

Mass, Action and Non Inertia

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ABSTRACT. Assimilated to inertia in Newton's law $\mathbf{F} = m\mathbf{a}$ and to energy in Einstein's formula $E = mc^2$, mass becomes a frequency with de Broglie's relation $h\nu = mc^2$. These very different aspects of mass can be reconciled if one comes back to the (Maupertuis) concept of action and asks for its physical meaning. We show that in the simple case of free colliding particles, action and mass are both related to the general notion of non inertial motion. In special relativity, this approach strongly suggests that the mass of a particle measures the frequency of non inertia of its internal processes. This idea, which is a reversal of the Newtonian conception, is supported at the quantum level by a simple analysis of the Dirac equation.

1 Introduction: What is inertial mass?

What is mass? This question has often been discussed in pedagogical papers (see e.g. [1]) or in historical reviews (see e.g. [2]), with the principal aim of distinguishing inertial mass, which enters for example Newton's law $\mathbf{F} = m\mathbf{a}$ where it characterizes the response (acceleration) of a body to a given external force, and gravitational masses, which are associated either to weight or to the source of gravitation. This distinction is important, but one must acknowledge that the fundamental question addressed to physicists in this field is why the ratio of inertial and gravitational masses is universal, suggesting that these masses are identical. Although we have it in mind, the subject of our paper is more restricted. We re-examine the notion of inertia and simply ask: is inertial mass what classical physics tells us? The Newtonian definition, which is still true in special relativity (SR) if one specifies

that all quantities are measured in the instantaneous rest frame of the body, has survived for ages in the physics community. Of course it has been enriched by Einstein's discovery that the inertial mass of a body is connected to its rest energy by the famous relation $E = mc^2$. As emphasized in most textbooks on relativity this relation tells that mass and energy at rest are identical concepts and allows to precise what mass is made of: the mass of a particle has different contributions among which the masses of its constituents and their kinetic and potential energies (divided by c^2).

But is this equivalence between inertia and energy the end of the story of inertial mass? Certainly not, because the quantum revolution forces us to think of energy in another way. Indeed Quantum Mechanics (QM) involves in a fundamental way the Planck constant h (or $\hbar = h/2\pi$). Similarly to $E = mc^2$, the Einstein relation $E = h\nu$ between the energy and the frequency of the light quanta, which is generalized¹ by the Schrödinger equation (SE) $H\psi = i\hbar\partial_t\psi$, tells us that energy and frequency must also be considered as two identical concepts. Then, since mass is energy, mass must be a frequency. At a dimensional level, this frequency is well known; it is nothing but the inverse of the Compton time τ_c associated with the Compton length $\lambda_c = c\tau_c = h/mc$ which Compton introduced in his analysis of the diffusion of X – rays by electrons. More generally, in relativistic quantum field theory (RQFT), \hbar like c being considered as dimensionless, mass and frequency have the same dimension. In the standard model, which accounts for most present particle interactions, masses are generated by the vacuum values of boson fields whose dimension is also $[T]^{-1}$ (like the electromagnetic potential vector).

However the above identification of the concepts of mass and frequency based on a dimensional argument is unsatisfactory. One would like to have a more intuitive and why not naïve way of talking of mass as frequency. For

¹ Let $\{\psi_\alpha\}$ be an arbitrary fixed orthonormal basis, the decomposition of any quantum state (even non stationary) satisfying the SE leads to the relation

$$\hbar \sum_\alpha |c_\alpha(t)|^2 \dot{\phi}_\alpha = (\psi, H\psi) \text{ which one can write whatever the basis } \hbar \langle \dot{\nu} \rangle = \langle H \rangle.$$

this, one must remember that h is a natural unit of action, as Planck already noted in 1900 long before the mathematical development of QM. Therefore a new look on inertial mass, even within classical physics (or better prequantum physics), can emerge from a physical insight about the concept of action. This implies than one stops to consider it as a pure mathematical quantity, and that (even in teaching) one gives to this notion in mechanics a central role (like that of entropy in thermodynamics). Restricting ourselves to the simple but not trivial case of free particles, for which the action is proportional to the mass, we shall show that there is a link between action and non inertial phenomena. Within Galilean relativity this link agrees with the traditional Newtonian idea of mass as inertia, but within SR (Einsteinian relativity), another interpretation emerges. According to it, the mass of a point like particle is a screen variable which is related to the non inertial processes which are internal to the particle: in the rest frame of the particle, the mass can be interpreted as the frequency of non inertia of these processes. This identification agrees with the idea that the mass of a particle is determined by its internal structure, independently of any precise model. From an epistemological point of view, we shall see that it also formulates in a new way the connection between time (frequency) and non inertial phenomena, as well as the question of what is inertial and what is not.

Let us emphasize that the idea of a relation between mass and action is not new. Up to our knowledge, it has occurred in the reflexion of physicists at least at two historical occasions. The first one was at the very birth of the idea of action. After having defined in 1744 the quantity of action by the product $v l$ of the speed of a body times the distance which it travels (in order to reconcile mechanical and optical laws of nature), Maupertuis added in 1746 that, when several bodies are present, one must take into account their mass (*i.e.* consider the product $m v l$). He then recovered in a simple case the conservation law (CL) of momentum in collisions from a minimum principle (“*Nature saves the quantity of action*”). The second one occurred at the beginning of the 20th century when physicists worked on an electromagnetic origin of mass [2,3]. In 1906, Poincaré [4] deduced the relativistic lagrangian $-m\sqrt{1-v^2}$ (he took $c=1$) from the invariance of the electromagnetic action of an extended charge under Lorentz transformations (LT) (see e.g. [5,6]). As Poincaré accompanied these transformations with dilations $t' = \gamma t$, $r' = \gamma r$, and discovered that they also leave the action invari-

ant, all quantities in his paper transform in the way prescribed by RQFT (where $[\hbar] = [c] = 1$)². If one has in mind that physics is born with Galilee's discovery of inertial frames and that most fundamental processes are described through least action principles (LAP), our proposition that action is related to the non inertia of these processes is not surprising from an historical point of view.

In section 2 we briefly recall how mass, inertia and energy occur in Newtonian and Einsteinian physics. Section 3 is concerned with the LAP for free classical particles. We show how the notions of action and mass are linked to that of non inertial motion and allow to derive in a simple way the CL of momentum and energy in elastic and inelastic collisions. It puts forward the importance of the proper time τ and provides a better understanding of the "non relativistic" lagrangian (kinetic – potential energies). The connection between action, or rather its opposite $mc^2\tau$ which we call "activity", and the non inertia of the internal dynamics of a particle, is introduced in section 4 within the historical context of prequantum physics and discussed in the light of present physics. Mass appears as a frequency of internal non inertia and is the inverse of Compton time. This image is justified by a simple analysis of the Dirac equation (DE) in Section 5.

2 Mass, inertia and energy in classical physics

Before briefly comparing Newtonian and Einsteinian physics it is important to recall that both need the introduction of inertial frames (solid bodies in inertial motion) and both are relativistic. The importance of these frames as reference frames in relation with symmetry properties and the invariance of physical laws was first noticed by Galilee. Galilean relativity simply differs from SR by the transformations laws of the spacetime interval (T, \mathbf{R}) between two events $(T = t_2 - t_1, \mathbf{R} = \mathbf{r}_2 - \mathbf{r}_1)$ when a frame R' moves with respect to R with velocity \mathbf{V} . They read $T' = T$ and $\mathbf{R}' = \mathbf{R} - \mathbf{V}T$ in the former, and lead to $\mathbf{v}' = \mathbf{v} - \mathbf{V}$ for the change of velocity; they keep

² Without knowing QM, he obtained for example $A' = l^{-1}A$ for the potential vector and $W' = l^{-1}W$ for the energy. But although he gave a prominent role to the least action principle in physics, Poincaré proposed no physical meaning to the concept of action.

$T^2 - \mathbf{R}^2 c^{-2}$ invariant in the latter and imply that c is an invariant velocity (we shall not need the explicit form of the LT). It is well known that Galilean relativity is a particular (limit) case of SR ($v \ll c, V \ll c$), but we insist that this remark implies that the understanding of Newtonian “everyday” physics must benefit from the knowledge of Einsteinian physics.

In Newtonian physics the inertial mass is often introduced through the relation $m\mathbf{a} = \mathbf{F}$ between the acceleration of a body and the external force. \mathbf{F} being fixed, the greater the mass is, the less the inertial motion of the body is modified, *i.e.* the less its velocity changes (in a given time interval). Another introduction which leads to the same picture of mass as inertia is the relation

$$m_1 \Delta \mathbf{v}_1 = -m_2 \Delta \mathbf{v}_2 \quad (1)$$

for an isolated system of two particles. Its advantage is that it avoids to speak of the notion of force (a concept whose interest is the description of interactions), and that it introduces a more fundamental quantity, the momentum $\mathbf{p} = m\mathbf{v}$ and its CL. In the following we restrict ourselves to the study of collisions of (approximately) point like particles. If the indexes i and f denote the initial and final particles in the collision, the CL of momentum reads:

$$\Delta \mathbf{p} = \sum \mathbf{p}_f - \sum \mathbf{p}_i = \mathbf{0}. \quad (2)$$

It is invariant (frame independent) provided that mass is also conserved (Lavoisier’s law)

$$\Delta m = \sum m_f - \sum m_i = 0. \quad (3)$$

As a consequence of these CL, the variation $\Delta K = \sum K_f - \sum K_i = 0$ of the kinetic energy in a collision (with $K = m\mathbf{v}^2/2$ for each particle) is also invariant. It is zero for elastic collisions but non zero for inelastic ones, as for example those involved in chemical reactions. One recovers (and verifies experimentally) a CL for energy by attributing to each i or f particle a proper

(internal) energy $U_{i,f}$ which up to an additive constant is its binding energy.

Writing $E_{i,f} = K_{i,f} + U_{i,f}$, the CL of energy reads:

$$\Delta E = \sum E_f - \sum E_i = 0 \quad \text{or} \quad \Delta K = -\Delta U. \quad (4)$$

(The change in kinetic energy comes from that of the binding energies). U is like m frame independent, but in Newtonian physics there is no relation between them.

In Einsteinian physics the energy and the momentum of a particle are

$$E = \gamma mc^2, \quad \mathbf{p} = \gamma m \mathbf{v}, \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \quad (5)$$

and the invariance of their CL is ensured by the property that $(E, c\mathbf{p})$ transforms linearly like a quadrivector (*i.e.* like (cT, \mathbf{R})). Inertial mass is generally introduced, not directly from the idea of inertia, but by the remark that one recovers $\mathbf{p} = m\mathbf{v}$ in the Galilean limit. A new picture of the inertial mass comes from the relation $E = mc^2$ for $v = 0$ which allows to define the kinetic energy by $K = E - mc^2$ ($= mv^2/2$ in the Galilean limit). Then the CL of energy reads $\Delta K = -(\Delta m)c^2$: the released kinetic energy in a reaction corresponds to a mass defect ($\Delta K > 0$ if $\Delta m < 0$). If the binding energy of a particle is defined by the difference

$$U = mc^2 - m^*c^2 \quad (6)$$

where m^* is the total mass of its constituents (for example $m^*_H = m_p + m_e$ for the H atom), one recovers the relation $\Delta K = -\Delta U$ provided the constituents are preserved in the reaction. This brief summary shows that in Einsteinian physics “mass is rest energy”, and therefore the mass of a particle is intimately related to its internal structure. Is mass equivalent with inertia? Some physicists define inertia by the factor in front of \mathbf{v} in the expression of the momentum \mathbf{p} , a definition which goes back to the electromagnetic theory of the electron before 1905. Nowadays this

definition has less interest (excepted for special pedagogical purposes [7]) since in SR the difference between mass m and inertia γm amounts to that between rest energy and energy. In section 4 we will be more interested by the interpretation of the factor $\gamma^{-1}m$ which is proportional to the lagrangian (action per unit time) of a relativistic particle.

3 Mass, action and non inertial motions

Most physicists are convinced that the concept of action is a very fundamental one. LAP enter many domains of physics; the CL of energy and momentum are consequences of the symmetries of the action; action, assimilated to a phase is at the basis of Feynman's path approach to quantum physics... But their comments on this concept are generally more mathematical comments than physical ones. This section and the next one aim at giving a physical meaning to the notion of action in the most simple case, that of free particles, but in a non trivial situation since we consider the possibility of elastic or non elastic collisions.

We begin our discussion with the familiar action for a free particle in Galilean relativity [8]:

$$S = \int_A^B \frac{1}{2} m v^2 dt = m \times I . \quad (7)$$

$A(t_A, \mathbf{r}_A)$ and $B(t_B, \mathbf{r}_B)$ are respectively the initial and final events, at the beginning and the end of the considered motion. S contains a kinematical term I and a dynamical one m . I possesses two important properties: A and B being fixed, the difference $I_2 - I_1$ for two motions is invariant, and I is minimum for an inertial motion. The first property $I'_2 - I'_1 = I_2 - I_1$ is an immediate consequence of the Galilean law of transformation of the velocity. The second one is obtained in the particular frame R_0 where $\mathbf{r}_{0A} = \mathbf{r}_{0B}$, *i.e.* where the positions of A and B coincide; in this frame one has $I_0 \geq 0$ and $I_0 = 0$ if the particle is at rest. These two properties make I the right quantity to characterize an inertial motion and the natural one to compare non inertial ones for a given mass m : for A and B fixed, the greater I is, the more non inertial the motion is. The role of the dynamical factor m with respect to non inertia is illustrated by the study of the elastic collisions of

two particles $n = 1, 2$ (cf. figure 1). A_n are initial events (before the collision), A'_n are final events (after it) and C (with coordinates t, \mathbf{r}) is the collision event. The total action can be written

$$S_t = m_1(I_1 + I'_1) + m_2(I_2 + I'_2) \quad (8)$$

where $m_n I_n$ (resp. $m_n I'_n$) are the actions between A_n and C (resp. C and A'_n) for particle n . Under this form it appears as a combination with different “weights” m_n , of those quantities $I_n + I'_n$ which would have been minimized if the particles were independent. S_t looks like a cost function in economy and can serve as a natural starting point to introduce masses. Although non familiar, this introduction is in perfect accordance with the intuitive Newtonian idea of inertia: if for instance m_1 is much larger than m_2 , the minimization will lead to a result which is close to that of $I_1 + I'_1$, i.e. to an almost inertial motion for particle 1.

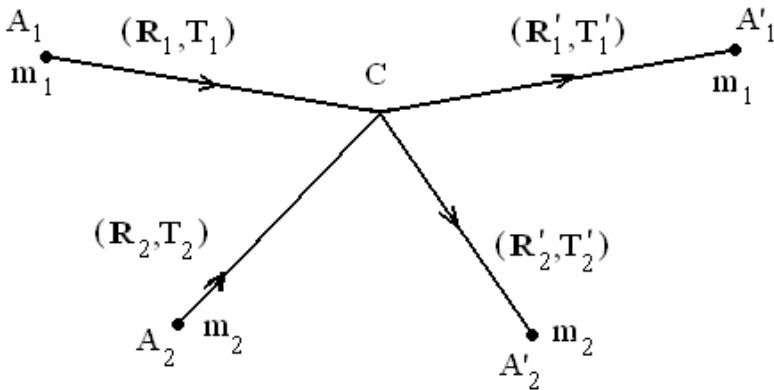


Figure 1

It remains to verify that the LAP effectively leads to the CL of momentum and energy in the same simple way as Fermat's principle leads to Snell's laws [9]. C being fixed, the minimization of S_t is obtained by iner-

tial motions of the particles before and after C . Then S_t reads (since the velocities are constant)

$$S_t = \frac{1}{2}m_1 \frac{\mathbf{R}_1^2}{T_1} + \frac{1}{2}m_2 \frac{\mathbf{R}_2^2}{T_2} + \frac{1}{2}m_1 \frac{\mathbf{R}'_1{}^2}{T'_1} + \frac{1}{2}m_2 \frac{\mathbf{R}'_2{}^2}{T'_2} \quad (9)$$

where $\mathbf{R}_n = \mathbf{r} - \mathbf{r}_n$, $T_n = t - t_n$, $\mathbf{R}'_n = \mathbf{r}'_n - \mathbf{r}$, $T'_n = t'_n - t$ are the travelled distances and travelling times before and after collision. (We do not need to suppose $t_1 = t_2$ and $t'_1 = t'_2$). The extremalization of this new expression of S_t with respect to the coordinates (t, \mathbf{r}) of C uses the easily verified relations

$$d\left(\frac{1}{2}m \frac{\mathbf{R}^2}{T}\right) = -EdT + \mathbf{p} \cdot d\mathbf{R} \quad (10)$$

where $E = m\mathbf{v}^2/2$ and $\mathbf{p} = m\mathbf{v}$ and the obvious ones:

$$dT_n = -dT'_n = dt; \quad d\mathbf{R}_n = -d\mathbf{R}'_n = d\mathbf{r}. \quad (11)$$

From the expression of dS_t

$$dS_t = -(E_1 + E_2 - E'_1 - E'_2)dt + (\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}'_1 - \mathbf{p}'_2) \cdot d\mathbf{r}, \quad (12)$$

one deduces that the extremal condition $dS_t = 0$ is equivalent to the CL of energy and momentum (dt and $d\mathbf{r}$ are arbitrary). The generalisation to inelastic collisions involves for each particle the proper energy U which is a constant and the action, which will be physically justified below

$$S = \int_A^B \left(\frac{1}{2}m\mathbf{v}^2 - U \right) dt. \quad (13)$$

Since it simply adds $-(U_1 T_1 + U_2 T_2 + U'_1 T'_1 + U'_2 T'_2)$ to the expression of S_t and $-(U_1 + U_2 - U'_1 - U'_2) dt$ to that of dS_t , it leads to the conservation of the total (kinetic + internal) energy.

In SR the standard action for a free particle is [10]:

$$S = -mc^2 \int_A^B \sqrt{1 - \frac{\mathbf{v}^2}{c^2}} dt = -mc^2 \tau. \quad (14)$$

Its kinematical part τ is the elapsed proper time of the particle between A and B . τ is an invariant quantity (like $dt^2 - d\mathbf{r}^2 c^{-2}$) and possesses the remarkable property, popularized by the Langevin's twins paradox, that it is maximum for an inertial motion. (This result is obvious in the frame R_0 where $\mathbf{r}_{0A} = \mathbf{r}_{0B}$ since the inertial motion then corresponds to $\mathbf{v}_0 = \mathbf{0}$). Therefore, because of the minus sign, S is minimum for such a motion. The factor c^2 gives to S the dimension of an action. Collisions and their CL can be treated like in Galilean relativity by demanding the minimization of the total action [11]:

$$S_t = -c^2 \left(\sum_i m_i \tau_i + \sum_f m_f \tau_f \right). \quad (15)$$

Here also S_t plays the role of a cost function with weights m_i, m_f and elastic collisions can be used to define masses in SR, but it will be important for the following to remark that we now allow different initial and final particles. Let A_i and A_f be respectively the initial and final events associated with them, and let $C(t, \mathbf{r})$ be the collision event. C being fixed, the minimization of S_t imposes an inertial motion for all particles (before and after C). Then for each one τ reads (since \mathbf{v} is constant)

$$\tau = T \left(1 - \frac{\mathbf{v}^2}{c^2} \right)^{1/2} = \left(T^2 - \frac{\mathbf{R}^2}{c^2} \right)^{1/2} \quad (16)$$

where T and \mathbf{R} denote as before the travelling time and distance, and one has

$$mc^2 d\tau = EdT - \mathbf{p} \cdot d\mathbf{R} \tag{17}$$

where now E and \mathbf{p} are the relativistic energy and momentum ($E = \gamma mc^2$ and $\mathbf{p} = \gamma m\mathbf{v}$). With these relations, the stationary condition with respect to the coordinates of the collision event simply becomes

$$dS_t = -\left[\left(\sum E_i - \sum E_f \right) dt - \left(\sum \mathbf{p}_i - \sum \mathbf{p}_f \right) \cdot d\mathbf{r} \right] = 0 \tag{18}$$

and yields the CL of energy and momentum. The case of zero masses can be taken into account by considering the limit case of very small masses in the usual relation $E^2 - p^2 c^2 = m^2 c^4$ obtained after minimization³. The fact that the extremum is indeed a minimum is proved in appendix A. Finally, in the Galilean limit, taking into account the approximation $\gamma^{-1} mc^2 = mc^2 - m\mathbf{v}^2/2$ and the definition $U = (m - m^*)c^2$ of the binding energy ($|U| \ll mc^2$), one recovers the action of a free particle [11]:

$$S = -m^* c^2 (t_B - t_A) + \int_A^B dt \left(\frac{1}{2} m^* v^2 - U \right). \tag{19}$$

This taking into account of U which unfortunately is unusual in the literature is important not only because it allows to consider the case of inelastic collisions, but also because it answers (in part) a question which any student has asked himself before adopting the opinion that action is a pure mathematical entity, namely: why is the lagrangian of a non relativistic particle in

³ An other possibility is to replace in S_t the term $m\tau$ by $\lambda\tau$ (or $\lambda\tau^2$) where λ is a Lagrange multiplier associated with the condition $\tau = 0$.

an external potential equal to the difference $mv^2/2 - U(\mathbf{r})$? (We recall that the source of a potential is considered as a non dynamical system and that the proper energy of the whole “source + particle”, which depends on \mathbf{r} , is attributed to the particle). Therefore not only as concerns the notion of energy, but also as concerns that of action does Einsteinian physics provide a better understanding of Newtonian physics. This leads us to examine in more details the relativistic action.

4 Mass and internal non-inertia; a reversal of the Newtonian conception

We come back to the question of the minus sign in the definition of the relativistic action S , which has been historically introduced only to recover the Galilean limit. If one starts directly with SR, forgetting Galilean relativity (and Newtonian physics), the first important notion, as Einstein told us in 1905, is the proper time τ . Having noted that τ is maximum for an inertial motion, and having observed that in an elastic collision the greater the mass of a body is, the more inertial its motion is, the quantity $m\tau$ appears to be the natural one to describe inertial motions and collisions. Then one discovers that the principle of maximization of the sum of the quantities $m\tau$ for all particles, not only leads to the CL of energy and momentum, but noticeably also applies to inelastic collisions where the particles appear and disappear “miraculously”, *i.e.* it applies to cases where matter “changes its form or content”. Paraphrasing Maupertuis, one would conclude from these observations that “*nature spends the quantity $m\tau$* ”. So the question “what is action?” must be replaced by:

“what is $A = mc^2\tau$?”.

(We introduce arbitrarily c to keep contact with the previous sections and continue to speak of the principle of maximization of A as LAP). To answer this question, we recall that one main result of the LAP is that the mass of the particle is its rest energy and depends on its internal structure. Therefore instead of considering $m\tau$ as linked to the motion of the particle, it is natural to consider that $m\tau$ is in relation with this structure, *i.e.* with the dynamics of the constituents of the particle. This point of view is also strongly suggested by the observation that $m\tau$ keeps to be non zero when the particle is at rest and by the above remark that the LAP continues to apply even when particles are not conserved. Since the internal dynamics is certainly non inertial, the quantity $mc^2\tau$ is the natural candidate to measure the non iner-

tia of this dynamics. Since one needs a short word and a symbol for this “internal non-inertia”, we propose to call it also “activity” and denote it A . Of course these considerations do not pretend to bring new information on (or a new modelling of) the internal structure of particles, but are only another way to speak (or think) of well known physics.

Let us begin with general remarks which could have been formulated a century ago, just before the development of QM. The first one deals with the proportionality of the “activity” of a particle at rest in a given frame to the time variable of this frame. If “activity”, which now plays the same role as action, is a major concept of physics, mass appears as a “rate of activity” (activity per unit of time). If activity can be measured (it becomes a phase in quantum mechanics), a conventional unit mass can serve to define time, *i.e.* this mass provides a unit of frequency. From an epistemological point of view, this implies that time is intimately related to non inertial phenomena [12]. From a philosophical point of view, this means that the Newtonian idea that “time flows” is now replaced by the idea that *activity flows and time is the flow of activity corresponding to a unit mass*. These considerations can be made more precise if physics provides a privileged unit of “activity” (or of action since A and S have the same dimension). This unit is of course the constant h which Planck introduced in 1900 before realizing from 1905’s Einstein paper on relativity that it is frame invariant, and discovering in 1912 that it is the natural unit of area in phase space. If a physicist of this prequantum period had introduced h in the above discussion, he would certainly have related the proper time τ , the Compton time τ_c and the activity of a particle of mass m by:

$$\frac{A}{h} = \frac{\tau}{\tau_c} \quad \left(\tau_c = \frac{h}{mc^2} \right). \quad (20)$$

He would then have interpreted τ_c as a typical time interval during which the internal motion can be considered as inertial. More precisely he might have imagined that the above internal non inertia is a kind of random (Poisson) process whose frequency $\nu_c = \tau_c^{-1}$ of occurrence of non inertial events is given by the relation

$$h\nu_c = mc^2. \quad (21)$$

Then the probability for the internal dynamics of a particle at rest to be inertial up to time t would be related to the “activity” by

$$\exp\left(-\frac{\tau}{\tau_c}\right) = \exp\left(-\frac{A}{\hbar}\right). \quad (22)$$

Finally for two non interacting (*i.e.* independent) particles at rest, this probability would become

$$\exp\left(-\frac{t}{\tau_{c1}}\right) \cdot \exp\left(-\frac{t}{\tau_{c2}}\right) = \exp\left(-\frac{t}{\tau_c}\right), \quad (23)$$

allowing to interpret the addition of masses $m = m_1 + m_2$ as that of the frequencies of the two processes. Of course, quantum physicists today do not adhere to this image of discrete random processes. Following de Broglie, the frequency ν_c is interpreted as that of a continuous (still confined) harmonic process, and the above real exponential is replaced by an imaginary one:

$$\exp(-i2\pi\nu_c\tau) = \exp\left(-\frac{iA}{\hbar}\right) = \exp\left(\frac{iS}{\hbar}\right). \quad (24)$$

“Activity” (*i.e.* internal non inertia) has now become the quantum phase. Differences of activities are measured for example in Young interferences with massive particles⁴.

⁴ Let us note also that the frequencies $h^{-1}\varepsilon_n$ which occur in the bounded states of the non relativistic SE are nothing but the differences of Compton frequencies since $\varepsilon_n = (m_n - m^*)c^2$ (where m_n is the mass of the atom in the state ψ_n associated with ε_n).

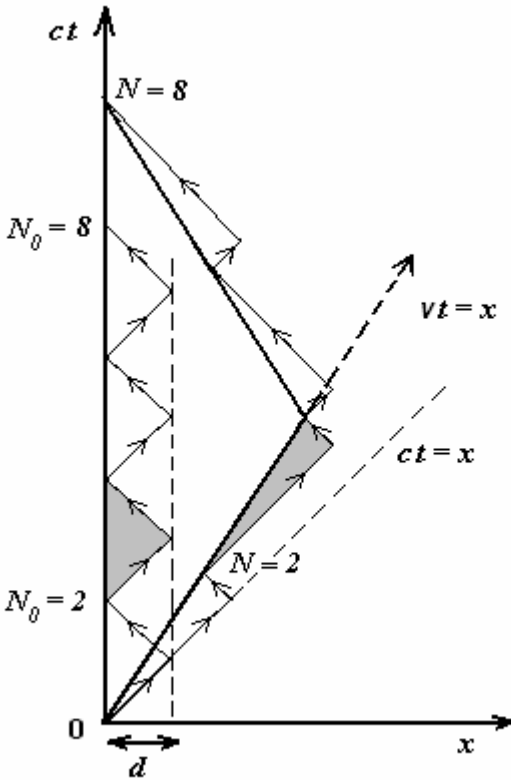


Figure 2

Our second remark concerns the presence of the proper time τ in the “activity” A and leads us to revisit Langevin’s twins paradox. If a particle makes a round trip during a time t (starting at $t=0$ and finishing at t at the same fixed place O of a given frame), then τ is inferior to t which is the proper time of a particle at rest at O . To understand why the “activity” is greater for the particle at rest, let us consider that each twin uses as a clock the bouncing of a light ray on two (confining) mirrors separated by a distance d . If

$N_0 = \frac{ct}{d}$ is the number of bouncing (non inertial events) for the clock at rest, this number for a moving clock is

$$N = \frac{c\tau}{d} = N_0 \sqrt{1 - \frac{v^2}{c^2}} \quad (25)$$

(by definition of the proper time) and therefore N is inferior to N_0 . On figure 2 we have represented the spacetime trajectories of the two twins and of the light pulses. In order to construct the spacetime trajectory of light for the moving mirrors, we have simply used the property that LT preserve the velocity c and the areas. Therefore without even knowing the explicit relation between t and τ , figure 2 clearly shows that τ is inferior to t because τ is measured by N whereas t is measured by N_0 . Although this illustration is not a model for particles, mass can be introduced from the relation $\tau_c = h/mc^2$ where τ_c is a typical time of non inertia. In the above pre-quantum approach, one could take for τ_c the period of the motion between mirrors, or more generally $\tau_c = 2d/c + \tau_r$, where τ_r is a delay introduced by the reflexion on the mirrors. This introduction of τ_r is made here to recall that even from a naïve point of view, the size of a particle may be quite different from its Compton length $c\tau_c$. In the waves approach ν_c would be the frequency of a stationary wave and would depend both on the distance between the mirrors and on their reflexion coefficients.

After these remarks it remains to discuss more precisely what the “activity” of a particle is made of. Generally, the energy of the particle at rest is written

$$mc^2 = \sum_i m_i c^2 + U \quad \text{with} \quad U = \sum_i K_i + E_{\text{pot}} \quad (26)$$

where m_i and K_i are the masses and kinetic energies of its constituents and E_{pot} is a potential energy. But here it is natural to introduce the proper activity of each constituent i during the time dt

$$m_i c^2 d\tau_i = E_i dt - \mathbf{p}_i \cdot d\mathbf{r}_i, \tag{27}$$

where $E_i = K_i + m_i c^2$ and \mathbf{p}_i are the energy and momentum of the constituent. Then the “activity” of the particle at rest can be rewritten as a sum of three contributions:

$$mc^2 dt = \sum_i m_i c^2 d\tau_i + \sum_i \mathbf{p}_i \cdot \mathbf{v}_i dt + E_{\text{pot}} dt. \tag{28}$$

The first one corresponds to the internal dynamics of the constituents. It disappears if all masses m_i are zero. The second one is associated with the motion of the constituents; it is very similar to a pressure contribution. Indeed if the constituents are considered as free particles occupying a volume V like in some bag models, one has for time averaged quantities (according to the virial theorem)

$$\langle \sum_i \mathbf{p}_i \cdot \mathbf{v}_i \rangle = - \langle \sum_i \mathbf{f}_i \cdot \mathbf{r}_i \rangle = \iiint Pr \cdot d\mathbf{S} = 3PV \tag{29}$$

where P is the kinetic pressure. This contribution is present even if the constituents are massless and then is equal to $\sum_i E_i dt$. The potential contribu-

tion $E_{\text{pot}} dt$ depends on the interactions.

More generally, the distribution of non inertia inside the particle depends on the model we have for it, which in turn depends on the experiments made on the particles. For example, in the diffusion $e^- + p \rightarrow e^- + X$ of high energy electrons by a proton (deep inelastic scattering (DIS) experiments), where X is any set of produced particles, very short scales are implied and the proton is seen as an infinity of quasi free and quasi massless punctual constituents called the partons (quarks, antiquarks and gluons) [13], whereas at lower energies it is seen as a continuous distribution of charges. In DIS only the second contribution is present. More precisely the relation

$$mc^2 = \sum_i E_i \text{ is replaced by } mc^2 = \sum_{\alpha} \int_0^1 (mc^2 x) q_{\alpha}(x) dx \text{ where } q_{\alpha}(x) dx$$

may be interpreted as the number of partons of type α with energy between

mc^2x and $mc^2(x+dx)$ ⁵. Experiments show that the q_α also depend on the scale at which the proton is seen.

5 Mass and the Dirac equation

In this section we do not consider the question of giving a physical meaning to the variational approach in QM, but we simply examine how the Dirac equation (DE) enlightens the above link between mass and non inertia. This equation, initially written (with $c = \hbar = 1$) as $i\partial_t\psi = (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)\psi$ where here $\mathbf{p} = -i\nabla$ is the momentum operator and α_i and β are 4x4 matrices, has been introduced in order to be both first order in time like the SE and to lead to the Klein-Gordon equation (KGE) $(\partial_t^2 - \Delta + m^2)\psi = 0$. $n = 4$ is the smallest dimension of the matrices such that these requirements may be satisfied. The interpretation of the mass m is more clear if one looks before for the simplest relativistic equations involving ∂_t . The lowest dimension is reduced to $n = 2$ and the equations only call for the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ (see appendix B):

$$(\partial_t - \boldsymbol{\sigma}\cdot\nabla)\psi_L = 0 \quad ; \quad (\partial_t + \boldsymbol{\sigma}\cdot\nabla)\psi_R = 0. \quad (30)$$

⁵ In full rigor the parameter x is defined from the relation, analogous to Compton

formula, $\lambda' - \lambda = \frac{1}{mx}(1 - \cos\theta)$ where λ and λ' are the wavelengths of the incoming and outgoing electron. The usual interpretation of x (justified by kinematics) is that in an infinite momentum frame where the proton has almost velocity 1 and a large momentum P , xP is the fraction of P taken by a parton. $q_\alpha(x)dx$ is interpreted as the number of partons of type α with momentum between xP and $(x+dx)P$ and it therefore verifies $\sum_{\alpha} \int_0^1 xq_\alpha(x)dx = 1$.

For a plane wave solution $\psi = a \exp -i(Et - \mathbf{p} \cdot \mathbf{r})$ the condition of solvability $\det(E \pm \boldsymbol{\sigma} \cdot \mathbf{p}) = 0$ gives $E^2 - \mathbf{p}^2 = 0$ and leads to:

$$(\boldsymbol{\sigma} \cdot \hat{\mathbf{p}})\psi_L = -\psi_L \quad ; \quad (\boldsymbol{\sigma} \cdot \hat{\mathbf{p}})\psi_R = -\psi_R \quad ; \quad \left(\hat{\mathbf{p}} = \frac{\mathbf{p}}{p} \right). \quad (31)$$

Therefore ψ_L (“left” spinor) and ψ_R (“right” spinor) describe zero mass particles with helicities (angular momentum along the direction $\hat{\mathbf{p}}$) respectively equal to $-1/2$ and $1/2$. For a non zero mass particle, the DE is recovered (in the Weyl form), by coupling ψ_L and ψ_R :

$$i(\partial_t - \boldsymbol{\sigma} \cdot \nabla)\psi_L = m\psi_R \quad ; \quad i(\partial_t + \boldsymbol{\sigma} \cdot \nabla)\psi_R = m\psi_L. \quad (32)$$

The 4-dimensional Dirac spinor (ψ_R, ψ_L) satisfies the KGE since $(\partial_t - \boldsymbol{\sigma} \cdot \nabla)(\partial_t + \boldsymbol{\sigma} \cdot \nabla) = \partial_t^2 - \Delta$. In order to examine how the coupling works, let us consider waves which depend only on t and z . Using the explicit form of σ_z and calling ψ_{1L}, ψ_{2L} and ψ_{1R}, ψ_{2R} the two components of ψ_L and ψ_R one immediately obtains:

$$\begin{aligned} i(\partial_t - \partial_z)\psi_{1L} &= m\psi_{1R} & ; & & i(\partial_t + \partial_z)\psi_{1R} &= m\psi_{1L} ; \\ i(\partial_t - \partial_z)\psi_{2R} &= m\psi_{2L} & ; & & i(\partial_t + \partial_z)\psi_{2L} &= m\psi_{2R}. \end{aligned} \quad (33)$$

For $m = 0$, ψ_{1R} and ψ_{2L} propagate in the z direction and ψ_{2R} and ψ_{1L} propagate in the $-z$ direction. If the particle is massive, the first set of equations couples ψ_{1L} and ψ_{1R} and the second one couples ψ_{2L} and ψ_{2R} . The usual interpretation in particle physics is that the presence of mass couples the two helicities. But noting that for example ψ_{1L} and ψ_{1R} correspond to a same angular momentum along z ($1/2$ since they are the first component of ψ_L and ψ_R), one can also say that mass is the expres-

sion of a coupling between the two opposite directions of propagation $-z$ and $+z$. Indeed for a plane wave solution (ψ_{1L} and ψ_{1R} behaving as $\exp -i(Et - pz)$ with $E > 0$), the two amplitudes are related by:

$$\psi_{1L} = \frac{m}{E + p} \psi_{1R}. \quad (34)$$

The greatest amplitude is the one which for $m = 0$ was associated with the direction of motion of the massive particle (for example $|\psi_{1R}| > |\psi_{1L}|$ if $p > 0$). More precisely, if $|\psi_{1R}|^2$ is the probability of finding the velocity 1 ($c = 1$) and $|\psi_{1L}|^2$ the same for -1 , the average velocity is given by (with $|\psi_{1L}|^2 + |\psi_{1R}|^2 = 1$):

$$\left(|\psi_{1R}|^2 - |\psi_{1L}|^2 \right) = \frac{p}{E} = v. \quad (35)$$

It is nothing but the speed of the massive particle. These probabilistic results are identical to those one would obtain from the (naïve) bouncing of a zero mass particle between two mirrors moving at the velocity v (figure 2).

This zigzag scenario associated with mass is also present in an interesting result due to Feynman and analysed in [14] which concerns the path integral formulation of the Green function $G(t_2 - t_1, z_2 - z_1)$ of the one dimensional DE (the above equations for ψ_{1L}, ψ_{1R}). In this formulation each path in the (t, z) plane between events (t_1, z_1) and (t_2, z_2) is a zigzag trajectory with velocities $+1$ and -1 and m^{-1} appears to be a typical time during which the velocity keeps its value⁶. When they speak of this motion at velocity $\pm c$, the authors attribute it to the massive particle. In his guide to the laws

⁶ When $|z_2 - z_1| \ll t_2 - t_1$ the zigzag motion becomes a diffusion process and the authors recover the diffusion constant $D \approx c \lambda_c \approx \hbar/m$ of the SE.

of the universe [15] Penrose speaks of different massless zig and zag particles with respective helicities $-1/2$ and $1/2$. Another interpretation is that the equations for ψ_L and ψ_R without mass, whose solutions propagate at the velocity c , describe inertial phenomena whereas their coupling, which is necessary to introduce m , describes non inertial ones. Let us recall that in the standard model, the basic fields ψ are those of massless fermions which acquire their mass $g \langle \varphi \rangle$ from a Yukawa interaction $g \bar{\psi} \psi \varphi$, with the Higgs field φ whose mean value is $\langle \varphi \rangle$. Since $|\psi|^2 d^3r$ is a probability of presence of the “matter particles”, the product $\langle \varphi \rangle |\psi|^2$ may be considered as a non inertia per unit of time and volume.

6 Conclusion; non inertia and gravitation

In this paper, we have suggested to change our point of view on the inertial mass of a body. The Newtonian one identified mass with inertia *i.e.* with the tendency of a massive body to keep an inertial motion (a constant velocity), and the LAP for free and colliding particles then appeared to be its best mathematical translation. In SR where the inertial motion of a body corresponds to a maximum of the proper time, we have seen that the natural quantity to consider is the opposite of the traditional action, namely what we have called the “activity” $mc^2 \tau$. As we have emphasized, this “activity” and its interpretation as measuring the non inertia of the internal motions present in a massive system (with the Compton time as a typical time of non inertia) could have been considered within classical physics just after physicists have realized that the constant h is an invariant unit of action. In quantum physics this activity becomes the phase and we have seen that whenever mass is present (Young’s experiment, DE, particle physics), it can naturally be associated with a frequency. Of course, mass in physics is first of all a phenomenological quantity, and we have not pretended to provide a model of particles.

The reader will have noted that our change of point of view on inertial mass is a reversal of what is inertial and what is not. It raises the question of which references one must take in order to claim that a motion is inertial. In the standard approach the reference motions are those of the “inertial frames” whose importance for the description of phenomena has been revealed by Galilee. For Galilee, an inertial motion is a motion at constant velocity and a modification of it involves the acceleration; then mass mani-

feels itself through Newton's law or the study of collisions. Let us note however that mass was already implicitly present in the definition of inertial frames since in practice these frames can be realized only by massive systems. In the new approach, reference motions are those at velocity c as the DE seems to suggest (or more simply motions which corresponds to zero "activity"). A modification of such a motion does not imply the acceleration but the velocity; if the velocity v of a system is different from c ($v < c$), there is some non inertia in it, to which one can associate a frequency, the mass of the system. As we have seen, the idea that zero mass particles are more fundamental than massive ones seems also to be a lesson of particle physics.

Finally, although this was not the theme of our paper, let us say a few words on the (unavoidable) question of the identity of inertial and gravitational masses. This identity which has led Einstein to connect matter and space time properties, seems miraculous if one thinks of gravitation as an interaction. On the contrary in the approach of section 4, the concepts of inertial mass and time are intimately related from the beginning: mass is a frequency of "activity" (non inertia) and time is the flow of activity of a unit mass at rest. In this approach the addition $m_1 + m_2$ of the masses of two bodies at rest results from a hypothesis of independence of the corresponding non inertial phenomena. But such phenomena are not punctual; they have a spatial extension. Therefore this independence hypothesis can be rigorously satisfied only if the bodies are infinitely distant. One may imagine that there is a length scale, Planck's scale \sqrt{G} , at which it is impossible to separate non inertial phenomena, or a time scale \sqrt{G} at which non inertia has no meaning. From this point of view gravitation is a correction to the previous addition of inertial masses and involves them (besides G and the distance r between the bodies). Consequently *gravitational masses and inertial ones must be equal simply because they deal with the same problem of how to evaluate non inertia*⁷.

⁷ When the correction is a small one, physics tells us that it is simply the Newtonian gravitational energy $-Gm_1m_2r^{-1}$ ($c = 1$). If m_2 is a test mass, this energy is attributed to it, and the activity of m_2 becomes $m_2\tau$ with $\tau = t\left(1 - Gm_1r^{-1}\right)$. One recovers Einstein's interpretation that the proper time of mass m_2 is modified by the presence of mass m_1 .

7 Appendix A. Extrema and minima of action

In this appendix we prove that the extremum of the total action S_t introduced in section 3 for the description of collisions is a minimum. We recall that an extremum of a function $f(\underline{x})$ is a minimum if the second term of its Taylor series near the extremum is positive:

$$\delta^2 f = \frac{1}{2} \sum_{\alpha, \beta} \frac{\partial^2 f}{\partial x_\alpha \partial x_\beta} dx_\alpha dx_\beta = \frac{1}{2} \sum_\alpha d \left(\frac{\partial f}{\partial x_\alpha} \right) dx_\alpha > 0. \quad (36)$$

In our case the variables x_α are the travelling times T_α and distances \mathbf{R}_α of each initial or final particle α and the function f which we consider is $S_t = \sum_\alpha s(T_\alpha, \mathbf{R}_\alpha)$ with:

$$s(T, \mathbf{R}) = \frac{1}{2} m \frac{\mathbf{R}^2}{T} \text{ or } s(T, \mathbf{R}) = -mc^2 \sqrt{T^2 - \frac{\mathbf{R}^2}{c^2}} \quad (37)$$

Since in both cases, one has $ds = \mathbf{p} \cdot d\mathbf{R} - EdT$, the second variation of s reads:

$$\delta^2 s = \frac{1}{2} (\mathbf{dp} \cdot d\mathbf{R} - dEdT). \quad (38)$$

Using the relation $dE = \mathbf{v} \cdot d\mathbf{p}$ (also valid in both relativities) $\delta^2 s$ becomes:

$$\delta^2 s = \frac{1}{2} T (\mathbf{dp} \cdot d\mathbf{v}) \quad \left(\mathbf{v} = \frac{\mathbf{R}}{T} \right) \quad (39)$$

It is then a simple task to verify that for $\mathbf{p} = m\mathbf{v}$ or $\mathbf{p} = \gamma m\mathbf{v}$ one has $\delta^2 s > 0$ and therefore $\delta^2 S_t > 0$. The inequality becomes $\delta^2 s \geq 0$ if m goes to zero.

8 Appendix B. Pauli matrices and the Dirac Equation.

The Pauli matrices defined by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (40)$$

allow to write a general 2x2 hermitian matrix X in the form:

$$X = T + \boldsymbol{\sigma} \cdot \mathbf{R} = \begin{pmatrix} T + Z & X - iY \\ X + iY & T - Z \end{pmatrix}. \quad (41)$$

Using $\det X = T^2 - \mathbf{R}^2$ and interpreting T, \mathbf{R} as a space-time interval, one deduces that the transformations defined by $X' = MXM^+$, $\det M = 1$ conserve $T^2 - \mathbf{R}^2$ and therefore correspond for (T, \mathbf{R}) to Lorentz Transformations. Since $i(\partial_t, -\vec{\nabla})$ is like (T, \mathbf{R}) a quadrivector (for example it leads to (ω, \mathbf{k}) when applied to a plane wave), one is naturally led to introduce the equation:

$$i(\partial_t - \boldsymbol{\sigma} \cdot \vec{\nabla})\psi_L = 0. \quad (42)$$

Since $M(\partial_t - \boldsymbol{\sigma} \cdot \vec{\nabla})M^+ = (\partial_{t'} - \boldsymbol{\sigma} \cdot \vec{\nabla}')$ one deduces that it is invariant (equivalent to $(\partial_{t'} - \boldsymbol{\sigma} \cdot \vec{\nabla}')\psi'_L = 0$) provided that ψ_L transforms as $\psi'_L = M^{+ -1}\psi_L$. The same can be shown for the equation

$$i(\partial_t + \boldsymbol{\sigma} \cdot \vec{\nabla})\psi_R = 0 \quad (43)$$

with $\psi'_R = M\psi_R$ (the demonstration lies on the remark that $\tilde{X} = T - \boldsymbol{\sigma} \cdot \mathbf{R} = (T^2 - \mathbf{R}^2)X^{-1}$ transforms as $\tilde{X}' = M^{+ -1}\tilde{X}M^{-1}$). The sets of matrices $M^{+ -1}$ and M are the two n=2 inequivalent representations of

the Lorentz group. The above equations describe zero mass particles (see section 5). If one wants to introduce a mass parameter, one could think of equations such as $i(\partial_t \pm \boldsymbol{\sigma} \cdot \nabla)\psi = m\psi$ but they are not invariant. One must couple the spinors ψ_L and ψ_R :

$$i(\partial_t - \boldsymbol{\sigma} \cdot \nabla)\psi_L = m\psi_R \quad ; \quad i(\partial_t + \boldsymbol{\sigma} \cdot \nabla)\psi_R = m\psi_L. \quad (44)$$

The invariance of the first equation for example follows from:

$$i(\partial_{t'} - \boldsymbol{\sigma} \cdot \nabla')\psi'_L = M i(\partial_t - \boldsymbol{\sigma} \cdot \nabla)M^+ M^{-1}\psi_L = i m M \psi_R = i m \psi'_R. \quad (45)$$

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