D_0^\nu) = 0$. It is the same for D_3

$$D_3 = \phi_0\sigma_3\phi_0^\dagger \tag{121}$$

D_1 and D_2, on the contrary, are variable. We let

$$d_1 = \phi_0\sigma_1\phi_0^\dagger \ ; \quad d_2 = \phi_0\sigma_2\phi_0^\dagger \tag{122}$$

which gives

$$\begin{aligned} D_1 &= \cos(2\varphi)d_1 + \sin(2\varphi)d_2 \\ D_2 &= -\sin(2\varphi)d_1 + \cos(2\varphi)d_2 \end{aligned} \tag{123}$$

We also have

$$D_0 = \phi_0\phi_0^\dagger = v\widehat{\phi}_0\phi_0^\dagger = v(\Omega_1 - i\Omega_2) \tag{124}$$

But D_0 is a vector, so we get

$$\Omega_2 = 0 \ ; \quad D_0 = v\Omega_1. \tag{125}$$

And $D_0^0 > 0$, so we have two cases, one with positive energy

$$\Omega_1 > 0 \ ; \quad v_0 = \sqrt{1 + \vec{v}^2} \tag{126}$$

the other with negative energy

$$\Omega_1 < 0 \ ; \quad v_0 = -\sqrt{1 + \vec{v}^2} \tag{127}$$

In any case as D_0 and D_3 are fixed we get

$$\boldsymbol{\partial}_\nu(D_0^\xi) = \boldsymbol{\partial}_\nu(D_3^\xi) = 0 \tag{128}$$

$$\Gamma^\beta_{0\nu} = \Gamma^\beta_{3\nu} = 0. \tag{129}$$

With D_1 and D_2 we get

$$\partial_\tau(D_1^\xi) = \partial_\tau[\cos(2\varphi)d_1^\xi + \sin(2\varphi)d_2^\xi] = 2mv_\tau D_2^\xi$$

$$\partial_\tau(D_2^\xi) = \partial_\tau[-\sin(2\varphi)d_1^\xi + \cos(2\varphi)d_2^\xi] = -2mv_\tau D_1^\xi$$

$$\boldsymbol{\partial}_\nu(D_1^\xi) = D_\nu^\tau\partial_\tau(D_1^\xi) = 2mD_\nu^\tau v_\tau D_2^\xi = 2m(D_\nu \cdot v)D_2^\xi \tag{130}$$

$$\boldsymbol{\partial}_\nu(D_2^\xi) = D_\nu^\tau\partial_\tau(D_2^\xi) = -2mD_\nu^\tau v_\tau D_1^\xi = -2m(D_\nu \cdot v)D_1^\xi \tag{131}$$

But (125) implies

$$D_\nu \cdot v = \frac{1}{\Omega_1} D_\nu \cdot D_0 = \Omega_1 \delta^0_\nu. \tag{132}$$

Therefore we get

$$\Gamma^\beta_{11} = \Gamma^\beta_{12} = \Gamma^\beta_{13} = \Gamma^\beta_{21} = \Gamma^\beta_{22} = \Gamma^\beta_{23} = 0. \tag{133}$$

With (130) we get

$$\Gamma^\beta_{10} = \frac{2m}{\Omega_1} D^\xi_2 \overline{D}^\beta_\xi \ ; \quad \Gamma^\beta_{20} = -\frac{2m}{\Omega_1} D^\xi_1 \overline{D}^\beta_\xi \tag{134}$$

which gives

$$\begin{aligned}
\Gamma^2_{10} &= \frac{2m}{\Omega_1} D^\xi_2 \overline{D}^2_\xi = \frac{2m}{\Omega_1}(D^0_2 \overline{D}^2_0 + D^1_2 \overline{D}^2_1 + D^2_2 \overline{D}^2_2 + D^3_2 \overline{D}^2_3) \\
&= \frac{2m}{\Omega_1}(-D^0_2 D^0_2 + D^1_2 D^1_2 + D^2_2 D^2_2 + D^3_2 D^3_2) \\
&= \frac{2m}{\Omega_1}(-D_2 \cdot D_2) = 2m\Omega_1.
\end{aligned} \tag{135}$$

We get also

$$\begin{aligned}
\Gamma^0_{10} &= \frac{2m}{\Omega_1}(D_2 \cdot D_0) = 0 \\
\Gamma^3_{10} &= \frac{2m}{\Omega_1}(-D_2 \cdot D_3) = 0 \\
\Gamma^1_{10} &= \frac{2m}{\Omega_1}(-D_2 \cdot D_1) = 0
\end{aligned} \tag{136}$$

Similarly for the Γ^β_{20} we get

$$\Gamma^1_{20} = -2m\Omega_1 \ ; \quad \Gamma^0_{20} = \Gamma^2_{20} = \Gamma^3_{20} = 0. \tag{137}$$

To resume, amongst the 64 $\Gamma^\beta_{\mu\nu}$ terms, 62 terms are zero. Two terms are not zero :

$$\Gamma^2_{10} = -\Gamma^1_{20} = 2m\Omega_1. \tag{138}$$

Therefore the torsion has two interesting components :

$$\frac{1}{2}(\Gamma^2_{10} - \Gamma^2_{01}) = m\Omega_1 \tag{139}$$

$$\frac{1}{2}(\Gamma^1_{20} - \Gamma^1_{02}) = -m\Omega_1 \tag{140}$$

As the non vanishing $\Gamma^{\beta}_{\mu\nu}$ terms are fixed, the curvature tensor cancels. So we can see that the intrinsic manifold S_w linked to a plane wave of the Dirac equation is without curvature, but with a fixed torsion linked to the mass term. It is easy to predict the torsion coming with a plane wave in the case of the first equation (111) :

$$\frac{1}{2}(\Gamma^2_{20} - \Gamma^2_{02}) = m\Omega_1 \tag{141}$$

$$\frac{1}{2}(\Gamma^1_{30} - \Gamma^1_{03}) = -m\Omega_1 \tag{142}$$

and in the case of the second equation (111) :

$$\frac{1}{2}(\Gamma^2_{30} - \Gamma^2_{03}) = m\Omega_1 \tag{143}$$

$$\frac{1}{2}(\Gamma^1_{10} - \Gamma^1_{01}) = -m\Omega_1 \tag{144}$$

In the three cases, the torsion is in the spin plane.

Concluding remarks.

The Dirac theory may be read with two frames. We have used here the Cl_3 frame. Nearly eighty years ago, Dirac used a more complicated frame, with both the Pauli algebra Cl_3 and new 4×4 complex matrices. We have presented these two frames so as to legitimate the use of our Cl_3 frame.

But we think that these two frames are not at all equivalent. The classical frame extends the $\mathcal{L}^{\uparrow}_{+}$ invariance to the total Lorentz group, with the P and T transformations, forgetting completely that the theory is not invariant under the restricted Lorentz group $\mathcal{L}^{\uparrow}_{+}$, but under its covering group $SL(2,\mathbb{C})$. The Cl_3 frame extends the true invariance under $SL(2,\mathbb{C})$ to an invariance under Cl_3^*, which conserves the time's arrow and the space's orientation. The P and T transformations are unknown here, and that is appropriate with the experimental facts concerning the P and T violations by weak interactions.

The difference between the two frames may also be seen with the tensors of the theory. $K = D_3$ is here a vector which transforms as the three other D_μ. With the Dirac matrices, the same K is seen as the dual vector of an antisymmetric tensor. If we do not restrict the invariance group to $r = 1$, (7) and (9) imply that R^{ν}_{μ} contains a r^2 factor. So an antisymmetric tensor with rank 3 has a r^6 factor. It is the same with the bivectors which have a r^4 factor with the old formalism, and only a r^2 factor with (53). The old formalism was not able to see J and K as two vectors of an orthogonal basis [3] , and so was not able to see all the geometry linked to the wave, the intrinsic spacetime manifold and its torsion.

The "internal symetries" developed to understand modern physics may be linked to the symetries of that intrinsic spacetime. If it is true, then the gravitation, which is linked to the geometry of S_{obs}, is necessarily outside of those internal symetries.

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