Differential Forms on Riemannian (Lorentzian) and Riemann-Cartan Structures and Some Applications to Physics

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ABSTRACT. In this paper after recalling some essential tools concerning the theory of differential forms in the Cartan, Hodge and Clifford bundles over a Riemannian or Riemann-Cartan space or a Lorentzian or Riemann-Cartan spacetime we solve with details several exercises involving different grades of difficult. One of the problems is to show that a recent formula given in [10] for the exterior covariant derivative of the Hodge dual of the torsion 2-forms is simply wrong. We believe that the paper will be useful for students (and eventually for some experts) on applications of differential geometry to some physical problems. A detailed account of the issues discussed in the paper appears in the table of contents.

Contents

1 Introduction

In this paper we first recall some essential tools concerning the theory of differential forms in the *Cartan*, *Hodge* and *Clifford* bundles over a n-dimensional manifold M equipped with a metric tensor $\boldsymbol{g} \in \sec T_2^0 M$ of arbitrary signature (p,q), $p+q=n$ and also equipped with metric compatible connections, the Levi-Civita ($\mathring{D}$) and a general Riemann-Cartan (D) one[1]. After that we solved with details some exercises involving different grades of difficult, ranging depending on the readers knowledge from kindergarten, intermediate to advanced levels. In particular we show how to express the *derivative* (d) and *coderivative* (δ) operators as functions of operators related to the Levi-Civita or a Riemann-Cartan connection defined on a manifold, namely the *standard* Dirac operator ($\mathring{\boldsymbol{\partial}}$) and general Dirac operator ($\boldsymbol{\partial}$). Those operators are then used to express Maxwell equations in both a Lorentzian and a Riemann-Cartan spacetime. We recall also important formulas (not well known as they deserve to be) for the square of the general Dirac and standard Dirac operators showing their relation with the *Hodge D'Alembertian* ($\lozenge$), the *covariant D' Alembertian* ($\mathring{\square}$) and the *Ricci operators* ($\mathring{\mathcal{R}}^{\mathbf{a}}$, $\mathcal{R}^{\mathbf{a}}$) and *Einstein* operator ($\mathring{\blacksquare}$) and the use of these operators in the *Einstein-Hilbert* gravitational theory. Finally, we study the *Bianchi* identities. Recalling that the first Bianchi identity is $D\mathcal{T}^{\mathbf{a}} = \mathcal{R}_{\mathbf{b}}^{\mathbf{a}} \wedge \theta^{\mathbf{b}}$, where $\mathcal{T}^{\mathbf{a}}$ and $\mathcal{R}_{\mathbf{b}}^{\mathbf{a}}$ are respectively the torsion and the curvature 2-forms and $\{\theta^{\mathbf{b}}\}$ is a cotetrad we ask the question: Who is $D \star \mathcal{T}^{\mathbf{a}}$? We find the correct answer (Eq.(218)) using the tools introduced in previous sections of the paper. Our result shows explicitly that the formula "$D{\star}\mathcal{T}^{\mathbf{a}} = {\star}\mathcal{R}_{\mathbf{b}}^{\mathbf{a}} \wedge \theta^{\mathbf{b}}$" recently found in [10] and claimed to imply a contradiction in Einstein-Hilbert gravitational theory is *wrong*. Two very simple counterexamples contradicting the wrong formula for $D \star \mathcal{T}^{\mathbf{a}}$ are presented. A detailed account of the issues discussed in the paper appears in the table of contents[2]. We call also the reader attention that in the physical applications we use natural units for which the numerical values of c,h and the gravitational constant k (appearing in Einstein equations) are equal to 1.

[1] A spacetime is a special structure where the manifold is 4-dimensional, the metric has signature $(1,3)$ and which is equipped with a Levi-Civita or a Riemann-Cartan connection, orientability and time orientation. See below and, e.g., [22, 26] for more details, if needed.

[2] More on the subject may be found in, e.g., [22] and recent advanced material may be found in several papers of the author posted on the arXiv.

2 Classification of Metric Compatible Structures $(M, \boldsymbol{g}, D)$

Let M denotes a n-dimensional manifold[3]. We denote as usual by T_xM and T_x^*M respectively the tangent and the cotangent spaces at $x \in M$. By $TM = \bigcup_{x\in M} T_xM$ and $T^*M = \bigcup_{x\in M} T_x^xM$ respectively the tangent and cotangent bundles. By T_s^rM we denote the bundle of r-contravariant and s-covariant tensors and by $\mathcal{T}M = \bigoplus_{r,s=0}^{\infty} T_s^rM$ the tensor bundle. By $\bigwedge^r TM$ and $\bigwedge^r T^*M$ denote respectively the bundles of r-multivector fields and of r-form fields. We call $\bigwedge TM = \bigoplus_{r=0}^{r=n} \bigwedge^r TM$ the bundle of (non homogeneous) multivector fields and call $\bigwedge T^*M = \bigoplus_{r=0}^{r=n} \bigwedge^r T^*M$ the exterior algebra (Cartan) bundle. Of course, it is the bundle of (non homogeneous) form fields. Recall that the real vector spaces are such that $\dim \bigwedge^r T_xM = \dim \bigwedge^r T_x^*M = \binom{n}{r}$ and $\dim \bigwedge T^*M = 2^n$. Some *additional* structures will be introduced or mentioned below when needed. Let[4] $\boldsymbol{g} \in \sec T_2^0M$ a metric of signature (p,q) and D an arbitrary metric compatible connection on M, i.e., $D\boldsymbol{g} = 0$. We denote by $\mathbf{R}$ and $\mathbf{T}$ respectively the (Riemann) curvature and torsion tensors[5] of the connection D, and recall that in general a given manifold given some additional conditions may admit many different metrics and many different connections.

Given a triple $(M, \boldsymbol{g}, D)$:

(a) it is called a Riemann-*Cartan space* if and only if

$$D\boldsymbol{g} = 0 \qquad \text{and} \qquad \mathbf{T} \neq 0. \tag{1}$$

(b) it is called *Weyl space if and only if*

$$D\boldsymbol{g} \neq 0 \qquad \text{and} \qquad \mathbf{T} = 0. \tag{2}$$

(c) it is called a *Riemann space* if and only if

$$D\boldsymbol{g} = 0 \qquad \text{and} \qquad \mathbf{T} = 0, \tag{3}$$

[3]We left the toplogy of M unspecified for a while.

[4]We denote by $\sec(X(M))$ the space of the sections of a bundle $X(M)$. Note that all functions and differential forms are supposed smooth, unless we explicitly say the contrary.

[5]The precise definitions of those objects will be recalled below.

and in that case the pair $(D, \mathbf{g})$ is called *Riemannian structure.*

*(*d) it is called *Riemann-Cartan-Weyl space* if and only if

$$D\mathbf{g} \neq 0 \qquad \text{and} \qquad \mathbf{T} \neq 0. \tag{4}$$

(e) it is called *(*Riemann*)* flat if and only if

$$D\mathbf{g} = 0 \qquad \text{and} \qquad \mathbf{R} = 0,$$

(f) it is called teleparallel if and only if

$$D\mathbf{g} = 0, \quad \mathbf{T} \neq 0 \quad \text{and} \quad \mathbf{R} = 0. \tag{5}$$

2.1 Levi-Civita and Riemann-Cartan Connections

For each metric tensor defined on the manifold M there exists one and only one connection in the conditions of Eq.(3). It is is called *Levi-Civita connection* of the metric considered, and is denoted in what follows by $\mathring{D}$. A connection satisfying the properties in (a) above is called a Riemann-Cartan connection. In general both connections may be defined in a given manifold and they are related by well established formulas recalled below. A connection defines a rule for the parallel transport of vectors (more generally tensor fields) in a manifold, something which is conventional [20], and so the question concerning which one is more important is according to our view meaningless[6]. The author knows that this assertion may surprise some readers, but he is sure that they will be convinced of its correctness after studying Section 15. More on the subject in [22]. For implementations of these ideas for the theory of gravitation see [18]

2.2 Spacetime Structures

Remark 1 *When* $\dim M = 4$ *and the metric* $\mathbf{g}$ *has signature* $(1,3)$ *we sometimes substitute Riemann by Lorentz in the previous definitions* (c),(e) *and* (f).

Remark 2 *In order to represent a spacetime structure a Lorentzian or a Riemann-Cartan structure* $(M, \mathbf{g}, D)$ *need be such that* M *is connected and paracompact* [11] *and equipped with an orientation defined by*

[6] Even if it is the case, that a particular one may be more convenient than others for some purposes. See the example of the Nunes connections in Section 15.

the volume element $\tau_{\mathbf{g}} \in \sec \bigwedge^4 T^*M$ *and a time orientation denoted by* $\uparrow$*. We omit here the details and ask to the interested reader to consult, e.g.,* [22]*. A general spacetime will be represented by a pentuple* $(M, \mathbf{g}, D, \tau_{\mathbf{g}}, \uparrow)$*.*

3 Absolute Differential and Covariant Derivatives

Given a differentiable manifold M, let $X, Y \in \sec TM$ be any vector fields and $\alpha \in \sec T^*M$ any covector field . Let $\mathcal{T}M = \bigoplus_{r,s=0}^{\infty} T_s^r M$ be the tensor bundle of M and $\mathbf{P} \in \sec \mathcal{T}M$ any general tensor field.

We now describe the main properties of a general connection D (also called absolute differential operator). We have

$$\begin{aligned} D &: \sec TM \times \sec \mathcal{T}M \to \sec \mathcal{T}M, \\ (X, \mathbf{P}) &\mapsto D_X \mathbf{P}, \end{aligned} \tag{6}$$

where D_X the covariant derivative in the direction of the vector field X satisfy the following properties: Given, differentiable functions $f, g : M \to \mathbb{R}$, vector fields $X, Y \in \sec TM$ and $\mathbf{P}, \mathbf{Q} \in \sec \mathcal{T}M$ we have

$$\begin{aligned} D_{fX+gY}\mathbf{P} &= fD_X\mathbf{P} + gD_Y\mathbf{P}, \\ D_X(\mathbf{P} + \mathbf{Q}) &= D_X\mathbf{P} + D_X\mathbf{Q}, \\ D_X(f\mathbf{P}) &= fD_X(\mathbf{P}) + X(f)\mathbf{P}, \\ D_X(\mathbf{P} \otimes \mathbf{Q}) &= D_X\mathbf{P} \otimes \mathbf{Q} + \mathbf{P} \otimes D_X\mathbf{Q}. \end{aligned} \tag{7}$$

Given $\mathbf{Q} \in \sec T_s^r M$ the relation between $D\mathbf{Q}$, the *absolute differential* of $\mathbf{Q}$ and $D_X\mathbf{Q}$ the covariant derivative of $\mathbf{Q}$ in the direction of the vector filed X is given by

$$\begin{aligned} &D\colon \sec T_s^r M \to \sec T_{s+1}^r M, \\ &\quad D\mathbf{Q}(X, X_1, ..., X_s, \alpha_1, ..., \alpha_r) \\ &\quad = D_X\mathbf{Q}(X_1, ..., X_s, \alpha_1, ..., \alpha_r), \\ &X_1, ..., X_s \in \sec TM, \alpha_1, ...\alpha_r \in \sec T^*M. \end{aligned} \tag{8}$$

Let $U \subset M$ and consider a chart of the maximal atlas of M covering U coordinate functions[7] $\{\mathbf{x}^\mu\}$. Let $\mathbf{g} \in \sec T_2^0 M$ be a metric field for

[7] If $e \in M$, then $\mathbf{x}^\mu(e) = x^\mu$ is the μ coordinate of e in the given chart.

M. Let $\{\boldsymbol{\partial}_\mu\}$ be a basis for TU, $U \subset M$ and let $\{\theta^\mu = dx^\mu\}$ be the dual basis of $\{\boldsymbol{\partial}_\mu\}$. The reciprocal basis of $\{\theta^\mu\}$ is denoted $\{\theta_\mu\}$, and $\mathsf{g}(\theta^\mu, \theta_\nu) := \theta^\mu \cdot \theta_\nu = \delta^\mu_\nu$. Introduce next a set of differentiable functions $q^{\mathbf{a}}_\mu, q^\nu_{\mathbf{b}} : U \to \mathbb{R}$ such that :

$$q^\mu_{\mathbf{a}} q^{\mathbf{b}}_\mu = \delta^{\mathbf{b}}_{\mathbf{a}} \ , \qquad q^\mu_{\mathbf{a}} q^{\mathbf{a}}_\nu = \delta^\mu_\nu \ . \tag{9}$$

It is trivial to verify the formulas

$$\begin{aligned} g_{\mu\nu} &= q^{\mathbf{a}}_\mu q^{\mathbf{b}}_\nu \eta_{\mathbf{ab}} \ , \qquad & g^{\mu\nu} &= q^\mu_{\mathbf{a}} q^\nu_{\mathbf{b}} \eta^{\mathbf{ab}}, \\ \eta_{\mathbf{ab}} &= q^\mu_{\mathbf{a}} q^\nu_{\mathbf{b}} g_{\mu\nu}, \qquad & \eta^{\mathbf{ab}} &= q^{\mathbf{a}}_\mu q^{\mathbf{b}}_\nu g^{\mu\nu}, \end{aligned} \tag{10}$$

with

$$\eta_{\mathbf{ab}} = \mathrm{diag}(\underbrace{1, ..., 1}_{p \ \text{ times}} \underbrace{-1, ... -1}_{q \ \text{ times}}) \ . \tag{11}$$

Moreover, defining

$$\mathbf{e_b} = q^\nu_{\mathbf{b}} \boldsymbol{\partial}_\nu$$

the set $\{\mathbf{e_a}\}$ with $\mathbf{e_a} \in \sec TM$ is an orthonormal basis for TU. The dual basis of TU is $\{\theta^{\mathbf{a}}\}$, with $\theta^{\mathbf{a}} = q^{\mathbf{a}}_\mu dx^\mu$. Also, $\{\theta_{\mathbf{b}}\}$ is the reciprocal basis of $\{\theta^{\mathbf{a}}\}$, i.e. $\theta^{\mathbf{a}} \cdot \theta_{\mathbf{b}} = \delta^{\mathbf{a}}_{\mathbf{b}}$.

Remark 3 *When* $\dim M = 4$ *the basis* $\{\mathbf{e_a}\}$ *of* TU *is called a tetrad and the (dual) basis* $\{\theta^{\mathbf{a}}\}$ *of* T^*U *is called a cotetrad. The names are appropriate ones if we recall the Greek origin of the word.*

The connection coefficients associated to the respective covariant derivatives in the respective basis will be denoted as:

$$D_{\boldsymbol{\partial}_\mu} \boldsymbol{\partial}_\nu = \Gamma^\rho_{\mu\nu} \boldsymbol{\partial}_\rho, \quad D_{\boldsymbol{\partial}_\sigma} \boldsymbol{\partial}^\mu = -\Gamma^\mu_{\sigma\alpha} \boldsymbol{\partial}^\alpha, \tag{12}$$

$$D_{\,\mathbf{e_a}} \mathbf{e_b} = \omega^{\mathbf{c}}_{\mathbf{ab}} \ \mathbf{e_c}, \qquad D_{\,\mathbf{e_a}} \ \mathbf{e^b} = -\omega^{\mathbf{b}}_{\mathbf{ac}} \ \mathbf{e^c}, \ \ D_{\boldsymbol{\partial}_\mu} \ \mathbf{e_b} = \omega^{\mathbf{c}}_{\mu\mathbf{b}} \ \mathbf{e_c},$$

$$D_{\boldsymbol{\partial}_\mu} dx^\nu = -\Gamma^\nu_{\mu\alpha} dx^\alpha, \quad D_{\boldsymbol{\partial}_\mu} \theta_\nu = \Gamma^\rho_{\mu\nu} \theta_\rho, \tag{13}$$

$$D_{\,\mathbf{e_a}} \theta^{\mathbf{b}} = -\omega^{\mathbf{b}}_{\mathbf{ac}} \theta^{\mathbf{c}}, \quad D_{\boldsymbol{\partial}_\mu} \theta^{\mathbf{b}} = -\omega^{\mathbf{b}}_{\mu\mathbf{a}} \theta^{\mathbf{a}} \tag{14}$$

$$\begin{aligned} & D_{\,\mathbf{e_a}} \theta^{\mathbf{b}} = -\omega_{\mathbf{cab}} \theta^{\mathbf{c}}, \\ & \quad \omega_{\mathbf{abc}} = \eta_{\mathbf{ad}} \omega^{\mathbf{d}}_{\mathbf{bc}} = -\omega_{\mathbf{cba}}, \quad \omega^{\mathbf{bc}}_{\mathbf{a}} = \eta^{\mathbf{bk}} \omega_{\mathbf{kal}} \eta^{\mathbf{cl}}, \quad \omega^{\mathbf{bc}}_{\mathbf{a}} = -\omega^{\mathbf{cb}}_{\mathbf{a}} \\ & \qquad \textit{etc...} \end{aligned} \tag{15}$$

Remark 4 *The connection coefficients of the Levi-Civita Connection in a coordinate basis are called Christoffel symbols. We write in what follows*

$$\overset{\circ}{D}_{\boldsymbol{\partial}_\mu}\boldsymbol{\partial}_\nu = \overset{\circ}{\Gamma}{}^{\rho}_{\mu\nu}\boldsymbol{\partial}_\rho, \overset{\circ}{D}_{\boldsymbol{\partial}_\mu} dx^\nu = -\overset{\circ}{\Gamma}{}^{\nu}_{\mu\rho}dx^\rho \ . \tag{16}$$

To understood how D works, consider its action, e.g., on the sections of $T^1_1 M = TM \otimes T^*M$.

$$D(X \otimes \alpha) = (DX) \otimes \alpha + X \otimes D\alpha. \tag{17}$$

For every vector field $V \in \sec TU$ and a covector field $C \in \sec T^*U$ we have

$$D_{\boldsymbol{\partial}_\mu} V = D_{\boldsymbol{\partial}_\mu}(V^\alpha \boldsymbol{\partial}_\alpha), \qquad D_{\boldsymbol{\partial}_\mu} C = D_{\boldsymbol{\partial}_\mu}(C_\alpha \theta^\alpha) \tag{18}$$

and using the properties of a covariant derivative operator introduced above, $D_{\boldsymbol{\partial}_\mu} V$ can be written as:

$$\begin{aligned} D_{\boldsymbol{\partial}_\mu} V &= D_{\boldsymbol{\partial}_\mu}(V^\alpha \boldsymbol{\partial}_\alpha) = (D_{\boldsymbol{\partial}_\mu} V)^\alpha \boldsymbol{\partial}_\alpha \\ &= (\boldsymbol{\partial}_\mu V^\alpha)\boldsymbol{\partial}_\alpha + V^\alpha D_{\boldsymbol{\partial}_\mu}\boldsymbol{\partial}_\alpha \\ &= \left(\frac{\partial V^\alpha}{\partial x^\mu} + V^\rho \Gamma^\alpha_{\mu\rho}\right)\boldsymbol{\partial}_\alpha := (D^+_\mu V^\alpha)\boldsymbol{\partial}_\alpha, \end{aligned} \tag{19}$$

where it is to be kept in mind that the symbol $D^+_\mu V^\alpha$ is a short notation for

$$D^+_\mu V^\alpha := (D_{\boldsymbol{\partial}_\mu} V)^\alpha \tag{20}$$

Also, we have

$$\begin{aligned} D_{\boldsymbol{\partial}_\mu} C &= D_{\boldsymbol{\partial}_\mu}(C_\alpha \theta^\alpha) = (D_{\boldsymbol{\partial}_\mu} C)_\alpha \theta^\alpha \\ &= \left(\frac{\partial C_\alpha}{\partial x^\mu} - C_\beta \Gamma^\beta_{\mu\alpha}\right)\theta^\alpha, \\ &:= (D^-_\mu C_\alpha)\theta^\alpha \end{aligned} \tag{21}$$

where it is to be kept in mind that [8] that the symbol $D^-_\mu C_\alpha$ is a short notation for

$$D^-_\mu C_\alpha := (D_{\boldsymbol{\partial}_\mu} C)_\alpha. \tag{22}$$

[8]Recall that other authors prefer the notations $(D_{\boldsymbol{\partial}_\mu} V)^\alpha := V^\alpha_{:\mu}$ and $(D_{\boldsymbol{\partial}_\mu} C)_\alpha := C_{\alpha:\mu}$. What is important is always to have in mind the meaning of the symbols.

Remark 5 *The necessity of precise notation becomes obvious when we calculate*

$$D^{-}_{\mu} q^{\mathbf{a}}_{\nu} := (D_{\boldsymbol{\partial}_{\mu}} \theta^{\mathbf{a}})_{\nu} = (D_{\boldsymbol{\partial}_{\mu}} q^{\mathbf{a}}_{\nu} dx^{\nu})_{\nu} = \partial_{\mu} q^{\mathbf{a}}_{\nu} - \Gamma^{\rho}_{\mu\nu} q^{\mathbf{a}}_{\rho} = \omega^{\mathbf{a}}_{\mu\mathbf{b}} q^{\mathbf{b}}_{\nu},$$
$$D^{+}_{\mu} q^{\mathbf{a}}_{\nu} := (D_{\boldsymbol{\partial}_{\mu}} q^{\mathbf{a}}_{\nu} \mathbf{e_a})^{\mathbf{a}} = \partial_{\mu} q^{\mathbf{a}}_{\nu} + \omega^{\rho}_{\mu\nu} q^{\mathbf{a}}_{\rho} = \Gamma^{\rho}_{\mu\nu} q^{\mathbf{a}}_{\rho},$$

thus verifying that $D^{-}_{\mu} q^{\mathbf{a}}_{\nu} \neq D^{+}_{\mu} q^{\mathbf{a}}_{\nu} \neq 0$ *and that*

$$\partial_{\mu} q^{\mathbf{a}}_{\nu} + \omega^{\mathbf{a}}_{\mu\mathbf{b}} q^{\mathbf{b}}_{\nu} - \Gamma^{\mathbf{a}}_{\mu\mathbf{b}} q^{\mathbf{b}}_{\nu} = 0. \tag{23}$$

Moreover, if we define the object

$$\mathbf{q} = \mathbf{e_a} \otimes \theta^{\mathbf{a}} = q^{\mathbf{a}}_{\mu} \; \mathbf{e_a} \otimes dx^{\mu} \in \sec T^1_1 U \subset \sec T^1_1 M, \tag{24}$$

which is clearly the identity endormorphism acting on sections of TU*, we find*

$$D_{\mu} q^{\mathbf{a}}_{\nu} := (D_{\boldsymbol{\partial}_{\mu}} \mathbf{q})^{\mathbf{a}}_{\nu} = \partial_{\mu} q^{\mathbf{a}}_{\nu} + \omega^{\mathbf{a}}_{\mu\mathbf{b}} q^{\mathbf{b}}_{\nu} - \Gamma^{\mathbf{a}}_{\mu\mathbf{b}} q^{\mathbf{b}}_{\nu} = 0. \tag{25}$$

Remark 6 *Some authors call* $\mathbf{q} \in \sec T^1_1 U$ *(a single object) a tetrad, thus forgetting the Greek meaning of that word. We shall avoid this nomenclature. Moreover,* Eq.(25) *is presented in many textbooks (see, e.g.,* [4, 13, 24]*) and articles under the name 'tetrad postulate' and it is said that the covariant derivative of the "tetrad" vanish. It is obvious that* Eq.(25) *it is not a postulate, it is a trivial (freshman) identity. In those books, since authors do not distinguish clearly the derivative operators* D^{+}, D^{-} *and* D*,* Eq.(25) *becomes sometimes misunderstood as meaning* $D^{-}_{\mu} q^{\mathbf{a}}_{\nu}$ *or* $D^{+}_{\mu} q^{\mathbf{a}}_{\nu}$*, thus generating a big confusion and producing errors (see below).*

4 Calculus on the Hodge Bundle $(\bigwedge T^{*}M, \cdot, \tau_{\boldsymbol{g}})$

We call in what follows Hodge bundle the quadruple $(\bigwedge T^{*}M, \wedge, \cdot, \tau_{\boldsymbol{g}})$. We now recall the meaning of the above symbols.

4.1 Exterior Product

We suppose in what follows that any reader of this paper knows the meaning of the exterior product of form fields and its main properties[9]. We simply recall here that if $\mathcal{A}_r \in \sec \bigwedge^r T^{*}M$, $\mathcal{B}_s \in \sec \bigwedge^s T^{*}M$ then

$$\mathcal{A}_r \wedge \mathcal{B}_s = (-1)^{rs} \mathcal{B}_s \wedge \mathcal{A}_r. \tag{26}$$

[9] We use the conventions of [22].

4.2 Scalar Product and Contractions

Let be $\mathcal{A}_r = a_1 \wedge ... \wedge a_r \in \sec \bigwedge^r T^*M$, $\mathcal{B}_r = b_1 \wedge ... \wedge b_r \in \sec \bigwedge^r T^*M$ where $a_i, b_j \in \sec \bigwedge^1 T^*M$ $(i, j = 1, 2, ..., r)$.

(i) The scalar product $\mathcal{A}_r \cdot \mathcal{B}_r$ is defined by

$$\mathcal{A}_r \cdot \mathcal{B}_r = (a_1 \wedge ... \wedge a_r) \cdot (b_1 \wedge ... \wedge b_r) = \begin{vmatrix} a_1 \cdot b_1 \; ....a_1 \cdot b_r \\ .... \\ a_r \cdot b_1 \; ....a_r \cdot b_r \end{vmatrix}. \tag{27}$$

where $a_i \cdot b_j := \mathtt{g}(a_i, b_j)$.

We agree that if $r = s = 0$, the scalar product is simple the ordinary product in the real field.

Also, if $r \neq s$, then $\mathcal{A}_r \cdot \mathcal{B}_s = 0$. Finally, the scalar product is extended by linearity for all sections of $\bigwedge T^*M$.

For $r \leq s$, $\mathcal{A}_r = a_1 \wedge ... \wedge a_r$, $\mathcal{B}_s = b_1 \wedge ... \wedge b_s$ we define the *left contraction* by

$$\lrcorner : (\mathcal{A}_r, \mathcal{B}_s) \mapsto \mathcal{A}_r \lrcorner \mathcal{B}_s = \sum_{i_1 < ... < i_r} \epsilon^{i_1 ..., i_s} (a_1 \wedge ... \wedge a_r) \cdot (b_{i_1} \wedge ... \wedge b_{i_r})^{\sim} b_{i_r+1} \wedge ... \wedge b_{i_s} \tag{28}$$

where $\sim$ is the reverse mapping (*reversion*) defined by

$$\sim : \sec \bigwedge^p T^*M \ni a_1 \wedge ... \wedge a_p \mapsto a_p \wedge ... \wedge a_1 \tag{29}$$

and extended by linearity to all sections of $\bigwedge T^*M$. We agree that for $\alpha, \beta \in \sec \bigwedge^0 T^*M$ the contraction is the ordinary (pointwise) product in the real field and that if $\alpha \in \sec \bigwedge^0 T^*M$, $\mathcal{A}_r \in \sec \bigwedge^r T^*M$, $\mathcal{B}_s \in \sec \bigwedge^s T^*M$ then $(\alpha \mathcal{A}_r) \lrcorner \mathcal{B}_s = \mathcal{A}_r \lrcorner (\alpha \mathcal{B}_s)$. Left contraction is extended by linearity to all pairs of elements of sections of $\bigwedge T^*M$, i.e., for $\mathcal{A}, \mathcal{B} \in \sec \bigwedge T^*M$

$$\mathcal{A} \lrcorner \mathcal{B} = \sum_{r,s} \langle \mathcal{A} \rangle_r \lrcorner \langle \mathcal{B} \rangle_s, \quad r \leq s, \tag{30}$$

where $\langle \mathcal{A} \rangle_r$ means the projection of $\mathcal{A}$ in $\bigwedge^r T^*M$.

It is also necessary to introduce the operator of *right contraction* denoted by $\llcorner$. The definition is obtained from the one presenting the left contraction with the imposition that $r \geq s$ and taking into account that now if $\mathcal{A}_r \in \sec \bigwedge^r T^*M$, $\mathcal{B}_s \in \sec \bigwedge^s T^*M$ then $\mathcal{B}_s \lrcorner \mathcal{A}_r = (-1)^{s(r-s)} \mathcal{A}_r \llcorner \mathcal{B}_s$.

4.3 Hodge Star Operator $\star$

The Hodge star operator is the mapping

$$\star : \sec \bigwedge^k T^*M \to \sec \bigwedge^{n-k} T^*M, \quad \mathcal{A}_k \mapsto \star\mathcal{A}_k$$

where for $\mathcal{A}_k \in \sec \bigwedge^k T^*M$

$$[\mathcal{B}_k \cdot \mathcal{A}_k]\tau_{\boldsymbol{g}} = \mathcal{B}_k \wedge \star\mathcal{A}_k, \quad \forall \mathcal{B}_k \in \sec \bigwedge^k T^*M \tag{31}$$

$\tau_{\boldsymbol{g}} \in \bigwedge^n T^*M$ is the *metric volume element*. Of course, the Hodge star operator is naturally extended to an isomorphism $\star : \sec \bigwedge T^*M \to \sec \bigwedge T^*M$ by linearity. The inverse $\star^{-1} : \sec \bigwedge^{n-r} T^*M \to \sec \bigwedge^r T^*M$ of the Hodge star operator is given by:

$$\star^{-1} = (-1)^{r(n-r)} \mathrm{sgn}\boldsymbol{g}\star, \tag{32}$$

where sgn $\boldsymbol{g} = \det \mathbf{g}/|\det \mathbf{g}|$ denotes the sign of the determinant of the matrix $(g_{\alpha\beta} = \ \mathbf{g}(e_\alpha, e_\beta))$, where $\{e_\alpha\}$ is an *arbitrary* basis of TU.

We can show that (see, e.g., [22]) that

$$\star\mathcal{A}_k = \widetilde{\mathcal{A}}_k \lrcorner \tau_{\boldsymbol{g}}, \tag{33}$$

where as noted before, in this paper $\widetilde{\mathcal{A}}_k$ denotes the *reverse* of $\mathcal{A}_k$.

Let $\{\vartheta^\alpha\}$ be the dual basis of $\{e_\alpha\}$ (i.e., it is a basis for $T^*U \equiv \bigwedge^1 T^*U$) then $\mathbf{g}(\vartheta^\alpha, \vartheta^\beta) = g^{\alpha\beta}$, with $g^{\alpha\beta} g_{\alpha\rho} = \delta^\beta_\rho$. Writing $\vartheta^{\mu_1 \cdots \mu_p} = \vartheta^{\mu_1} \wedge ... \wedge \vartheta^{\mu_p}$, $\vartheta^{\nu_{p+1} \cdots \nu_n} = \vartheta^{\nu_{p+1}} \wedge ... \wedge \vartheta^{\nu_n}$ we have from Eq.(33)

$$\star\, \theta^{\mu_1 \ldots \mu_p} = \frac{1}{(n-p)!} \sqrt{|\det \mathbf{g}|} g^{\mu_1 \nu_1} ... g^{\mu_p \nu_p} \epsilon_{\nu_1 \ldots \nu_n} \vartheta^{\nu_{p+1} \ldots \nu_n}. \tag{34}$$

Some identities (used below) involving the Hodge star operator, the exterior product and contractions are[10]:

$$\begin{array}{l} A_r \wedge \star B_s = B_s \wedge \star A_r; \quad r = s \\ A_r \cdot \star B_s = B_s \cdot \star A_r; \quad r + s = n \\ A_r \wedge \star B_s = (-1)^{r(s-1)} \star (\tilde{A}_r \lrcorner B_s); \quad r \leq s \\ A_r \lrcorner \star B_s = (-1)^{rs} \star (\tilde{A}_r \wedge B_s); \quad r + s \leq n \\ \star\tau_{\boldsymbol{g}} = \mathrm{sign} \quad \boldsymbol{g}; \quad \star 1 = \tau_{\boldsymbol{g}}. \end{array} \tag{35}$$

[10] See also the last formula in Eq.(45) which uses the Clifford product.

4.4 Exterior derivative d and Hodge coderivative δ

The *exterior derivative is a mapping*

$$d : \sec \bigwedge T^*M \to \sec \bigwedge T^*M,$$

satisfying:

$$\begin{array}{ll}\text{(i)} & d(A+B) = dA + dB; \\ \text{(ii)} & d(A \wedge B) = dA \wedge B + \bar{A} \wedge dB; \\ \text{(iii)} & df(v) = v(f); \\ \text{(iv)} & d^2 = 0,\end{array} \tag{36}$$

for every $A, B \in \sec \bigwedge T^*M$, $f \in \sec \bigwedge^0 T^*M$ and $v \in \sec TM$.

The *Hodge codifferential* operator in the Hodge bundle is the mapping $\delta : \sec \bigwedge^r T^*M \to \sec \bigwedge^{r-1} T^*M$, given for homogeneous multiforms, by:

$$\delta = (-1)^r \star^{-1} d\star, \tag{37}$$

where $\star$ is the Hodge star operator. The operator δ extends by linearity to all $\bigwedge T^*M$

The *Hodge Laplacian (or Hodge D'Alembertian)* operator is the mapping

$$\lozenge : \sec \bigwedge T^*M \to \sec \bigwedge T^*M$$

given by:

$$\lozenge = -(d\delta + \delta d). \tag{38}$$

The exterior derivative, the Hodge codifferential and the Hodge D' Alembertian satisfy the relations:

$$\begin{array}{l} dd = \delta\delta = 0; \quad \lozenge = (d-\delta)^2 \\ d\lozenge = \lozenge d; \quad \delta\lozenge = \lozenge\delta \\ \delta\star = (-1)^{r+1} \star d; \quad \star\delta = (-1)^r d\star \\ d\delta\star = \star\delta d; \quad \star d\delta = \delta d\star; \quad \star\lozenge = \lozenge\star . \end{array} \tag{39}$$

5 Clifford Bundles

Let $(M, \mathbf{g}, \nabla)$ be a Riemannian, Lorentzian or Riemann-Cartan structure[11]. As before let $\mathtt{g} \in \sec T_0^2 M$ be the metric on the cotangent bundle associated with $\mathbf{g} \in \sec T_2^0 M$. Then $T_x^* M \simeq \mathbb{R}^{p,q}$, where $\mathbb{R}^{p,q}$

[11] ∇ may be the Levi-Civita connection $\mathring{D}$ of $\mathbf{g}$ or an arbitrary Riemann-Cartan connection D.

is a vector space equipped with a scalar product $\bullet \equiv \mathtt{g}|_x$ of signature (p,q). The Clifford bundle of differential forms $\mathcal{C}\ell(M,\mathtt{g})$ is the bundle of algebras, i.e., $\mathcal{C}\ell(M,\mathtt{g}) = \cup_{x\in M}\mathcal{C}\ell(T_x^*M,\bullet)$, where $\forall x \in M$, $\mathcal{C}\ell(T_x^*M,\bullet) = \mathbb{R}_{p,q}$, a real Clifford algebra. When the structure $(M,\boldsymbol{g},\nabla)$ is part of a Lorentzian or Riemann-Cartan spacetime $\mathcal{C}\ell(T_x^*M,\bullet) = \mathbb{R}_{1,3}$ the so called *spacetime algebra*. Recall also that $\mathcal{C}\ell(M,\mathtt{g})$ is a vector bundle associated with the $\mathtt{g}$-*orthonormal coframe bundle* $\mathbf{P}_{\mathrm{SO}^e_{(p,q)}}(M,\mathtt{g})$, i.e., $\mathcal{C}\ell(M,\mathtt{g}) = P_{\mathrm{SO}^e_{(p,q)}}(M,\mathtt{g})\times_{ad}\mathbb{R}_{1,3}$ (see more details in, e.g., [16, 22]). For any $x \in M$, $\mathcal{C}\ell(T_x^*M,\bullet)$ is a linear space over the real field $\mathbb{R}$. Moreover, $\mathcal{C}\ell(T_x^*M)$ is isomorphic as a real vector space to the Cartan algebra $\bigwedge T_x^*M$ of the cotangent space. Then, sections of $\mathcal{C}\ell(M,\mathtt{g})$ can be represented as a *sum* of non homogeneous differential forms. Let now $\{\mathbf{e_a}\}$ be an orthonormal basis for TU and $\{\theta^{\mathbf{a}}\}$ its dual basis. Then, $\mathtt{g}(\theta^{\mathbf{a}},\theta^{\mathbf{b}}) = \eta^{\mathbf{ab}}$.

5.1 Clifford Product

The fundamental *Clifford product* (in what follows to be denoted by juxtaposition of symbols) is generated by

$$\theta^{\mathbf{a}}\theta^{\mathbf{b}} + \theta^{\mathbf{b}}\theta^{\mathbf{a}} = 2\eta^{\mathbf{ab}} \tag{40}$$

and if $\mathcal{C} \in \mathcal{C}\ell(M,\mathtt{g})$ we have

$$\mathcal{C} = s + v_{\mathbf{a}}\theta^{\mathbf{a}} + \frac{1}{2!}b_{\mathbf{ab}}\theta^{\mathbf{a}}\theta^{\mathbf{b}} + \frac{1}{3!}a_{\mathbf{abc}}\theta^{\mathbf{a}}\theta^{\mathbf{b}}\theta^{\mathbf{c}} + p\theta^{n+1}\ , \tag{41}$$

where $\tau_{\boldsymbol{g}} := \theta^{n+1} = \theta^0\theta^1\theta^2\theta^3...\theta^n$ is the volume element and s, $v_{\mathbf{a}}$, $b_{\mathbf{ab}}$, $a_{\mathbf{abc}}$, $p \in \sec\bigwedge^0 T^*M \hookrightarrow \sec\mathcal{C}\ell(M,\mathtt{g})$.

Let $\mathcal{A}_r, \in \sec\bigwedge^r T^*M \hookrightarrow \sec\mathcal{C}\ell(M,\mathtt{g})$, $\mathcal{B}_s \in \sec\bigwedge^s T^*M \hookrightarrow \sec\mathcal{C}\ell(M,\mathtt{g})$. For $r = s = 1$, we define the *scalar product* as follows:

For $a, b \in \sec\bigwedge^1 T^*M \hookrightarrow \sec\mathcal{C}\ell(M,\mathtt{g})$,

$$a\cdot b = \frac{1}{2}(ab + ba) = \mathtt{g}(a,b). \tag{42}$$

We identify the *exterior product* $((\forall r, s = 0,1,2,3,...,n)$ of homogeneous forms (already introduced above) by

$$\mathcal{A}_r \wedge \mathcal{B}_s = \langle\mathcal{A}_r\mathcal{B}_s\rangle_{r+s}, \tag{43}$$

where $\langle\rangle_k$ is the *component* in $\bigwedge^k T^*M$ (projection) of the Clifford field. The exterior product is extended by linearity to all sections of $\mathcal{C}\ell(M,\mathtt{g})$.

The scalar product, the left and the right are defined for homogeneous form fields that are sections of the Clifford bundle in exactly the same way as in the Hodge bundle and they are extended by linearity for all sections of $\mathcal{C}\ell(M,\mathbf{g})$.

In particular, for $\mathcal{A},\mathcal{B} \in \sec \mathcal{C}\ell(M,\mathbf{g})$ we have

$$\mathcal{A} \lrcorner \mathcal{B} = \sum_{r,s} \langle \mathcal{A} \rangle_r \lrcorner \langle \mathcal{B} \rangle_s, \quad r \leq s. \tag{44}$$

The main formulas used in the present paper can be obtained (details may be found in [22]) from the following ones (where $a \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\mathbf{g})$):

$$\begin{aligned}
a\mathcal{B}_s &= a \lrcorner \mathcal{B}_s + a \wedge \mathcal{B}_s, \quad \mathcal{B}_s a = \mathcal{B}_s \llcorner a + \mathcal{B}_s \wedge a,\\
a \lrcorner \mathcal{B}_s &= \frac{1}{2}(a\mathcal{B}_s - (-1)^s \mathcal{B}_s a),\\
\mathcal{A}_r \lrcorner \mathcal{B}_s &= (-1)^{r(s-r)} \mathcal{B}_s \llcorner \mathcal{A}_r,\\
a \wedge \mathcal{B}_s &= \frac{1}{2}(a\mathcal{B}_s + (-1)^s \mathcal{B}_s a),\\
\mathcal{A}_r \mathcal{B}_s &= \langle \mathcal{A}_r \mathcal{B}_s \rangle_{|r-s|} + \langle \mathcal{A}_r \mathcal{B}_s \rangle_{|r-s|+2} + \ldots + \langle \mathcal{A}_r \mathcal{B}_s \rangle_{|r+s|}\\
&= \sum_{k=0}^{m} \langle \mathcal{A}_r \mathcal{B}_s \rangle_{|r-s|+2k}\\
\mathcal{A}_r \cdot \mathcal{B}_r &= \mathcal{B}_r \cdot \mathcal{A}_r = \widetilde{\mathcal{A}}_r \quad \lrcorner \mathcal{B}_r = \mathcal{A}_r \llcorner \widetilde{\mathcal{B}}_r = \langle \widetilde{\mathcal{A}}_r \mathcal{B}_r \rangle_0 = \langle \mathcal{A}_r \widetilde{\mathcal{B}}_r \rangle_0,\\
\star \mathcal{A}_k &= \widetilde{\mathcal{A}}_k \lrcorner \tau_{\boldsymbol{g}} = \widetilde{\mathcal{A}}_k \tau_{\boldsymbol{g}}.
\end{aligned} \tag{45}$$

Two other important identities to be used below are:

$$a \lrcorner (\mathcal{X} \wedge \mathcal{Y}) = (a \lrcorner \mathcal{X}) \wedge \mathcal{Y} + \hat{\mathcal{X}} \wedge (a \lrcorner \mathcal{Y}), \tag{46}$$

for any $a \in \sec \bigwedge^1 T^*M$ and $\mathcal{X},\mathcal{Y} \in \sec \bigwedge T^*M$, and

$$A \lrcorner (B \lrcorner C) = (A \wedge B) \lrcorner C, \tag{47}$$

for any $A, B, C \in \sec \bigwedge T^*M \hookrightarrow \mathcal{C}\ell(M,\mathbf{g})$

5.2 Dirac Operators Acting on Sections of a Clifford Bundle $\mathcal{C}\ell(M,\mathbf{g})$

5.2.1 The Dirac Operator $\boldsymbol{\partial}$ Associated to D

The Dirac operator associated to a general Riemann-Cartan structure $(M,\mathbf{g},D)$ acting on sections of $\mathcal{C}\ell(M,\mathtt{g})$ is the invariant first order differential operator

$$\boldsymbol{\partial} = \theta^{\mathbf{a}} D_{\mathbf{e_a}} = \vartheta^{\alpha} D_{e_\alpha}. \tag{48}$$

For any $\mathcal{A} \in \sec \bigwedge T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\mathtt{g})$ we define

$$\begin{aligned} \boldsymbol{\partial}\mathcal{A} &= \boldsymbol{\partial} \wedge \mathcal{A} + \boldsymbol{\partial} \lrcorner \mathcal{A} \\ \boldsymbol{\partial} \wedge \mathcal{A} &= \theta^{\mathbf{a}} \wedge (D_{\mathbf{e_a}}\mathcal{A}), \quad \boldsymbol{\partial} \lrcorner \mathcal{A} = \theta^{\mathbf{a}} \lrcorner (D_{\mathbf{e_a}}\mathcal{A}). \end{aligned} \tag{49}$$

5.2.2 Clifford Bundle Calculation of $D_{\mathbf{e_a}}\mathcal{A}$

Recall that the *reciprocal* basis of $\{\theta^{\mathbf{b}}\}$ is denoted $\{\theta_{\mathbf{a}}\}$ with $\theta_{\mathbf{a}} \cdot \theta_{\mathbf{b}} = \eta_{\mathbf{ab}}$ ($\eta_{\mathbf{ab}} = \mathrm{diag}(1, ..., 1, -1, ..., -1)$) and that

$$D_{\mathbf{e_a}}\theta^{\mathbf{b}} = -\omega^{\mathbf{b}}_{\mathbf{ac}}\theta^{\mathbf{c}} = -\omega^{\mathbf{bc}}_{\mathbf{a}}\theta_{\mathbf{c}}, \tag{50}$$

with $\omega^{\mathbf{bc}}_{\mathbf{a}} = -\omega^{\mathbf{cb}}_{\mathbf{a}}$, and $\omega^{\mathbf{bc}}_{\mathbf{a}} = \eta^{\mathbf{bk}}\omega_{\mathbf{kal}}\eta^{\mathbf{cl}}$, $\omega_{\mathbf{abc}} = \eta_{\mathbf{ad}}\omega^{\mathbf{d}}_{\mathbf{bc}} = -\omega_{\mathbf{cba}}$. Defining

$$\omega_{\mathbf{a}} = \frac{1}{2}\omega^{\mathbf{bc}}_{\mathbf{a}}\theta_{\mathbf{b}} \wedge \theta_{\mathbf{c}} \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\mathtt{g}), \tag{51}$$

we have (by linearity) that [16] for any $\mathcal{A} \in \sec \bigwedge T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\mathtt{g})$

$$D_{\mathbf{e_a}}\mathcal{A} = \partial_{\mathbf{e_a}}\mathcal{A} + \frac{1}{2}[\omega_{\mathbf{a}}, \mathcal{A}], \tag{52}$$

where $\partial_{\mathbf{e_a}}$ is the Pfaff derivative, i.e., for any $A = \frac{1}{p!}A_{\mathbf{i}_1...\mathbf{i}_p}\theta^{\mathbf{i}_1}...\theta^{\cdot\mathbf{i}_p} \in \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M,\mathtt{g})$ it is:

$$\partial_{\mathbf{e_a}}A = \frac{1}{p!}[\mathbf{e_a}(A_{\mathbf{i}_1...\mathbf{i}_p})]\theta^{\mathbf{i}_1}...\theta^{\cdot\mathbf{i}_p} \quad . \tag{53}$$

5.2.3 The Dirac Operator $\boldsymbol{\partial}$ Associated to $\mathring{D}$

Using Eq.(52) we can show that for the case of a Riemannian or Lorentzian structure $(M, \boldsymbol{g}, \mathring{D})$ the standard Dirac operator defined by:

$$\begin{aligned} \boldsymbol{\partial} &= \theta^{\mathbf{a}} \mathring{D}_{\mathbf{e_a}} = \vartheta^{\alpha} \mathring{D}_{e_\alpha}, \\ \boldsymbol{\partial}\mathcal{A} &= \boldsymbol{\partial} \wedge \mathcal{A} + \boldsymbol{\partial} \lrcorner \mathcal{A} \end{aligned} \tag{54}$$

for any $\mathcal{A} \in \sec \bigwedge T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ is such that

$$\boldsymbol{\partial} \wedge \mathcal{A} = d\mathcal{A} \ , \quad \boldsymbol{\partial} \lrcorner \mathcal{A} = -\delta \mathcal{A} \tag{55}$$

i.e.,

$$\boldsymbol{\partial} = d - \delta \tag{56}$$

6 Torsion, Curvature and Cartan Structure Equations

As we said in the beginning of Section 1 a given structure $(M, \boldsymbol{g})$ may admit many different metric compatible connections. Let then $\mathring{D}$ be the Levi-Civita connection of $\boldsymbol{g}$ and D a Riemann-Cartan connection acting on the tensor fields defined on M.

Let $U \subset M$ and consider a chart of the maximal atlas of M covering U with arbitrary coordinates $\{x^\mu\}$. Let $\{\boldsymbol{\partial}_\mu\}$ be a basis for TU, $U \subset M$ and let $\{\theta^\mu = dx^\mu\}$ be the dual basis of $\{\boldsymbol{\partial}_\mu\}$. The reciprocal basis of $\{\theta^\mu\}$ is denoted $\{\theta^\mu\}$, and $\mathtt{g}(\theta^\mu, \theta_\nu) := \theta^\mu \cdot \theta_\nu = \delta^\mu_\nu$.

Let also $\{\mathbf{e_a}\}$ be an orthonormal basis for $TU \subset TM$ with $\mathbf{e_b} = q^\nu_{\mathbf{b}} \boldsymbol{\partial}_\nu$. The dual basis of TU is $\{\theta^{\mathbf{a}}\}$, with $\theta^{\mathbf{a}} = q^{\mathbf{a}}_\mu dx^\mu$. Also, $\{\theta_{\mathbf{b}}\}$ is the reciprocal basis of $\{\theta^{\mathbf{a}}\}$, i.e. $\theta^{\mathbf{a}} \cdot \theta_{\mathbf{b}} = \delta^{\mathbf{a}}_{\mathbf{b}}$. An arbitrary frame on $TU \subset TM$, coordinate or orthonormal will be denote by $\{e_\alpha\}$. Its dual frame will be denoted by $\{\vartheta^\rho\}$ (i.e., $\vartheta^\rho(e_\alpha) = \delta^\rho_\alpha$).

6.1 Torsion and Curvature Operators

Definition 7 *The torsion and curvature operators τ and ρ of a connection D, are respectively the mappings:*

$$\tau(\mathbf{u}, \mathbf{v}) = D_{\mathbf{u}}\mathbf{v} - D_{\mathbf{v}}\mathbf{u} - [\mathbf{u}, \mathbf{v}], \tag{57}$$

$$\rho(\mathbf{u}, \mathbf{v}) = D_{\mathbf{u}} D_{\mathbf{v}} - D_{\mathbf{v}} D_{\mathbf{u}} - D_{[\mathbf{u}, \mathbf{v}]}, \tag{58}$$

for every $\mathbf{u}, \mathbf{v} \in \sec TM$.

6.2 Torsion and Curvature Tensors

Definition 8 *The torsion andcurvature tensors of a connection D, are respectively the mappings:*

$$\mathbf{T}(\alpha, \mathbf{u}, \mathbf{v}) = \alpha\left(\tau(\mathbf{u}, \mathbf{v})\right), \tag{59}$$
$$\mathbf{R}(\mathbf{w}, \alpha, \mathbf{u}, \mathbf{v}) = \alpha(\rho(\mathbf{u}, \mathbf{v})\mathbf{w}), \tag{60}$$

for every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \sec TM$ *and* $\alpha \in \sec \bigwedge^1 T^*M$.

We recall that for any differentiable functions f, g and h we have

$$\begin{aligned} \tau(g\mathbf{u}, h\mathbf{v}) &= gh\tau(\mathbf{u}, \mathbf{v}), \\ \rho(g\mathbf{u}, h\mathbf{v}) f\mathbf{w} &= ghf\rho(\mathbf{u}, \mathbf{v})\mathbf{w} \end{aligned} \tag{61}$$

6.2.1 Properties of the Riemann Tensor for a Metric Compatible Connection

Note that it is quite obvious that

$$\mathbf{R}(\mathbf{w}, \alpha, \mathbf{u}, \mathbf{v}) = \mathbf{R}(\mathbf{w}, \alpha, \mathbf{v}, \mathbf{u}). \tag{62}$$

Define the tensor field $\mathbf{R}'$ as the mapping such that for every $\mathbf{a}, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \sec TM$ and $\alpha \in \sec \bigwedge^1 T^*M$.

$$\mathbf{R}'(\mathbf{w}, \mathbf{a}, \mathbf{u}, \mathbf{v}) = \mathbf{R}(\mathbf{w}, \alpha, \mathbf{v}, \mathbf{u}). \tag{63}$$

It is quite ovious that

$$\mathbf{R}'(\mathbf{w}, \mathbf{a}, \mathbf{u}, \mathbf{v}) = \mathbf{a}\cdot(\rho(\mathbf{u}, \mathbf{v})\mathbf{w}), \tag{64}$$

where

$$\alpha = \boldsymbol{g}(\mathbf{a}, \ ) \ , \quad \mathbf{a} = \mathtt{g}(\alpha, \ ) \tag{65}$$

We now show that for any structure $(M, \boldsymbol{g}, D)$ such that $D\boldsymbol{g} = 0$ we have for $\mathbf{c}, \mathbf{u}, \mathbf{v} \in \sec TM$,

$$\mathbf{R}'(\mathbf{c}, \mathbf{c}, \mathbf{u}, \mathbf{v}) = \mathbf{c}\cdot(\rho(\mathbf{u}, \mathbf{v})\mathbf{c}) = 0. \tag{66}$$

We start recalling that for every metric compatible connection it holds:

$$\begin{aligned}\mathbf{u}(\mathbf{v}(\mathbf{c}\cdot\mathbf{c}) &= \mathbf{u}(D_{\mathbf{v}}\mathbf{c}\cdot\mathbf{c}+\mathbf{c}\cdot D_{\mathbf{v}}\mathbf{c}) = 2\mathbf{u}(D_{\mathbf{v}}\mathbf{c}\cdot\mathbf{c}) \\ &= 2(D_{\mathbf{u}}D_{\mathbf{v}}\mathbf{c})\cdot\mathbf{c}+2(D_{\mathbf{u}}\mathbf{c})\cdot\mathbf{D}_{\mathbf{v}}\mathbf{c},\end{aligned} \tag{67}$$

Exachanging $\mathbf{u}\leftrightarrow\mathbf{v}$ in the last equation we get

$$\mathbf{v}(\mathbf{u}(\mathbf{c}\cdot\mathbf{c}) = 2(D_{\mathbf{v}}D_{\mathbf{u}}\mathbf{c})\cdot\mathbf{c}+2(D_{\mathbf{v}}\mathbf{c})\cdot\mathbf{D}_{\mathbf{u}}\mathbf{c}. \tag{68}$$

Subtracting Eq.(67) from Eq.(68) we have

$$[\mathbf{u},\mathbf{v}](\mathbf{c}\cdot\mathbf{c}) = 2([D_{\mathbf{u}},D_{\mathbf{v}}]\mathbf{c})\cdot\mathbf{c} \tag{69}$$

But since

$$[\mathbf{u},\mathbf{v}](\mathbf{c}\cdot\mathbf{c}) = D_{[\mathbf{u},\mathbf{v}]}(\mathbf{c}\cdot\mathbf{c}) = 2(D_{[\mathbf{u},\mathbf{v}]}\mathbf{c})\cdot\mathbf{c}, \tag{70}$$

we have from Eq.(69) that

$$([D_{\mathbf{u}},D_{\mathbf{v}}]\mathbf{c}-D_{[\mathbf{u},\mathbf{v}]}\mathbf{c})\cdot\mathbf{c}=0\ , \tag{71}$$

and it follows that $\mathbf{R}'(\mathbf{c},\mathbf{c},\mathbf{u},\mathbf{v})=0$ as we wanted to show.

Exercise 9 *Prove that for any metric compatible connection,*

$$\mathbf{R}'(\mathbf{c},\mathbf{d},\mathbf{u},\mathbf{v})=\mathbf{R}'(\mathbf{d},\mathbf{c},\mathbf{u},\mathbf{v}). \tag{72}$$

Given an arbitrary frame $\{e_\alpha\}$ on $TU\subset TM$, let $\{\vartheta^\rho\}$ be the *dual frame.* We write:

$$\begin{aligned}[e_\alpha,e_\beta] &= c_{\alpha\beta}^{\rho}e_\rho \\ D_{e_\alpha}e_\beta &= \mathbf{L}_{\alpha\beta}^{\rho}e_\rho,\end{aligned} \tag{73}$$

where $c_{\alpha\beta}^{\rho}$ are the *structure coefficients*of the frame $\{e_\alpha\}$ and $\mathbf{L}_{\alpha\beta}^{\rho}$ are the *connection coefficients*in this frame. Then, the components of the torsion and curvature tensors are given, respectively, by:

$$\begin{aligned}\mathbf{T}(\vartheta^\rho,e_\alpha,e_\beta) &= T_{\alpha\beta}^{\rho}=\mathbf{L}_{\alpha\beta}^{\rho}-\mathbf{L}_{\beta\alpha}^{\rho}-c_{\alpha\beta}^{\rho} \\ \mathbf{R}(e_\mu,\vartheta^\rho,e_\alpha,e_\beta) &= R_{\mu}{}^{\rho}{}_{\alpha\beta} \\ &= e_\alpha(\mathbf{L}_{\beta\mu}^{\rho})-e_\beta(\mathbf{L}_{\alpha\mu}^{\rho})+\mathbf{L}_{\alpha\sigma}^{\rho}\mathbf{L}_{\beta\mu}^{\sigma}-\mathbf{L}_{\beta\sigma}^{\rho}\mathbf{L}_{\alpha\mu}^{\sigma}-c_{\alpha\beta}^{\sigma}\mathbf{L}_{\sigma\mu}^{\rho}.\end{aligned} \tag{74}$$

It is important for what follows to keep in mind the definition of the (symmetric) Ricci tensor, here denoted $\mathbf{Ric} \in \sec T_2^0 M$ and which in an arbitrary basis is written as

$$\mathbf{Ric} = R_{\mu\nu}\vartheta^\mu \otimes \vartheta^\nu := R_\mu{}^\rho{}_{\rho\nu}\vartheta^\mu \otimes \vartheta^\nu \tag{75}$$

It is crucial here to take into account the *place* where the contraction in the Riemann tensor takes place according to our conventions.

We also have:

$$\begin{aligned} d\vartheta^\rho &= -\tfrac{1}{2}c^\rho_{\alpha\beta}\vartheta^\alpha \wedge \vartheta^\beta \\ D_{e_\alpha}\vartheta^\rho &= -\mathbf{L}^\rho_{\alpha\beta}\vartheta^\beta \end{aligned} \tag{76}$$

where $\omega^\rho_\beta \in \sec \bigwedge^1 T^*M$ are the *connection 1-forms,* $\mathbf{L}^\rho_{\alpha\beta}$ are said to be the connection coefficients in the given basis, and the $\mathcal{T}^\rho \in \sec \bigwedge^2 T^*M$ are the *torsion 2-forms* and the $\mathcal{R}^\rho_\beta \in \sec \bigwedge^2 T^*M$ are the *curvature 2-forms*, given by:

$$\begin{aligned} \omega^\rho_\beta &= \mathbf{L}^\rho_{\alpha\beta}\vartheta^\alpha, \\ \mathcal{T}^\rho &= \frac{1}{2}T^\rho_{\alpha\beta}\vartheta^\alpha \wedge \theta^\beta \\ \mathcal{R}^\rho_\mu &= \frac{1}{2}R_\mu{}^\rho{}_{\alpha\beta}\vartheta^\alpha \wedge \vartheta^\beta. \end{aligned} \tag{77}$$

Multiplying Eqs.(74) by $\frac{1}{2}\vartheta^\alpha \wedge \vartheta^\beta$ and using Eqs.(76) and (77), we get:

6.3 Cartan Structure Equations

$$\begin{aligned} d\vartheta^\rho + \omega^\rho_\beta \wedge \vartheta^\beta &= \mathcal{T}^\rho, \\ d\omega^\rho_\mu + \omega^\rho_\beta \wedge \omega^\beta_\mu &= \mathcal{R}^\rho_\mu. \end{aligned} \tag{78}$$

We can show that the torsion and (Riemann) curvature tensors can be written as

$$\mathbf{T} = e_\alpha \otimes \mathcal{T}^\alpha, \tag{79}$$

$$\mathbf{R} = e_\rho \otimes e^\mu \otimes \mathcal{R}^\rho_\mu. \tag{80}$$

7 Exterior Covariant Derivative D

Sometimes, Eqs.(78) are written by some authors [27] as:

$$\mathbf{D}\vartheta^{\rho} = \mathcal{T}^{\rho}, \tag{81}$$

$$\text{“}\ \mathbf{D}\omega_{\mu}^{\rho} = \mathcal{R}_{\mu}^{\rho}.\ \text{”} \tag{82}$$

and $\mathbf{D} : \sec \bigwedge T^*M \to \sec \bigwedge T^*M$ is said to be the *exterior covariant derivative*related to the connection D. Now, Eq.(82) has been printed with quotation marks due to the fact that it is an *incorrect* equation. Indeed, a *legitimate* exterior covariant derivative operator[12] is a concept that can be defined for $(p+q)$-indexed r-form fields[13] as follows. Suppose that $X \in \sec T_p^{r+q}M$ and let

$$X_{\nu_1....\nu_q}^{\mu_1....\mu_p} \in \sec \bigwedge^r T^*M, \tag{83}$$

such that for $v_i \in \sec TM$, $i = 0, 1, 2, .., r$,

$$X_{\nu_1....\nu_q}^{\mu_1....\mu_p}(v_1, ..., v_r) = X(v_1, ..., v_r, e_{\nu_1}, ..., e_{\nu_q}, \vartheta^{\mu_1}, ..., \vartheta^{\mu_p}). \tag{84}$$

The exterior covariant differential $\mathbf{D}$ of $X_{\nu_1....\nu_q}^{\mu_1....\mu_p}$ on a manifold with a general connection D is the mapping:

$$\mathbf{D} : \sec \bigwedge^r T^*M \to \sec \bigwedge^{r+1} T^*M\ ,\ \ 0 \leq r \leq 4, \tag{85}$$

such that[14]

$$\begin{aligned} &(r+1)\mathbf{D}X_{\nu_1....\nu_q}^{\mu_1....\mu_p}(v_0, v_1, ..., v_r) \\ &= \sum_{\nu=0}^{r}(-1)^{\nu} D_{\mathbf{e}_\nu} X(v_0, v_1, ..., \check{v}_\nu, ...v_r, e_{\nu_1}, ..., e_{\nu_q}, \vartheta^{\mu_1}, ..., \vartheta^{\mu_p}) \\ &- \sum_{0 \leq \lambda, \varsigma \leq r} (-1)^{\nu+\varsigma} X(\mathbf{T}(v_\lambda, v_\varsigma), v_0, v_1, ..., \check{v}_\lambda, ..., \check{v}_\varsigma, ..., v_r, e_{\nu_1}, ..., e_{\nu_q}, \vartheta^{\mu_1}, ..., \vartheta^{\mu_p}). \end{aligned} \tag{86}$$

[12] Sometimes also called exterior covariant differential.

[13] Which is not the case of the connection 1-forms ω_{β}^{α}, despite the name. More precisely, the ω_{β}^{α} are not true indexed forms, i.e., there does not exist a tensor field ω such that $\omega(e_i, e_\beta, \vartheta^\alpha) = \omega_{\beta}^{\alpha}(e_i)$.

[14] As usual the inverted hat over a symbol (in Eq.(86)) means that the corresponding symbol is missing in the expression.

Then, we may verify that

$$\begin{aligned}\mathbf{D}X^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} &= dX^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} + \omega^{\mu_1}_{\mu_s}\wedge X^{\mu_s\ldots\mu_p}_{\nu_1\ldots\nu_q} + \ldots + \quad \omega^{\mu_1}_{\mu_s}\wedge X^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} \\ &\quad - \omega^{\nu_s}_{\nu_1}\wedge X^{\mu_1\ldots\mu_p}_{\nu_s\ldots\nu_q} - \ldots - \quad \omega^{\mu_1}_{\mu_s}\wedge X^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_s}.\end{aligned} \tag{87}$$

Remark 10 *Note that if* Eq.(87) *is applied on any one of the connection* 1*-forms* ω^μ_ν *we would get* $\mathbf{D}\omega^\mu_\nu = d\omega^\mu_\nu + \omega^\mu_\alpha\wedge\omega^\alpha_\nu - \omega^\alpha_\nu\wedge\omega^\mu_\alpha$*. So, we see that the symbol* $\mathbf{D}\omega^\mu_\nu$ *in* Eq.(82)*, supposedly defining the curvature* 2*-forms is simply wrong despite this being an equation printed in many Physics textbooks and many professional articles*[15]*! .*

7.1 Properties of **D**

The exterior covariant derivative **D** satisfy the following properties:

(a) For any $X^J \in \sec\bigwedge^r T^*M$ and $Y^K \in \sec\bigwedge^s T^*M$ are sets of indexed forms[16], then

$$\mathbf{D}(X^J\wedge Y^K) = \mathbf{D}X^J\wedge Y^K + (-1)^{rs}X^J\wedge\mathbf{D}Y^K. \tag{88}$$

(b) For any $X^{\mu_1\ldots\mu_p} \in \sec\bigwedge^r T^*M$ then

$$\mathbf{DD}X^{\mu_1\ldots\mu_p} = dX^{\mu_1\ldots\mu_p} + \mathcal{R}^{\mu_1}_{\mu_s}\wedge X^{\mu_s\ldots\mu_p} + \ldots\mathcal{R}^{\mu_p}_{\mu_s}\wedge X^{\mu_1\ldots\mu_s}. \tag{89}$$

(c) For any metric-compatible connection D if $g = g_{\mu\nu}\vartheta^\mu\otimes\vartheta^\nu$ then,

$$\mathbf{D}g_{\mu\nu} = 0. \tag{90}$$

7.2 Formula for Computation of the Connection 1*- Forms* $\omega^{\mathbf{a}}_{\mathbf{b}}$

In an orthonormal cobasis $\{\theta^{\mathbf{a}}\}$ we have (see, e.g., [22]) for the connection 1-forms

$$\omega^{\mathbf{cd}} = \frac{1}{2}\left[\theta^{\mathbf{d}}\lrcorner d\theta^{\mathbf{c}} - \theta^{\mathbf{c}}\lrcorner d\theta^{\mathbf{d}} + \theta^{\mathbf{c}}\lrcorner(\theta^{\mathbf{d}}\lrcorner d\theta_{\mathbf{a}})\theta^{\mathbf{a}}\right], \tag{91}$$

or taking into account that $d\theta^{\mathbf{a}} = -\frac{1}{2}c^{\mathbf{a}}_{\mathbf{jk}}\theta^{\mathbf{j}}\wedge\theta^{\mathbf{k}}$,

$$\omega^{\mathbf{cd}} = \frac{1}{2}(-c^{\mathbf{c}}_{\mathbf{jk}}\eta^{\mathbf{dj}} + c^{\mathbf{d}}_{\mathbf{jk}}\eta^{\mathbf{cj}} - \eta^{\mathbf{ca}}\eta_{\mathbf{bk}}\eta^{\mathbf{dj}}c^{\mathbf{b}}_{\mathbf{ja}})\theta^{\mathbf{k}}. \tag{92}$$

[15] The authors of reference [27] knows exactly what they are doing and use "$\mathbf{D}\omega^\rho_\mu = \mathcal{R}^\rho_\mu$" only as a short notation. Unfortunately this is not the case for some other authors.

[16] Multi indices are here represented by J and K.

8 Relation Between the Connections $\mathring{D}$ and D

As we said above a given structure $(M, \mathbf{g})$ in general admits many different connections. Let then $\mathring{D}$ and D be the Levi-Civita connection of $\mathbf{g}$ on M and D and arbitrary Riemann-Cartan connection. Given an arbitrary basis $\{e_\alpha\}$ on $TU \subset TM$, let $\{\vartheta^\rho\}$ be the dual frame. We write for the connection coefficients of the Riemann-Cartan and the Levi-Civita connections in the arbitrary bases $\{e_\alpha\}$,$\{\vartheta^\rho\}$:

$$
\begin{aligned}
D_{e_\alpha} e_\beta &= \mathbf{L}^{\rho}_{\alpha\beta} e_\rho, \quad D_{e_\alpha} \vartheta^\rho = -\mathbf{L}^{\rho}_{\alpha\beta} \vartheta^\beta, \\
\mathring{D}_{e_\alpha} e_\beta &= \mathring{\mathbf{L}}^{\rho}_{\alpha\beta} e_\rho, \quad \mathring{D}_{e_\alpha} \vartheta^\rho = -\mathring{\mathbf{L}}^{\rho}_{\alpha\beta} \vartheta^\beta.
\end{aligned} \tag{93}
$$

Moreover, the structure coefficients of the arbitrary basis $\{e_\alpha\}$ are:

$$
[e_\alpha, e_\beta] = c^{\rho}_{\alpha\beta} e_\rho. \tag{94}
$$

Let moreover,

$$
b^{\rho}_{\alpha\beta} = -(\pounds_{e^\rho} \mathbf{g})_{\alpha\beta}, \tag{95}
$$

where $\pounds_{e^\rho}$ is the Lie derivative in the direction of the vector field e^ρ. Then, we have the noticeable formula (for a proof, see, e.g., [22]):

$$
\mathbf{L}^{\rho}_{\alpha\beta} = \mathring{\mathbf{L}}^{\rho}_{\alpha\beta} + \frac{1}{2} T^{\rho}_{\alpha\beta} + \frac{1}{2} S^{\rho}_{\alpha\beta}, \tag{96}
$$

where the tensor $S^{\rho}_{\alpha\beta}$ is called the strain tensor of the connection and can be decomposed as:

$$
S^{\rho}_{\alpha\beta} = \breve{S}^{\rho}_{\alpha\beta} + \frac{2}{n} s^\rho g_{\alpha\beta} \tag{97}
$$

where $\breve{S}^{\rho}_{\alpha\beta}$ is its traceless part, is called the *shear* of the connection, and

$$
s^\rho = \frac{1}{2} g^{\mu\nu} S^{\rho}_{\mu\nu} \tag{98}
$$

is its trace part, is called the *dilation* of the connection. We also have that connection coefficients of the Levi-Civita connection can be written as:

$$
\mathring{\mathbf{L}}^{\rho}_{\alpha\beta} = \frac{1}{2}(b^{\rho}_{\alpha\beta} + c^{\rho}_{\alpha\beta}). \tag{99}
$$

Moreover, we introduce the *contorsion tensor* whose components in an arbitrary basis are defined by

$$K^{\rho}_{\alpha\beta} = \mathbf{L}^{\rho}_{\alpha\beta} - \mathring{\mathbf{L}}^{\rho}_{\alpha\beta} = \frac{1}{2}(T^{\rho}_{\alpha\beta} + S^{\rho}_{\alpha\beta}), \tag{100}$$

and which can be written as

$$K^{\rho}_{\alpha\beta} = -\frac{1}{2}g^{\rho\sigma}(g_{\mu\alpha}T^{\mu}_{\sigma\beta} + g_{\mu\beta}T^{\mu}_{\sigma\alpha} - g_{\mu\sigma}T^{\mu}_{\alpha\beta}). \tag{101}$$

We now present the relation between the Riemann curvature tensor $R_{\mu}{}^{\rho}{}_{\alpha\beta}$ associated with the Riemann-Cartan connection D and the Riemann curvature tensor $\mathring{R}_{\mu}{}^{\rho}{}_{\alpha\beta}$ of the Levi-Civita connection $\mathring{D}$.

$$R_{\mu}{}^{\rho}{}_{\alpha\beta} = \mathring{R}_{\mu}{}^{\rho}{}_{\alpha\beta} + J_{\mu}{}^{\rho}{}_{[\alpha\beta]}, \tag{102}$$

where:

$$J_{\mu}{}^{\rho}{}_{\alpha\beta} = \mathring{D}_{\alpha}K^{\rho}_{\beta\mu} - K^{\rho}_{\beta\sigma}K^{\sigma}_{\alpha\mu} = D_{\alpha}K^{\rho}_{\beta\mu} - K^{\rho}_{\alpha\sigma}K^{\sigma}_{\beta\mu} + K^{\sigma}_{\alpha\beta}K^{\rho}_{\sigma\mu}. \tag{103}$$

Multiplying both sides of Eq.(102) by $\frac{1}{2}\theta^{\alpha}\wedge\theta^{\beta}$ we get:

$$\mathcal{R}^{\rho}_{\mu} = \mathring{\mathcal{R}}^{\rho}_{\mu} + \mathfrak{J}^{\rho}_{\mu}, \tag{104}$$

where

$$\mathfrak{J}^{\rho}_{\mu} = \frac{1}{2}J_{\mu}{}^{\rho}{}_{[\alpha\beta]}\theta^{\alpha}\wedge\theta^{\beta}. \tag{105}$$

From Eq.(102) we also get the relation between the Ricci tensors of the connections D and $\mathring{D}$. We write for the *Ricci tensor of* D

$$\begin{aligned} \mathbf{Ric} &= R_{\mu\alpha}dx^{\mu}\otimes dx^{\nu} \\ R_{\mu\alpha} &:= R_{\mu}{}^{\rho}{}_{\alpha\rho} \end{aligned} \tag{106}$$

Then, we have

$$R_{\mu\alpha} = \mathring{R}_{\mu\alpha} + J_{\mu\alpha}, \tag{107}$$

with

$$\begin{aligned} J_{\mu\alpha} &= \mathring{D}_{\alpha}K^{\rho}_{\rho\mu} - \mathring{D}_{\rho}K^{\rho}_{\alpha\mu} + K^{\rho}_{\alpha\sigma}K^{\sigma}_{\rho\mu} - K^{\rho}_{\rho\sigma}K^{\sigma}_{\alpha\mu} \\ &= D_{\alpha}K^{\rho}_{\rho\mu} - D_{\rho}K^{\rho}_{\alpha\mu} - K^{\rho}_{\sigma\alpha}K^{\sigma}_{\rho\mu} + K^{\rho}_{\rho\sigma}K^{\sigma}_{\alpha\mu}. \end{aligned} \tag{108}$$

Observe that since the connection D is arbitrary, its Ricci tensor will be *not* be symmetric in general. Then, since the Ricci tensor $\mathring{R}_{\mu\alpha}$ of $\mathring{D}$ is necessarily symmetric, we can split Eq.(107) into:

$$\begin{aligned} R_{[\mu\alpha]} &= J_{[\mu\alpha]}, \\ R_{(\mu\alpha)} &= \mathring{R}_{(\mu\alpha)} + J_{(\mu\alpha)}. \end{aligned} \tag{109}$$

9 Expressions for d and δ in Terms of Covariant Derivative Operators $\mathring{D}$ and D

We have the following noticeable formulas whose proof can be found in, e.g., [22]. Let $\mathcal{Q} \in \sec \bigwedge T^*M$. Then as we already know

$$
\begin{aligned}
d\mathcal{Q} &= \vartheta^\alpha \wedge (\mathring{D}_{e_\alpha}\mathcal{Q}) = \boldsymbol{\partial}\wedge\mathcal{Q}, \\
\delta\mathcal{Q} &= -\vartheta^\alpha \lrcorner (\mathring{D}_{e_\alpha}\mathcal{Q}) = \boldsymbol{\partial}\lrcorner\mathcal{Q}.
\end{aligned} \tag{110}
$$

We have also the important formulas

$$
\begin{aligned}
d\mathcal{Q} &= \vartheta^\alpha \wedge (D_{e_\alpha}\mathcal{Q}) - \mathcal{T}^\alpha \wedge (\vartheta_\alpha \lrcorner \mathcal{Q}) = \boldsymbol{\partial} \wedge \mathcal{Q} - \mathcal{T}^\alpha \wedge (\vartheta_\alpha \lrcorner \mathcal{Q}), \\
\delta\mathcal{Q} &= -\vartheta^\alpha \lrcorner (D_{e_\alpha}\mathcal{Q}) - \mathcal{T}^\alpha \lrcorner (\vartheta_\alpha \wedge \mathcal{Q}) = -\boldsymbol{\partial} \lrcorner \mathcal{Q} - \mathcal{T}^\alpha \lrcorner (\vartheta_\alpha \wedge \mathcal{Q}).
\end{aligned} \tag{111}
$$

10 Square of Dirac Operators and D' Alembertian, Ricci and Einstein Operators

We now investigate the square of a Dirac operator. We start recalling that the square of the standard Dirac operator can be identified with the Hodge D' Alembertian and that it can be separated in some interesting parts that we called in [22] the D'Alembertian, Ricci and Einstein operators of $(M, \mathbf{g}, \mathring{D})$.

10.1 The Square of the Dirac Operator $\boldsymbol{\partial}$ Associated to $\mathring{D}$

The square of standard Dirac operator $\boldsymbol{\partial}$ is the operator, $\boldsymbol{\partial}^2 = \boldsymbol{\partial}\boldsymbol{\partial}$: $\sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g}) \rightarrow \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathtt{g})$ given by:

$$
\boldsymbol{\partial}^2 = (\boldsymbol{\partial}\wedge + \boldsymbol{\partial}\lrcorner)(\boldsymbol{\partial}\wedge + \boldsymbol{\partial}\lrcorner) = (d - \delta)(d - \delta) \tag{112}
$$

It is quite obvious that

$$
\boldsymbol{\partial}^2 = -(d\delta + \delta d), \tag{113}
$$

and thus we recognize that $\boldsymbol{\partial}^2 \equiv \lozenge$ is the *Hodge D'Alembertian* of the manifold introduced by Eq.(38)

On the other hand, remembering the standard Dirac operator is $\boldsymbol{\partial} = \vartheta^\alpha \mathring{D}_{e_\alpha}$, where $\{\vartheta^\alpha\}$ is the dual basis of an arbitrary basis $\{e_\alpha\}$ on $TU \subset$

TM and $\mathring{D}$ is the Levi-Civita connection of the metric $\mathbf{g}$, we have:

$$\begin{aligned}\boldsymbol{\partial}^2 &= (\vartheta^\alpha \mathring{D}_{e_\alpha})(\vartheta^\beta \mathring{D}_{e_\beta}) = \vartheta^\alpha(\vartheta^\beta \mathring{D}_{e_\alpha}\mathring{D}_{e_\beta} + (\mathring{D}_{e_\alpha}\vartheta^\beta)\mathring{D}_{e_\beta})\\ &= g^{\alpha\beta}(\mathring{D}_{e_\alpha}\mathring{D}_{e_\beta} - \mathring{\mathbf{L}}^\rho_{\alpha\beta}\mathring{D}_{e_\rho}) + \vartheta^\alpha\wedge\vartheta^\beta(\mathring{D}_{e_\alpha}\mathring{D}_{e_\beta} - \mathring{\mathbf{L}}^\rho_{\alpha\beta}\mathring{D}_{\mathbf{e}_\rho}).\end{aligned}$$

Then defining the operators:

$$\begin{array}{ll}\text{(a)} & \boldsymbol{\partial}\cdot\boldsymbol{\partial} = g^{\alpha\beta}(\mathring{D}_{e_\alpha}\mathring{D}_{e_\beta} - \mathring{\mathbf{L}}^\rho_{\alpha\beta}\mathring{D}_{e_\rho})\\ \text{(b)} & \boldsymbol{\partial}\wedge\boldsymbol{\partial} = \vartheta^\alpha\wedge\vartheta^\beta(\mathring{D}_{e_\alpha}\mathring{D}_{e_\beta} - \mathring{\mathbf{L}}^\rho_{\alpha\beta}\mathring{D}_{e_\rho}),\end{array} \tag{114}$$

we can write:

$$\Diamond = \boldsymbol{\partial}^2 = \boldsymbol{\partial}\cdot\boldsymbol{\partial} + \boldsymbol{\partial}\wedge\boldsymbol{\partial} \tag{115}$$

or,

$$\begin{aligned}\boldsymbol{\partial}^2 &= (\boldsymbol{\partial}\lrcorner + \boldsymbol{\partial}\wedge)(\boldsymbol{\partial}\lrcorner + \boldsymbol{\partial}\wedge)\\ &= \boldsymbol{\partial}\lrcorner\,\boldsymbol{\partial}\wedge + \boldsymbol{\partial}\wedge\boldsymbol{\partial}\lrcorner\end{aligned} \tag{116}$$

It is important to observe that the operators $\boldsymbol{\partial}\cdot\boldsymbol{\partial}$ and $\boldsymbol{\partial}\wedge\boldsymbol{\partial}$ do not have anything analogous in the formulation of the differential geometry in the Cartan and Hodge bundles.

The operator $\boldsymbol{\partial}\cdot\boldsymbol{\partial}$ can also be written as:

$$\boldsymbol{\partial}\cdot\boldsymbol{\partial} = \frac{1}{2}g^{\alpha\beta}\left[\mathring{D}_{e_\alpha}\mathring{D}_{e_\beta} + \mathring{D}_{e_\beta}\mathring{D}_{e_\alpha} - b^\rho_{\alpha\beta}\mathring{D}_{e_\rho}\right]. \tag{117}$$

Applying this operator to the 1-forms of the frame $\{\theta^\alpha\}$, we get:

$$(\boldsymbol{\partial}\cdot\boldsymbol{\partial})\vartheta^\mu = -\frac{1}{2}g^{\alpha\beta}\mathring{M}_\rho{}^\mu{}_{\alpha\beta}\theta^\rho, \tag{118}$$

where:

$$\mathring{M}_\rho{}^\mu{}_{\alpha\beta} = e_\alpha(\mathring{\mathbf{L}}^\mu_{\beta\rho}) + e_\beta(\mathring{\mathbf{L}}^\mu_{\alpha\rho}) - \mathring{\mathbf{L}}^\mu_{\alpha\sigma}\mathring{\mathbf{L}}^\sigma_{\beta\rho} - \mathring{\mathbf{L}}^\mu_{\beta\sigma}\mathring{\mathbf{L}}^\sigma_{\alpha\rho} - b^\sigma_{\alpha\beta}\mathring{\mathbf{L}}^\mu_{\sigma\rho}. \tag{119}$$

The proof that an object with these components is a tensor may be found in [22]. In particular, for every r-form field $\omega\in\sec\bigwedge^r T^*M$, $\omega = \frac{1}{r!}\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}$, we have:

$$(\boldsymbol{\partial}\cdot\boldsymbol{\partial})\omega = \frac{1}{r!}g^{\alpha\beta}\mathring{D}_\alpha\mathring{D}_\beta\omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}, \tag{120}$$

where $\mathring{D}_\alpha \mathring{D}_\beta \omega_{\alpha_1 \ldots \alpha_r}$ are the components of the covariant derivative of ω, i.e., writing $\mathring{D}_{\mathbf{e}_\beta}\omega = \frac{1}{r!}\mathring{D}_\beta \omega_{\alpha_1\ldots\alpha_r}\theta^{\alpha_1}\wedge\ldots\wedge\theta^{\alpha_r}$, it is:

$$\mathring{D}_\beta \omega_{\alpha_1\ldots\alpha_r} = e_\beta(\omega_{\alpha_1\ldots\alpha_r}) - \mathring{\mathbf{L}}^\sigma_{\beta\alpha_1}\omega_{\sigma\alpha_2\ldots\alpha_r} - \cdots - \mathring{\mathbf{L}}^\sigma_{\beta\alpha_r}\omega_{\alpha_1\ldots\alpha_{r-1}\sigma}. \quad (121)$$

In view of Eq.(120), we call the operator $\mathring{\square} = \mathring{\boldsymbol{\partial}}\cdot\mathring{\boldsymbol{\partial}}$ the *covariant D'Alembertian.*

Note that the covariant D'Alembertian of the 1-forms ϑ^μ can also be written as:

$$(\mathring{\boldsymbol{\partial}}\cdot\mathring{\boldsymbol{\partial}})\vartheta^\mu = \mathring{g}^{\alpha\beta}\mathring{D}_\alpha\mathring{D}_\beta\delta^\mu_\rho\vartheta^\rho = \frac{1}{2}\mathring{g}^{\alpha\beta}(\mathring{D}_\alpha\mathring{D}_\beta\delta^\mu_\rho + \mathring{D}_\beta\mathring{D}_\alpha\delta^\mu_\rho)\vartheta^\rho$$

and therefore, taking into account the Eq.(118), we conclude that:

$$\mathring{M}_\rho{}^\mu{}_{\alpha\beta} = -(\mathring{D}_\alpha\mathring{D}_\beta\delta^\mu_\rho + \mathring{D}_\beta\mathring{D}_\alpha\delta^\mu_\rho). \quad (122)$$

By its turn, the operator $\mathring{\boldsymbol{\partial}}\wedge\mathring{\boldsymbol{\partial}}$ can also be written as:

$$\mathring{\boldsymbol{\partial}}\wedge\mathring{\boldsymbol{\partial}} = \frac{1}{2}\vartheta^\alpha\wedge\vartheta^\beta\left[\mathring{D}_\alpha\mathring{D}_\beta - \mathring{D}_\beta\mathring{D}_\alpha - c^\rho_{\alpha\beta}\mathring{D}_\rho\right]. \quad (123)$$

Applying this operator to the 1-forms of the frame $\{\vartheta^\mu\}$, we get:

$$(\mathring{\boldsymbol{\partial}}\wedge\mathring{\boldsymbol{\partial}})\vartheta^\mu = -\frac{1}{2}\mathring{R}_\rho{}^\mu{}_{\alpha\beta}(\vartheta^\alpha\wedge\vartheta^\beta)\vartheta^\rho = -\mathring{\mathcal{R}}^\mu_\rho\vartheta^\rho, \quad (124)$$

where $\mathring{R}_\rho{}^\mu{}_{\alpha\beta}$ are the components of the curvature tensor of the connection $\mathring{D}$. Then using the second formula in the first line of Eq.(45) we have

$$\mathring{\mathcal{R}}^\mu_\rho\theta^\rho = \mathring{\mathcal{R}}^\mu_\rho\llcorner\theta^\rho + \mathring{\mathcal{R}}^\mu_\rho\wedge\theta^\rho. \quad (125)$$

The second term in the r.h.s. of this equation is identically null because due to the first Bianchi identity which for the particular case of the Levi-Civita connection ($\mathcal{T}^\mu = 0$) is $\mathring{\mathcal{R}}^\mu_\rho\wedge\theta^\rho = 0$. The first term in Eq.(125) can be written

$$\begin{aligned}
\mathring{\mathcal{R}}^\mu_\rho\llcorner\theta^\rho &= \frac{1}{2}\mathring{R}_\rho{}^\mu{}_{\alpha\beta}(\theta^\alpha\wedge\theta^\beta)\llcorner\theta^\rho \\
&= \frac{1}{2}\mathring{R}_\rho{}^\mu{}_{\alpha\beta}\theta^\rho\lrcorner(\theta^\alpha\wedge\theta^\beta) \\
&= -\frac{1}{2}\mathring{R}_\rho{}^\mu{}_{\alpha\beta}(\mathring{g}^{\rho\alpha}\theta^\beta - \mathring{g}^{\rho\beta}\theta^\alpha) \\
&= -\mathring{g}^{\rho\alpha}\mathring{R}_\rho{}^\mu{}_{\alpha\beta}\theta^\beta = -\mathring{R}^\mu_\beta\theta^\beta, \quad (126)
\end{aligned}$$

where $\mathring{R}^{\mu}_{\beta}$ are the components of the Ricci tensor of the Levi-Civita connection $\mathring{D}$ of $\mathbf{g}$. Thus we have a really beautiful result:

$$(\boldsymbol{\partial}\wedge\boldsymbol{\partial})\theta^{\mu} = \mathring{\mathcal{R}}^{\mu}, \tag{127}$$

where $\mathring{\mathcal{R}}^{\mu} = \mathring{R}^{\mu}_{\beta}\theta^{\beta}$ are the Ricci 1-forms of the manifold. Because of this relation, we call the operator $\boldsymbol{\partial}\wedge\boldsymbol{\partial}$ the *Ricci operator* of the manifold associated to the Levi-Civita connection $\mathring{D}$ of $\mathbf{g}$.

We can show [22] that the Ricci operator $\boldsymbol{\partial}\wedge\boldsymbol{\partial}$ satisfies the relation:

$$\boldsymbol{\partial}\wedge\boldsymbol{\partial} = \mathring{\mathcal{R}}^{\sigma}\wedge\mathbf{i}_{\sigma} + \mathring{\mathcal{R}}^{\rho\sigma}\wedge\mathbf{i}_{\rho}\mathbf{i}_{\sigma}, \tag{128}$$

where the curvature 2-forms are $\mathring{\mathcal{R}}^{\rho\sigma} = \frac{1}{2}\mathring{R}^{\rho\sigma}{}_{\alpha\beta}\vartheta^{\alpha}\wedge\vartheta^{\beta}$ and

$$\mathbf{i}_{\sigma}\omega := \vartheta_{\sigma}\lrcorner\omega. \tag{129}$$

Observe that applying the operator given by the second term in the r.h.s. of Eq.(128) to the dual of the 1-forms ϑ^{μ}, we get:

$$\begin{aligned}\mathring{\mathcal{R}}^{\rho\sigma}\wedge\mathbf{i}_{\rho}\mathbf{i}_{\sigma}\star\vartheta^{\mu} &= \mathring{\mathcal{R}}_{\rho\sigma}\star\vartheta^{\rho}\lrcorner(\vartheta^{\sigma}\lrcorner\star\vartheta^{\mu}))\\ &= -\mathring{\mathcal{R}}_{\rho\sigma}\wedge\star(\vartheta^{\rho}\wedge\vartheta^{\sigma}\star\vartheta^{\mu})\\ &= \star(\mathring{\mathcal{R}}_{\rho\sigma}\lrcorner(\vartheta^{\rho}\wedge\vartheta^{\sigma}\wedge\vartheta^{\mu})),\end{aligned} \tag{130}$$

where we have used the Eqs.(35). Then, recalling the definition of the curvature forms and using the Eq.(28), we conclude that:

$$\mathring{\mathcal{R}}^{\rho\sigma}\wedge(\vartheta_{\rho}\lrcorner\vartheta_{\sigma}\lrcorner\star\vartheta^{\mu}) = 2\star(\mathring{\mathcal{R}}^{\mu} - \frac{1}{2}\mathring{R}\vartheta^{\mu}) = 2\star\mathring{\mathcal{G}}^{\mu}, \tag{131}$$

where $\mathring{R}$ is the scalar curvature of the manifold and the $\mathring{\mathcal{G}}^{\mu}$ may be called the *Einstein 1-form fields.*

That observation motivate us to introduce in [22] the *Einstein operator* of the Levi-Civita connection $\mathring{D}$ of $\mathbf{g}$ on the manifold M as the mapping $\mathring{\blacksquare} : \sec\mathcal{C}\ell(M,\mathtt{g}) \to \sec\mathcal{C}\ell(M,\mathtt{g})$ given by:

$$\mathring{\blacksquare} = \frac{1}{2}\star^{-1}(\mathring{\mathcal{R}}^{\rho\sigma}\wedge\mathbf{i}_{\rho}\mathbf{i}_{\sigma})\star. \tag{132}$$

Obviously, we have:

$$\mathring{\blacksquare}\theta^{\mu} = \mathring{\mathcal{G}}^{\mu} = \mathring{\mathcal{R}}^{\mu} - \frac{1}{2}\mathring{R}\vartheta^{\mu}. \tag{133}$$

In addition, it is easy to verify that $\star^{-1}(\mathring{\boldsymbol{\partial}}\wedge\mathring{\boldsymbol{\partial}})\star = -\mathring{\boldsymbol{\partial}}\wedge\mathring{\boldsymbol{\partial}}$ and $\star^{-1}(\mathring{\mathcal{R}}^{\sigma}\wedge\mathbf{i}_{\sigma})\star = \mathring{\mathcal{R}}^{\sigma}\lrcorner\mathbf{j}_{\sigma}$. Thus we can also write the Einstein operator as:

$$\mathring{\blacksquare} = -\frac{1}{2}(\mathring{\boldsymbol{\partial}}\wedge\mathring{\boldsymbol{\partial}}+\mathring{\mathcal{R}}^{\sigma}\lrcorner\mathbf{j}_{\sigma}), \tag{134}$$

where

$$\mathbf{j}_{\sigma}\mathcal{A} = \vartheta_{\sigma}\wedge\mathcal{A}, \tag{135}$$

for any $\mathcal{A}\in\sec\bigwedge T^{*}M\hookrightarrow\sec\mathcal{C}\ell(M,\mathtt{g})$.

We recall [22] that if $\mathring{\omega}_{\rho}^{\mu}$ are the Levi-Civita connection 1-forms fields in an arbitrary moving frame $\{\vartheta^{\mu}\}$ on $(M,\mathbf{g},\mathring{D})$ then:

$$\begin{array}{ll}\text{(a)} & (\mathring{\boldsymbol{\partial}}\cdot\mathring{\boldsymbol{\partial}})\vartheta^{\mu}=-(\mathring{\boldsymbol{\partial}}\cdot\mathring{\omega}_{\rho}^{\mu}-\mathring{\omega}_{\rho}^{\sigma}\cdot\mathring{\omega}_{\sigma}^{\mu})\vartheta^{\rho}\\ \text{(b)} & (\mathring{\boldsymbol{\partial}}\wedge\mathring{\boldsymbol{\partial}})\vartheta^{\mu}=-(\mathring{\boldsymbol{\partial}}\wedge\mathring{\omega}_{\rho}^{\mu}-\mathring{\omega}_{\rho}^{\sigma}\wedge\mathring{\omega}_{\sigma}^{\mu})\vartheta^{\rho},\end{array} \tag{136}$$

and

$$\mathring{\boldsymbol{\partial}}^{2}\vartheta^{\mu} = -(\mathring{\boldsymbol{\partial}}\mathring{\omega}_{\rho}^{\mu}-\mathring{\omega}_{\rho}^{\sigma}\mathring{\omega}_{\sigma}^{\mu})\vartheta^{\rho}. \tag{137}$$

Exercise 11 *Show that* $\vartheta_{\rho}\wedge\vartheta_{\sigma}\mathring{\mathcal{R}}^{\rho\sigma} = -\mathring{R}$, *where* $\mathring{R}$ *is the curvature scalar.*

10.2 The Square of the Dirac Operator $\boldsymbol{\partial}$ *Associated to* D

Consider the structure $(M,\mathbf{g},D)$, where D is an arbitrary Riemann-Cartan-Weyl connection and the Clifford algebra $\mathcal{C}\ell(M,\mathtt{g})$. Let us now compute the square of the (general) Dirac operator $\boldsymbol{\partial} = \vartheta^{\alpha}D_{e_{\alpha}}$. As in the earlier section, we have, by one side,

$$\begin{aligned}\boldsymbol{\partial}^{2} &= (\boldsymbol{\partial}\lrcorner+\ \boldsymbol{\partial}\wedge)(\boldsymbol{\partial}\lrcorner+\ \boldsymbol{\partial}\wedge)\\ &= \boldsymbol{\partial}\lrcorner\boldsymbol{\partial}\lrcorner+\ \boldsymbol{\partial}\lrcorner\boldsymbol{\partial}\wedge+\ \boldsymbol{\partial}\wedge\boldsymbol{\partial}\lrcorner+\ \boldsymbol{\partial}\wedge\boldsymbol{\partial}\wedge\end{aligned}$$

and we write $\boldsymbol{\partial}\lrcorner\boldsymbol{\partial}\lrcorner\equiv\boldsymbol{\partial}^{2}\lrcorner$, $\boldsymbol{\partial}\wedge\boldsymbol{\partial}\wedge\equiv\boldsymbol{\partial}^{2}\wedge$ and

$$\mathcal{L}_{+} = \boldsymbol{\partial}\lrcorner\boldsymbol{\partial}\wedge+\boldsymbol{\partial}\wedge\boldsymbol{\partial}\lrcorner, \tag{138}$$

so that:

$$\boldsymbol{\partial}^{2} = \boldsymbol{\partial}^{2}\lrcorner\ +\ \mathcal{L}_{+}\ +\ \boldsymbol{\partial}^{2}\wedge\quad. \tag{139}$$

The operator $\mathcal{L}_+$ when applied to scalar functions corresponds, for the case of a Riemann-Cartan space, to the wave operator introduced by Rapoport [23] in his theory of Stochastic Mechanics. Obviously, for the case of the standard Dirac operator, $\mathcal{L}_+$ reduces to the usual Hodge D' Alembertian of the manifold, which preserve graduation of forms. For more details see [18].

On the other hand, we have also:

$$\begin{aligned}\boldsymbol{\partial}^2 &= (\vartheta^\alpha D_{e_\alpha})(\vartheta^\beta D_{\mathbf{e}_\beta}) = \vartheta^\alpha(\vartheta^\beta D_{e_\alpha} D_{e_\beta} + (D_{e_\alpha}\vartheta^\beta)D_{e_\beta}) \\ &= g^{\alpha\beta}(D_{e_\alpha}D_{e_\beta} - \mathbf{L}^\rho_{\alpha\beta}D_{e_\rho}) + \vartheta^\alpha\wedge\vartheta^\beta(D_{e_\alpha}D_{e_\beta} - \mathbf{L}^\rho_{\alpha\beta}D_{e_\rho})\end{aligned}$$

and we can then define:

$$\begin{aligned}\boldsymbol{\partial}\cdot\boldsymbol{\partial} &= g^{\alpha\beta}(D_{e_\alpha}D_{e_\beta} - \mathbf{L}^\rho_{\alpha\beta}D_{e_\rho}) \\ \boldsymbol{\partial}\wedge\boldsymbol{\partial} &= \theta^\alpha\wedge\theta^\beta(D_{e_\alpha}D_{e_\beta} - \mathbf{L}^\rho_{\alpha\beta}D_{e_\rho})\end{aligned} \tag{140}$$

in order to have:

$$\boldsymbol{\partial}^2 = \boldsymbol{\partial}\boldsymbol{\partial} = \boldsymbol{\partial}\cdot\boldsymbol{\partial} + \boldsymbol{\partial}\wedge\boldsymbol{\partial} \quad . \tag{141}$$

The operator $\boldsymbol{\partial}\cdot\boldsymbol{\partial}$ can also be written as:

$$\begin{aligned}\boldsymbol{\partial}\cdot\boldsymbol{\partial} &= \frac{1}{2}\theta^\alpha\cdot\theta^\beta(D_{e_\alpha}D_{e_\beta} - \mathbf{L}^\rho_{\alpha\beta}D_{e_\rho}) + \frac{1}{2}\theta^\beta\cdot\theta^\alpha(D_{e_\beta}D_{e_\alpha} - \mathbf{L}^\rho_{\beta\alpha}D_{e_\rho}) \\ &= \frac{1}{2}g^{\alpha\beta}[D_{e_\alpha}D_{e_\beta} + D_{e_\beta}D_{e_\alpha} - (\mathbf{L}^\rho_{\alpha\beta} + \mathbf{L}^\rho_{\beta\alpha})D_{e_\rho}]\end{aligned} \tag{142}$$

or,

$$\boldsymbol{\partial}\cdot\boldsymbol{\partial} = \frac{1}{2}g^{\alpha\beta}(D_{e_\alpha}D_{e_\beta} + D_{e_\beta}D_{e_\alpha} - b^\rho_{\alpha\beta}D_{e_\rho}) - s^\rho D_{e_\rho}, \tag{143}$$

where s^ρ has been defined in Eq.(98).

By its turn, the operator $\boldsymbol{\partial}\wedge\boldsymbol{\partial}$ can also be written as:

$$\begin{aligned}\boldsymbol{\partial}\wedge\boldsymbol{\partial} &= \frac{1}{2}\vartheta^\alpha\wedge\vartheta^\beta(D_{e_\alpha}D_{e_\beta} - \mathbf{L}^\rho_{\alpha\beta}D_{e_\rho}) + \frac{1}{2}\vartheta^\beta\wedge\vartheta^\alpha(D_{e_\beta}D_{e_\alpha} - \mathbf{L}^\rho_{\beta\alpha}D_{e_\rho}) \\ &= \frac{1}{2}\vartheta^\alpha\wedge\vartheta^\beta[D_{e_\alpha}D_{e_\beta} - D_{e_\beta}D_{e_\alpha} - (\mathbf{L}^\rho_{\alpha\beta} - \mathbf{L}^\rho_{\beta\alpha})D_{e_\rho}]\end{aligned}$$

or,

$$\boldsymbol{\partial}\wedge\boldsymbol{\partial} = \frac{1}{2}\vartheta^\alpha\wedge\vartheta^\beta(D_{e_\alpha}D_{e_\beta} - D_{e_\beta}D_{\mathbf{e}_\alpha} - c^\rho_{\alpha\beta}D_{\mathbf{e}_\rho}) - \mathcal{T}^\rho D_{\mathbf{e}_\rho}. \tag{144}$$

Remark 12 *For the case of a Levi-Civita connection we have similar formulas for* $\mathring{\boldsymbol{\partial}}\cdot\mathring{\boldsymbol{\partial}}$ (Eq.(142)) *and* $\mathring{\boldsymbol{\partial}}\wedge\mathring{\boldsymbol{\partial}}$ (Eq.(144)) *with* $D\mapsto\mathring{D}$, *and of course,* $\mathcal{T}^\rho = 0$, *as follows directly from* Eq.(114).

11 Coordinate Expressions for Maxwell Equations on Lorentzian and Riemann-Cartan Spacetimes

11.1 Maxwell Equations on a Lorentzian Spacetime

We now take $(M, \boldsymbol{g})$ as a Lorentzian manifold, i.e., $\dim M = 4$ and the signature of $\boldsymbol{g}$ is $(1,3)$. We consider moreover a Lorentzian spacetime structure on $(M, \boldsymbol{g})$, i.e., the pentuple $(M, \boldsymbol{g}, \overset{\circ}{D}, \tau_{\mathbf{g}}, \uparrow)$ and a Riemann-Cartan spacetime structure $(M, \mathbf{g}, D, \tau_{\mathbf{g}}, \uparrow)$.

Now, in both spacetime structures, Maxwell equations in vacuum read:

$$d\mathbf{F} = 0, \qquad \delta\mathbf{F} = -\mathbf{J}, \tag{145}$$

where $\mathbf{F} \in \sec \bigwedge^2 T^*M$ is the Faraday tensor (electromagnetic field) and $\mathbf{J} \in \sec \bigwedge^1 T^*M$ is the current. We observe that writing

$$\mathbf{F} = \frac{1}{2}F_{\mu\nu}dx^{\mu} \wedge dx^{\nu} = \frac{1}{2}F_{\mu\nu}\theta^{\mu} \wedge \theta^{\nu} = \frac{1}{2}F_{\mu\nu}\theta^{\mu\nu}, \tag{146}$$

we have using Eq.(34) that

$$\star\mathbf{F} = \frac{1}{2}F_{\mu\nu}(\star\theta^{\mu\nu}) = \frac{1}{2}\ {}^{\star}\mathbf{F}_{\rho\sigma}\vartheta^{\rho\sigma} = \frac{1}{2}(F_{\mu\nu}\frac{1}{2}\sqrt{|\det \mathbf{g}|}g^{\mu\alpha}g^{\nu\beta}\epsilon_{\alpha\beta\rho\sigma})\vartheta^{\rho\sigma} \tag{147}$$

Thus

$${}^{\star}\mathbf{F}_{\rho\sigma} = (\star\mathbf{F})_{\rho\sigma} = \frac{1}{2}F_{\mu\nu}\sqrt{|\det \mathbf{g}|}g^{\mu\alpha}g^{\nu\beta}\epsilon_{\alpha\beta\rho\sigma}. \tag{148}$$

The homogeneous Maxwell equation $d\mathbf{F} = 0$ can be writing as $\delta\star\mathbf{F} = 0$. The proof follows at once from the definition of δ (Eq.(37)). Indeed, we can write

$$0 = d\mathbf{F} = \star\star^{-1} d\star\star^{-1}\mathbf{F} = \star\delta\star^{-1}\mathbf{F} = -\star\delta\star\mathbf{F} = 0.$$

Then $\star^{-1}\star\delta\star\mathbf{F} = 0$ and we end with

$$\delta\star\mathbf{F} = 0.$$

(a) We now express the equivalent equations $dF = 0$ and $\delta\star F = 0$ in arbitrary coordinates $\{x^{\mu}\}$ covering $U \subset M$ using first the Levi-Civita connection and noticeable formula in Eq.(110). We have

$$
\begin{aligned}
d\mathbf{F} &= \theta^\alpha \wedge (\mathring{D}_{\boldsymbol{\partial}_\alpha} F) \\
&= \frac{1}{2}\theta^\alpha \wedge \left[\mathring{D}_{\boldsymbol{\partial}_\alpha}(F_{\mu\nu}\theta^\mu \wedge \theta^\nu)\right] \\
&= \frac{1}{2}\theta^\alpha \wedge \left[(\partial_\alpha F_{\mu\nu})\theta^\mu \wedge \theta^\nu - F_{\mu\nu}\mathring{\Gamma}^{\mu}_{\alpha\rho}\theta^\rho \wedge \theta^\nu - F_{\mu\nu}\mathring{\Gamma}^{\nu}_{\alpha\rho}\theta^\mu \wedge \theta^\rho\right] \\
&= \frac{1}{2}\theta^\alpha \wedge \left[(\mathring{D}_\alpha F_{\mu\nu})\theta^\mu \wedge \theta^\nu\right] \\
&= \frac{1}{2} D_\alpha F_{\mu\nu}\theta^\alpha \wedge \theta^\mu \wedge \theta^\nu \\
&= \frac{1}{2}\left[\frac{1}{3}\mathring{D}_\alpha F_{\mu\nu}\theta^\alpha \wedge \theta^\mu \wedge \theta^\nu + \frac{1}{3}\mathring{D}_\mu F_{\nu\alpha}\theta^\mu \wedge \theta^\nu \wedge \theta^\alpha + \frac{1}{3}\mathring{D}_\nu F_{\alpha\mu}\theta^\nu \wedge \theta^\alpha \wedge \theta^\mu\right] \\
&= \frac{1}{2}\left[\frac{1}{3}\mathring{D}_\alpha F_{\mu\nu}\theta^\alpha \wedge \theta^\mu \wedge \theta^\nu + \frac{1}{3}\mathring{D}_\mu F_{\nu\alpha}\theta^\alpha \wedge \theta^\mu \wedge \theta^\nu + \frac{1}{3}\mathring{D}_\nu F_{\alpha\mu}\theta^\alpha \wedge \theta^\mu \wedge \theta^\nu\right] \\
&= \frac{1}{6}\left(\mathring{D}_\alpha F_{\mu\nu} + \mathring{D}_\mu F_{\nu\alpha} + \mathring{D}_\nu F_{\alpha\mu}\right)\theta^\alpha \wedge \theta^\mu \wedge \theta^\nu.
\end{aligned}
$$

So,

$$
d\mathbf{F} = 0 \Leftrightarrow \mathring{D}_\alpha F_{\mu\nu} + \mathring{D}_\mu F_{\nu\alpha} + \mathring{D}_\nu F_{\alpha\mu} = 0. \tag{149}
$$

If we calculate $d\mathbf{F} = 0$ using the definition of d we get:

$$
\begin{aligned}
d\mathbf{F} &= \frac{1}{2}(\partial_\alpha F_{\mu\nu})\theta^\alpha \wedge \theta^\mu \wedge \theta^\nu \\
&= \frac{1}{6}\left(\partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu}\right)\theta^\alpha \wedge \theta^\mu \wedge \theta^\nu,
\end{aligned} \tag{150}
$$

from where we get that

$$
d\mathbf{F} = 0 \Longleftrightarrow \partial_\alpha F_{\mu\nu} + \partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} = 0 \Longleftrightarrow \mathring{D}_\alpha F_{\mu\nu} + \mathring{D}_\mu F_{\nu\alpha} + \mathring{D}_\nu F_{\alpha\mu} = 0. \tag{151}
$$

Next we calculate $\delta \star F = 0$. We have

$$\begin{aligned}
\delta \star \mathbf{F} &= -\theta^{\alpha} \lrcorner (\mathring{D}_{\boldsymbol{\partial}_{\alpha}} \star \mathbf{F}) \\
&= -\frac{1}{2}\theta^{\alpha} \lrcorner \left\{ \mathring{D}_{\boldsymbol{\partial}_{\alpha}} [{}^{\star}F_{\mu\nu}\theta^{\mu} \wedge \theta^{\nu}] \right\} \\
&= -\frac{1}{2}\theta^{\alpha} \lrcorner \left\{ (\partial_{\alpha} {}^{\star}F_{\mu\nu})\theta^{\mu} \wedge \theta^{\nu} - {}^{\star}F_{\mu\nu}\mathring{\Gamma}^{\mu}_{\alpha\rho}\theta^{\rho} \wedge \theta^{\nu} - {}^{\star}F_{\mu\nu}\mathring{\Gamma}^{\nu}_{\alpha\rho}\theta^{\mu} \wedge \theta^{\rho} \right\} \\
&= -\frac{1}{2}\theta^{\alpha} \lrcorner \left\{ (\partial_{\alpha} {}^{\star}F_{\mu\nu})\theta^{\mu} \wedge \theta^{\nu} - {}^{\star}F_{\rho\nu}\mathring{\Gamma}^{\rho}_{\alpha\mu}\theta^{\mu} \wedge \theta^{\nu} - {}^{\star}F_{\mu\rho}\mathring{\Gamma}^{\rho}_{\alpha\nu}\theta^{\mu} \wedge \theta^{\nu} \right\} \\
&= -\frac{1}{2}\theta^{\alpha} \lrcorner \left\{ (\mathring{D}_{\alpha} {}^{\star}F_{\mu\nu})\theta^{\mu} \wedge \theta^{\nu} \right\} \\
&= -\frac{1}{2} \left\{ (\mathring{D}_{\alpha} {}^{\star}F_{\mu\nu})g^{\alpha\mu}\theta^{\nu} - (\mathring{D}_{\alpha} {}^{\star}F_{\mu\nu})g^{\alpha\nu}\theta^{\mu} \right\} \\
&= -(\mathring{D}_{\alpha} {}^{\star}F_{\mu\nu})g^{\alpha\mu}\theta^{\nu} \\
&= -[\mathring{D}_{\alpha}({}^{\star}F_{\mu\nu}g^{\alpha\mu})]\theta^{\nu} \\
&= -(\mathring{D}_{\alpha} {}^{\star}F^{\alpha}_{\nu})]\theta^{\nu}. \qquad (152)
\end{aligned}$$

Then we get that

$$\mathring{D}_{\alpha}F_{\mu\nu} + \mathring{D}_{\mu}F_{\nu\alpha} + \mathring{D}_{\nu}F_{\alpha\mu} = 0 \Leftrightarrow d\mathbf{F} = 0 \Leftrightarrow \delta \star \mathbf{F} = 0 \Longleftrightarrow \mathring{D}_{\alpha} {}^{\star}F^{\alpha}_{\nu} = 0. \qquad (153)$$

(b) Also, the non homogenoeous Maxwell equation $\delta\mathbf{F} = -J$ can be written using the definition of δ (Eq.(37)) as $d \star \mathbf{F} = - \star \mathbf{J}$:

$$\begin{aligned}
\delta\mathbf{F} &= -\mathbf{J}, \\
(-1)^2 \star^{-1} d \star \mathbf{F} &= -\mathbf{J}, \\
\star\star^{-1} d \star \mathbf{F} &= - \star \mathbf{J}, \\
d \star \mathbf{F} &= - \star \mathbf{J}. \qquad (154)
\end{aligned}$$

We now express $\delta\mathbf{F} = -\,\mathbf{J}$ in arbitrary coordinates[17] using first the Levi-Civita connection. We have following the same steps as in Eq.(152)

$$\begin{aligned}
\delta\mathbf{F} + \mathbf{J} &= -\frac{1}{2}\theta^{\alpha} \lrcorner \left\{ \mathring{D}_{\boldsymbol{\partial}_{\alpha}} [F_{\mu\nu}\theta^{\mu} \wedge \theta^{\nu}] \right\} + J_{\nu}\theta^{\nu} \qquad (155) \\
&= (-\mathring{D}_{\alpha}F^{\alpha}_{\nu} + J_{\nu})\theta^{\nu}.
\end{aligned}$$

[17]We observe that in terms of the "classical" charge and "vector" current densities we have $\mathbf{J} = \rho\theta^0 - j_i\theta^i$.

Then

$$\delta\mathbf{F} + \mathbf{J} = 0 \Leftrightarrow \mathring{D}_\alpha F^{\alpha\nu} = J^\nu. \tag{156}$$

We also observe that using the symmetry of the connection coefficients and the antisymmetry of the $F^{\alpha\nu}$ that $\mathring{\Gamma}^{\nu}_{\alpha\rho} F^{\alpha\rho} = -\mathring{\Gamma}^{\nu}_{\alpha\rho} F^{\alpha\rho} = 0$. Also,

$$\mathring{\Gamma}^{\alpha}_{\alpha\rho} = \partial_\rho \ln \sqrt{|\det \mathbf{g}|} = \frac{1}{\sqrt{|\det \mathbf{g}|}} \partial_\rho \left|\det \mathbf{g}\right|,$$

and

$$\begin{aligned}
\mathring{D}_\alpha F^{\alpha\nu} &= \partial_\alpha F^{\alpha\nu} + \mathring{\Gamma}^{\alpha}_{\alpha\rho} F^{\rho\nu} + \mathring{\Gamma}^{\nu}_{\alpha\rho} F^{\alpha\rho} \\
&= \partial_\alpha F^{\alpha\nu} + \mathring{\Gamma}^{\alpha}_{\alpha\rho} F^{\rho\nu} \\
&= \partial_\rho F^{\rho\nu} + \frac{1}{\sqrt{|\det \mathbf{g}|}} \partial_\rho (\sqrt{|\det \mathbf{g}|}) F^{\rho\nu}.
\end{aligned}$$

Then

$$\begin{aligned}
\mathring{D}_\alpha F^{\alpha\nu} &= J^\nu, \\
\sqrt{|\det \mathbf{g}|}\partial_\rho F^{\rho\nu} + \partial_\rho(\sqrt{|\det \mathbf{g}|}) F^{\rho\nu} &= \sqrt{|\det \mathbf{g}|} J^\nu, \\
\partial_\rho(\sqrt{|\det \mathbf{g}|} F^{\rho\nu}) &= \sqrt{|\det \mathbf{g}|} J^\nu, \\
\frac{1}{\sqrt{|\det \mathbf{g}|}} \partial_\rho(\sqrt{|\det \mathbf{g}|} F^{\rho\nu}) &= J^\nu,
\end{aligned} \tag{157}$$

and

$$\delta\mathbf{F} = 0 \Leftrightarrow \mathring{D}_\alpha F^{\alpha\nu} = J^\nu \Leftrightarrow \frac{1}{\sqrt{|\det \mathbf{g}|}} \partial_\rho(\sqrt{|\det \mathbf{g}|} F^{\rho\nu}) = J^\nu. \tag{158}$$

Exercise 13 *Show that in a Lorentzian spacetime Maxwell equations become Maxwell equation, i.e.,*

$$\boldsymbol{\partial}\mathbf{F} = \mathbf{J}. \tag{159}$$

11.2 Maxwell Equations on Riemann-Cartan Spacetime

From time to time we see papers (e.g., [19, 25]) writing Maxwell equations in a Riemann-Cartan spacetime using arbitrary coordinates and (of course) the Riemann-Cartan connection. As we shall see such enterprises are simple exercises, if we make use of the noticeable formulas of Eq.(111). Indeed, the homogeneous Maxwell equation $d\mathbf{F} = 0$ reads

$$d\mathbf{F} = \theta^\alpha \wedge (D_{\boldsymbol{\partial}_\alpha} \mathbf{F}) - \mathcal{T}^\alpha \wedge (\theta_\alpha \lrcorner \mathbf{F}) = 0 \tag{160}$$

or

$$
\begin{aligned}
&\frac{1}{6}(D_\alpha F_{\mu\nu} + D_\mu F_{\nu\alpha} + D_\nu F_{\alpha\mu})\theta^\alpha \wedge \theta^\mu \wedge \theta^\nu \\
&- \frac{1}{2}\frac{1}{2}T^\alpha_{\rho\sigma}\theta^\rho \wedge \theta^\sigma \wedge [\theta_\alpha \lrcorner F_{\mu\nu}(\theta^\mu \wedge \theta^\nu)] \\
&= \frac{1}{6}(D_\alpha F_{\mu\nu} + D_\mu F_{\nu\alpha} + D_\nu F_{\alpha\mu})\theta^\alpha \wedge \theta^\mu \wedge \theta^\nu \\
&- \frac{1}{2}T^\alpha_{\rho\sigma}F_{\mu\nu}\theta^\rho \wedge \theta^\sigma \wedge \delta^\mu_\alpha\theta^\nu \\
&= \frac{1}{6}(D_\alpha F_{\mu\nu} + D_\mu F_{\nu\alpha} + D_\nu F_{\alpha\mu})\theta^\alpha \wedge \theta^\mu \wedge \theta^\nu \\
&- \frac{1}{2}T^\sigma_{\alpha\mu}F_{\sigma\nu}\theta^\alpha \wedge \theta^\mu \wedge \theta^\nu \\
&= \frac{1}{6}(D_\alpha F_{\mu\nu} + D_\mu F_{\nu\alpha} + D_\nu F_{\alpha\mu})\theta^\alpha \wedge \theta^\mu \wedge \theta^\nu \\
&- \frac{1}{6}(T^\sigma_{\alpha\mu}F_{\sigma\nu} + T^\sigma_{\mu\nu}F_{\sigma\alpha} + T^\sigma_{\nu\alpha}F_{\sigma\mu})\theta^\alpha \wedge \theta^\mu \wedge \theta^\nu \\
&= \frac{1}{6}(D_\alpha F_{\mu\nu} + D_\mu F_{\nu\alpha} + D_\nu F_{\alpha\mu})\theta^\alpha \wedge \theta^\mu \wedge \theta^\nu \\
&+ \frac{1}{6}(F_{\alpha\sigma}T^\sigma_{\mu\nu} + F_{\mu\sigma}T^\sigma_{\nu\alpha} + F_{\nu\sigma}T^\sigma_{\alpha\mu})\theta^\alpha \wedge \theta^\mu \wedge \theta^\nu.
\end{aligned}
$$

i.e.,

$$
d\mathbf{F} = 0 \Longleftrightarrow D_\alpha F_{\mu\nu} + D_\mu F_{\nu\alpha} + D_\nu F_{\alpha\mu} + F_{\sigma\alpha}T^\sigma_{\mu\nu} + F_{\mu\sigma}T^\sigma_{\nu\alpha} + F_{\nu\sigma}T^\sigma_{\alpha\mu} = 0. \tag{161}
$$

Also, taking into account that $d\mathbf{F} = 0 \Longleftrightarrow \delta \star \mathbf{F} = 0$ we have using the second noticeable formula in Eq.(111) that

$$
\delta \star \mathbf{F} = -\theta^\alpha \lrcorner (D_{e_\alpha} \star \mathbf{F}) - \mathcal{T}^\alpha \lrcorner (\theta_\alpha \wedge \star\mathbf{F}) = 0. \tag{162}
$$

Now,

$$
\theta^\alpha \lrcorner (D_{e_\alpha} \star \mathbf{F}) = (D_\alpha {}^\star F^\alpha_{\ \nu})\theta^\nu = (D_\alpha {}^\star F^{\alpha\nu})\theta_\nu \tag{163}
$$

and

$$
\begin{aligned}
&\mathcal{T}^{\alpha}\lrcorner(\theta_{\alpha}\wedge\star\mathbf{F})\\
&=\frac{1}{4}T^{\alpha}_{\beta\rho}(\theta^{\beta}\wedge\theta^{\rho})\lrcorner(\theta_{\alpha}\wedge({}^{\star}F_{\mu\nu}\theta^{\mu}\wedge\theta^{\nu})\\
&=\frac{1}{4}T^{\alpha}_{\beta\rho}{}^{\star}F_{\mu\nu}(\theta^{\beta}\wedge\theta^{\rho})\lrcorner(\theta_{\alpha}\wedge\theta^{\mu}\wedge\theta^{\nu})\\
&=\frac{1}{4}T^{\alpha}_{\beta\rho}{}^{\star}F^{\mu\nu}\theta^{\beta}\lrcorner[\theta^{\rho}\lrcorner(\theta_{\alpha}\wedge\theta_{\mu}\wedge\theta_{\nu})]\\
&=\frac{1}{4}T^{\alpha}_{\beta\rho}{}^{\star}F^{\mu\nu}\theta^{\beta}\lrcorner(\delta^{\rho}_{\alpha}\theta_{\mu}\wedge\theta_{\nu}-\delta^{\rho}_{\mu}\theta_{\alpha}\wedge\theta_{\nu}+\delta^{\rho}_{\nu}\theta_{\alpha}\wedge\theta_{\mu})\\
&=\frac{1}{4}T^{\alpha}_{\beta\alpha}{}^{\star}F^{\mu\nu}\theta^{\beta}\lrcorner(\theta_{\mu}\wedge\theta_{\nu})-\frac{1}{4}T^{\alpha}_{\beta\mu}{}^{\star}F^{\mu\nu}\theta^{\beta}\lrcorner(\theta_{\alpha}\wedge\theta_{\nu})+\frac{1}{4}T^{\alpha}_{\beta\nu}{}^{\star}F^{\mu\nu}\theta^{\beta}\lrcorner(\theta_{\alpha}\wedge\theta_{\mu})\\
&=\frac{1}{4}T^{\alpha}_{\beta\alpha}{}^{\star}F^{\mu\nu}\theta^{\beta}\lrcorner(\theta_{\mu}\wedge\theta_{\nu})-\frac{1}{4}T^{\mu}_{\beta\rho}{}^{\star}F^{\rho\nu}\theta^{\beta}\lrcorner(\theta_{\mu}\wedge\theta_{\nu})+\frac{1}{4}T^{\mu}_{\beta\rho}{}^{\star}F^{\nu\rho}\theta^{\beta}\lrcorner(\theta_{\mu}\wedge\theta_{\nu})\\
&=\frac{1}{4}(T^{\alpha}_{\beta\alpha}{}^{\star}F^{\mu\nu}-T^{\mu}_{\beta\rho}{}^{\star}F^{\rho\nu}+T^{\mu}_{\beta\rho}{}^{\star}F^{\nu\rho})\theta^{\beta}\lrcorner(\theta_{\mu}\wedge\theta_{\nu})\\
&=\frac{1}{4}(T^{\alpha}_{\beta\alpha}{}^{\star}F^{\mu\nu}-T^{\mu}_{\beta\rho}{}^{\star}F^{\rho\nu}+T^{\mu}_{\beta\rho}{}^{\star}F^{\nu\rho})(\delta^{\beta}_{\mu}\theta_{\nu}-\delta^{\beta}_{\nu}\theta_{\mu})\\
&=\frac{1}{4}(T^{\alpha}_{\mu\alpha}{}^{\star}F^{\mu\nu}-T^{\mu}_{\mu\rho}{}^{\star}F^{\rho\nu}+T^{\mu}_{\mu\rho}{}^{\star}F^{\nu\rho})\theta_{\nu}-\frac{1}{4}(T^{\alpha}_{\mu\alpha}{}^{\star}F^{\nu\mu}-T^{\nu}_{\mu\rho}{}^{\star}F^{\rho\mu}+T^{\nu}_{\mu\rho}{}^{\star}F^{\mu\rho})\theta_{\nu}\\
&=\frac{1}{2}(T^{\alpha}_{\mu\alpha}{}^{\star}F^{\mu\nu}-T^{\mu}_{\mu\rho}{}^{\star}F^{\rho\nu}+T^{\nu}_{\mu\rho}{}^{\star}F^{\mu\rho})\theta_{\nu} \qquad (164)
\end{aligned}
$$

Using Eqs.(163) and (164) in Eq.(162) we get

$$D_{\alpha}{}^{\star}F^{\alpha\nu}+\frac{1}{2}(T^{\alpha}_{\mu\alpha}{}^{\star}F^{\mu\nu}-T^{\mu}_{\mu\rho}{}^{\star}F^{\rho\nu}+T^{\nu}_{\mu\rho}{}^{\star}F^{\mu\rho})=0 \qquad (165)$$

and we have

$$d\mathbf{F}=0\Leftrightarrow\delta\star\mathbf{F}=0\Leftrightarrow D_{\alpha}{}^{\star}F^{\alpha\nu}+\frac{1}{2}(T^{\alpha}_{\mu\alpha}{}^{\star}F^{\mu\nu}-T^{\mu}_{\mu\rho}{}^{\star}F^{\rho\nu}+T^{\nu}_{\mu\rho}{}^{\star}F^{\mu\rho})=0. \qquad (166)$$

Finally we express the non homogenous Maxwell equation $\delta\mathbf{F}=-\mathbf{J}$ in arbitrary coordinates using the Riemann-Cartan connection. We have

$$
\begin{aligned}
\delta\mathbf{F}&=-\theta^{\alpha}\lrcorner(D_{e_{\alpha}}\mathbf{F})-\mathcal{T}^{\alpha}\lrcorner(\theta_{\alpha}\wedge\mathbf{F})\\
&=-[D_{\alpha}F^{\alpha\nu}+\frac{1}{2}(T^{\alpha}_{\mu\alpha}{}^{\star}F^{\mu\nu}-T^{\mu}_{\mu\rho}{}^{\star}F^{\rho\nu}+T^{\nu}_{\mu\rho}{}^{\star}F^{\mu\rho})]\theta_{\nu}=-J^{\nu}\theta_{\nu},
\end{aligned}
\qquad (167)
$$

i.e.,

$$D_\alpha F^{\alpha\nu} + \frac{1}{2}(T^{\alpha}_{\mu\alpha}\,{}^{\star}F^{\mu\nu} - T^{\mu}_{\mu\rho}\,{}^{\star}F^{\rho\nu} + T^{\nu}_{\mu\rho}\,{}^{\star}F^{\mu\rho}) = J^\nu. \tag{168}$$

Exercise 14 *Show* (*use* Eq.(111)) *that in a Riemann-Cartan spacetime Maxwell equations become Maxwell equation, i.e.,*

$$\boldsymbol{\partial}\mathbf{F} = \mathbf{J} + \mathcal{T}^{\mathbf{a}} \lrcorner (\theta_{\mathbf{a}} \wedge \mathbf{F}) - \mathcal{T}^{\mathbf{a}} \wedge (\theta_{\mathbf{a}} \lrcorner \mathbf{F}). \tag{169}$$

12 Bianchi Identities

We rewrite Cartan's structure equations *for an arbitrary Riemann-Cartan structure* $(M, \mathbf{g}, D, \tau_{\mathbf{g}})$ where $\dim M = n$ and $\mathbf{g}$ is a metric of signature (p, q), with $p + q = n$ using an arbitrary cotetrad $\{\theta^{\mathbf{a}}\}$ as

$$\begin{aligned} \mathcal{T}^{\mathbf{a}} &= d\theta^{\mathbf{a}} + \omega^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}} = \mathbf{D}\theta^{\mathbf{a}}, \\ \mathcal{R}^{\mathbf{a}}_{\mathbf{b}} &= d\omega^{\mathbf{a}}_{\mathbf{b}} + \omega^{\mathbf{a}}_{\mathbf{c}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}} \end{aligned} \tag{170}$$

where

$$\omega^{\mathbf{a}}_{\mathbf{b}} = \omega^{\mathbf{a}}_{\mathbf{cb}}\theta^{\mathbf{c}},$$

$$\mathcal{T}^{\mathbf{a}} = \frac{1}{2}T^{\mathbf{a}}_{\mathbf{bc}}\theta^{\mathbf{b}} \wedge \theta^{\mathbf{c}} \tag{171}$$

$$\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} = \frac{1}{2}R_{\mathbf{b}}{}^{\mathbf{a}}{}_{\mathbf{cd}}\theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}}. \tag{172}$$

Since the $\mathcal{T}^{\mathbf{a}}$ and the $\mathcal{R}^{\mathbf{a}}_{\mathbf{b}}$ are *index form fields* we can apply to those objects the exterior covariant differential (Eq.(87)). We get

$$\begin{aligned} \mathbf{D}\mathcal{T}^{\mathbf{a}} &= d\mathcal{T}^{\mathbf{a}} + \omega^{\mathbf{a}}_{\mathbf{b}} \wedge \mathcal{T}^{\mathbf{b}} = d^2\theta^{\mathbf{a}} + d(\omega^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}}) + \omega^{\mathbf{a}}_{\mathbf{b}} \wedge \mathcal{T}^{\mathbf{b}} \\ &= d\omega^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}} - \omega^{\mathbf{a}}_{\mathbf{b}} \wedge d\theta^{\mathbf{b}} + \omega^{\mathbf{a}}_{\mathbf{b}} \wedge \mathcal{T}^{\mathbf{b}} \\ &= d\omega^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}} - \omega^{\mathbf{a}}_{\mathbf{b}} \wedge (\mathcal{T}^{\mathbf{b}} - \omega^{\mathbf{b}}_{\mathbf{c}} \wedge \theta^{\mathbf{c}}) + \omega^{\mathbf{a}}_{\mathbf{b}} \wedge \mathcal{T}^{\mathbf{b}} \\ &= (d\omega^{\mathbf{a}}_{\mathbf{b}} + \omega^{\mathbf{a}}_{\mathbf{c}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}}) \wedge \theta^{\mathbf{b}} \\ &= \mathcal{R}^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}} \end{aligned} \tag{173}$$

Also,

$$
\begin{aligned}
\mathbf{D}\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} &= d\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} + \omega^{\mathbf{a}}_{\mathbf{c}} \wedge \mathcal{R}^{\mathbf{c}}_{\mathbf{b}} - \omega^{\mathbf{c}}_{\mathbf{b}} \wedge \mathcal{R}^{\mathbf{a}}_{\mathbf{c}} \\
&= d^2\omega^{\mathbf{a}}_{\mathbf{b}} + d\omega^{\mathbf{a}}_{\mathbf{c}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}} - d\omega^{\mathbf{c}}_{\mathbf{b}} \wedge \omega^{\mathbf{a}}_{\mathbf{c}} - \mathcal{R}^{\mathbf{a}}_{\mathbf{c}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}} + \mathcal{R}^{\mathbf{c}}_{\mathbf{b}} \wedge \omega^{\mathbf{a}}_{\mathbf{c}} \\
&= d\omega^{\mathbf{a}}_{\mathbf{c}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}} - (d\omega^{\mathbf{a}}_{\mathbf{c}} + \omega^{\mathbf{a}}_{\mathbf{d}} \wedge \omega^{\mathbf{d}}_{\mathbf{c}}) \wedge \omega^{\mathbf{c}}_{\mathbf{b}} - d\omega^{\mathbf{a}}_{\mathbf{c}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}} + (d\omega^{\mathbf{c}}_{\mathbf{b}} + \omega^{\mathbf{c}}_{\mathbf{d}} \wedge \omega^{\mathbf{d}}_{\mathbf{b}}) \wedge \omega^{\mathbf{a}}_{\mathbf{c}} \\
&= -\omega^{\mathbf{a}}_{\mathbf{d}} \wedge \omega^{\mathbf{d}}_{\mathbf{c}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}} + \omega^{\mathbf{c}}_{\mathbf{d}} \wedge \omega^{\mathbf{d}}_{\mathbf{b}} \wedge \omega^{\mathbf{a}}_{\mathbf{c}} \\
&= -\omega^{\mathbf{a}}_{\mathbf{d}} \wedge \omega^{\mathbf{d}}_{\mathbf{c}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}} + \omega^{\mathbf{d}}_{\mathbf{c}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}} \wedge \omega^{\mathbf{a}}_{\mathbf{d}} \\
&= -\omega^{\mathbf{a}}_{\mathbf{d}} \wedge \omega^{\mathbf{d}}_{\mathbf{c}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}} + \omega^{\mathbf{a}}_{\mathbf{d}} \wedge \omega^{\mathbf{d}}_{\mathbf{c}} \wedge \omega^{\mathbf{c}}_{\mathbf{b}} = 0.
\end{aligned} \tag{174}
$$

So, we have the general Bianchi identities which are valid for any one of the metrical compatible structures[18] classified in Section 2,

$$
\begin{aligned}
\mathbf{D}\mathcal{T}^{\mathbf{a}} &= \mathcal{R}^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}}, \\
\mathbf{D}\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} &= 0.
\end{aligned} \tag{175}
$$

12.1 Coordinate Expressions of the First Bianchi Identity

Taking advantage of the calculations we done for the coordinate expressions of Maxwell equations we can write in a while:

$$
\begin{aligned}
\mathbf{D}\mathcal{T}^{\mathbf{a}} &= d\mathcal{T}^{\mathbf{a}} + \omega^{\mathbf{a}}_{\mathbf{b}} \wedge \mathcal{T}^{\mathbf{b}} \\
&= \frac{1}{3!}\left(\partial_\mu T^{\mathbf{a}}_{\alpha\beta} + \omega^{\mathbf{a}}_{\mu\mathbf{b}} T^{\mathbf{b}}_{\alpha\beta} + \partial_\alpha T^{\mathbf{a}}_{\beta\mu} + \omega^{\mathbf{a}}_{\alpha\mathbf{b}} T^{\mathbf{b}}_{\beta\mu} + \partial_\beta T^{\mathbf{a}}_{\mu\alpha} + \omega^{\mathbf{a}}_{\beta\mathbf{b}} T^{\mathbf{b}}_{\mu\alpha}\right)\theta^\mu \wedge \theta^\alpha \wedge \theta^\beta.
\end{aligned} \tag{176}
$$

Now,

$$
\partial_\mu T^{\mathbf{a}}_{\alpha\beta} = (\partial_\mu q^{\mathbf{a}}_\rho) T^\rho_{\alpha\beta} + q^{\mathbf{a}}_\rho \partial_\mu T^\rho_{\alpha\beta}, \tag{177}
$$

and using the freshman identity (Eq.(23)) we can write

$$
\omega^{\mathbf{a}}_{\mu\mathbf{b}} T^{\mathbf{b}}_{\alpha\beta} = \omega^{\mathbf{a}}_{\mu\mathbf{b}} q^{\mathbf{b}}_\rho T^\rho_{\alpha\beta} = L^{\mathbf{a}}_{\mu\mathbf{b}} q^{\mathbf{b}}_\rho T^\rho_{\alpha\beta} - (\partial_\mu q^{\mathbf{a}}_\rho) T^\rho_{\alpha\beta}. \tag{178}
$$

So,

$$
\begin{aligned}
&\partial_\mu T^{\mathbf{a}}_{\alpha\beta} + \omega^{\mathbf{a}}_{\mu\mathbf{b}} T^{\mathbf{b}}_{\alpha\beta} \\
&= q^{\mathbf{a}}_\rho \partial_\mu T^\rho_{\alpha\beta} + \Gamma^{\mathbf{a}}_{\mu\mathbf{b}} q^{\mathbf{b}}_\rho T^\rho_{\alpha\beta} \\
&= q^{\mathbf{a}}_\rho (D_\mu T^\rho_{\alpha\beta} + \Gamma^\kappa_{\mu\alpha} T^\rho_{\kappa\beta} + \Gamma^\kappa_{\mu\beta} T^\rho_{\alpha\kappa}).
\end{aligned} \tag{179}
$$

[18] For non metrical compatible structures we have more general equations than the Cartan structure equations and thus more general identities, see [22].

Now, recalling that $T^{\kappa}_{\mu\alpha} = \Gamma^{\kappa}_{\mu\alpha} - \Gamma^{\kappa}_{\alpha\mu}$ we can write

$$q^{\mathbf{a}}_{\rho}(\Gamma^{\kappa}_{\mu\alpha}T^{\rho}_{\kappa\beta} + \Gamma^{\kappa}_{\mu\beta}T^{\rho}_{\alpha\kappa})\theta^{\mu} \wedge \theta^{\alpha} \wedge \theta^{\beta} \\ = q^{\mathbf{a}}_{\rho}T^{\kappa}_{\mu\alpha}T^{\rho}_{\kappa\beta}\theta^{\mu} \wedge \theta^{\alpha} \wedge \theta^{\beta}. \tag{180}$$

Using these formulas we can write

$$\mathbf{D}\mathcal{T}^{\mathbf{a}} = \\ \frac{1}{3!}q^{\mathbf{a}}_{\rho}\left\{D_{\mu}T^{\rho}_{\alpha\beta} + D_{\alpha}T^{\rho}_{\beta\mu} + D_{\beta}T^{\rho}_{\mu\alpha} + T^{\kappa}_{\mu\alpha}T^{\rho}_{\kappa\beta} + T^{\kappa}_{\alpha\beta}T^{\rho}_{\kappa\mu} + T^{\kappa}_{\beta\mu}T^{\rho}_{\kappa\alpha}\right\}\theta^{\mu} \wedge \theta^{\alpha} \wedge \theta^{\beta}. \tag{181}$$

Now, the coordinate representation of $\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}}$ is:

$$\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}} = \frac{1}{3!}q^{\mathbf{a}}_{\rho}(R_{\mu}{}^{\rho}{}_{\alpha\beta} + R_{\alpha}{}^{\rho}{}_{\beta\mu} + R_{\beta}{}^{\rho}{}_{\mu\alpha})\theta^{\mu} \wedge \theta^{\alpha} \wedge \theta^{\beta}, \tag{182}$$

and thus the coordinate expression of the first Bianchi identity is:

$$D_{\mu}T^{\rho}_{\alpha\beta} + D_{\alpha}T^{\rho}_{\beta\mu} + D_{\beta}T^{\rho}_{\mu\alpha} \\ = (R_{\mu}{}^{\rho}{}_{\alpha\beta} + R_{\alpha}{}^{\rho}{}_{\beta\mu} + R_{\beta}{}^{\rho}{}_{\mu\alpha}) - (T^{\kappa}_{\mu\alpha}T^{\rho}_{\kappa\beta} + T^{\kappa}_{\alpha\beta}T^{\rho}_{\kappa\mu} + T^{\kappa}_{\beta\mu}T^{\rho}_{\kappa\alpha}), \tag{183}$$

which we can write as

$$\sum_{(\mu\alpha\beta)} R_{\mu}{}^{\rho}{}_{\alpha\beta} = \sum_{(\mu\alpha\beta)} \left(D_{\mu}T^{\rho}_{\alpha\beta} - T^{\kappa}_{\mu\beta}T^{\rho}_{\kappa\alpha}\right), \tag{184}$$

with $\sum_{(\mu\alpha\beta)}$ denoting as usual the sum over cyclic permutation of the indices $(\mu\alpha\beta)$. For the particular case of a Levi-Civita connection $\mathring{D}$ since the $T^{\rho}_{\alpha\beta} = 0$ we have the standard form of the first Bianchi identity in classical Riemannian geometry, i.e.,

$$R_{\mu}{}^{\rho}{}_{\alpha\beta} + R_{\alpha}{}^{\rho}{}_{\beta\mu} + R_{\beta}{}^{\rho}{}_{\mu\alpha} = 0. \tag{185}$$

If we now recall the steps that lead us to Eq.(166) we can write for the torsion 2-form fields $\mathcal{T}^{\mathbf{a}}$,

$$d\mathcal{T}^{\mathbf{a}} = \star\star^{-1} d \star\star^{-1}\mathcal{T}^{\mathbf{a}} \\ = (-1)^{n-2} \star \delta \star^{-1} \mathcal{T}^{\mathbf{a}} = (-1)^{n-2}(-1)^{n-2}\operatorname{sgn}\mathbf{g} \star \delta \star \mathcal{T}^{\mathbf{a}} \\ = (-1)^{n-2} \star^{-1} \delta \star \mathcal{T}^{\mathbf{a}}. \tag{186}$$

with $\text{sgn}\mathbf{g} = \det \mathbf{g}/\left|\det \mathbf{g}\right|$. Then we can write the first Bianchi identity as

$$\delta \star \mathcal{T}^{\mathbf{a}} = (-1)^{n-2} \star [\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}} - \omega^{\mathbf{a}}_{\mathbf{b}} \wedge \mathcal{T}^{\mathbf{b}}], \tag{187}$$

and taking into account that

$$\begin{aligned} \star(\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}}) &= \star(\theta^{\mathbf{b}} \wedge \mathcal{R}^{\mathbf{a}}_{\mathbf{b}}) = \theta^{\mathbf{b}} \lrcorner \star \mathcal{R}^{\mathbf{a}}_{\mathbf{b}}, \\ \star(\omega^{\mathbf{a}}_{\mathbf{b}} \wedge \mathcal{T}^{\mathbf{b}}) &= \omega^{\mathbf{a}}_{\mathbf{b}} \lrcorner \star \mathcal{T}^{\mathbf{b}}, \end{aligned} \tag{188}$$

we end with

$$\delta \star \mathcal{T}^{\mathbf{a}} = (-1)^{n-2} (\theta^{\mathbf{b}} \lrcorner \star \mathcal{R}^{\mathbf{a}}_{\mathbf{b}} - \omega^{\mathbf{a}}_{\mathbf{b}} \lrcorner \star \mathcal{T}^{\mathbf{b}}). \tag{189}$$

This is the first Bianchi identity written in terms of duals. To calculate its coordinate expression, we recall the steps that lead us to Eq.(166) and write directly for the torsion 2-form fields $\mathcal{T}^{\mathbf{a}}$

$$\begin{aligned} &\delta \star \mathcal{T}^{\mathbf{a}} \\ &= -(D_\alpha {}^\star T^{\mathbf{a}\alpha\nu} + \frac{1}{2}(T^{\alpha}_{\mu\alpha} {}^\star T^{\mathbf{a}\mu\nu} - T^{\mu}_{\mu\rho} {}^\star T^{\mathbf{a}\rho\nu} + T^{\nu}_{\mu\rho} {}^\star T^{\mathbf{a}\mu\rho}))\theta_\nu. \end{aligned} \tag{190}$$

Also, writing

$$\star \mathcal{R}^{\mathbf{a}}_{\mathbf{b}} = \frac{1}{2} {}^* R_{\mathbf{b}\ \mathbf{cd}}^{\ \mathbf{a}} \theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}}, \tag{191}$$

we have:

$$\begin{aligned} \star(\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}}) &= \theta^{\mathbf{b}} \lrcorner \star \mathcal{R}^{\mathbf{a}}_{\mathbf{b}} \\ &= \frac{1}{2} \theta^{\mathbf{b}} \lrcorner ({}^\star R_{\mathbf{b}\ \mathbf{cd}}^{\ \mathbf{a}} \theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}}) \\ &= {}^\star R_{\mathbf{b}\ \mathbf{cd}}^{\ \mathbf{a}} \eta^{\mathbf{bc}} \theta^{\mathbf{d}} \\ &= {}^\star R^{\mathbf{ca}}_{\ \ \mathbf{cd}} \theta^{\mathbf{d}} = {}^\star R^{\mathbf{ca}\ \mathbf{d}}_{\ \ \mathbf{c}} \theta_{\mathbf{d}} = {}^\star R^{\mathbf{ca}\ \mathbf{d}}_{\ \ \mathbf{c}} q^{\nu}_{\mathbf{d}} \theta_\nu. \end{aligned} \tag{192}$$

On the other hand we can also write:

$$
\begin{aligned}
\star(\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}}) &= \theta^{\mathbf{b}} \lrcorner \star \mathcal{R}^{\mathbf{a}}_{\mathbf{b}} \\
&= \frac{1}{2}\vartheta^{\mathbf{b}} \lrcorner (\frac{1}{(n-2)!} R_{\mathbf{b}}^{\ \mathbf{akl}} \epsilon_{\mathbf{klmn}} \theta^{\mathbf{m}} \wedge \theta^{\mathbf{n}}) \\
&= \frac{1}{2}\frac{1}{(n-2)!}(R_{\mathbf{b}}^{\ \mathbf{akl}} \epsilon_{\mathbf{klmn}} \eta^{\mathbf{bm}} \wedge \theta^{\mathbf{n}} - R_{\mathbf{b}}^{\ \mathbf{akl}} \epsilon_{\mathbf{klmn}} \eta^{\mathbf{bn}} \wedge \theta^{\mathbf{m}}) \\
&= \frac{1}{(n-2)!} R_{\mathbf{b}}^{\ \mathbf{akl}} \epsilon_{\mathbf{klmn}} \eta^{\mathbf{bm}} \theta^{\mathbf{n}} = \frac{1}{(n-2)!} R^{\mathbf{makl}} \epsilon_{\mathbf{klmn}} \theta^{\mathbf{n}} \\
&= \frac{1}{(n-2)!} R_{\mathbf{m}}^{\ \mathbf{akl}} \epsilon_{\mathbf{kl}}^{\ \ \mathbf{mn}} \theta_{\mathbf{n}} \\
&= \frac{1}{(n-2)!} R_{\mathbf{m}}^{\ \mathbf{akl}} \epsilon_{\mathbf{kl}}^{\ \ \mathbf{mn}} q^{\nu}_{\mathbf{n}} \theta_{\nu}.
\end{aligned}
$$

from where we get in agreement with Eq.(34) the formula

$$
{}^{\star}R^{\mathbf{ca}}_{\ \ \ \mathbf{cd}} = \frac{1}{(n-2)!} R^{\mathbf{makl}} \epsilon_{\mathbf{mkld}}, \tag{193}
$$

which shows explicitly that ${}^{\star}R^{\mathbf{ca}}_{\ \ \ \mathbf{cd}}$ are not the components of the *Ricci* tensor.

Moreover,

$$
\begin{aligned}
&\omega^{\mathbf{a}}_{\mathbf{b}} \lrcorner \star \mathcal{T}^{\mathbf{b}} \\
&= \frac{1}{2}\omega^{\mathbf{a}}_{\alpha\mathbf{b}} \theta^{\alpha} \lrcorner ({}^{\star}T^{\mathbf{b}\mu\nu} \theta_{\mu} \wedge \theta_{\nu}) \\
&= {}^{\star}T^{\mathbf{b}\mu\nu} \omega^{\mathbf{a}}_{\alpha\mathbf{b}} \theta_{\nu}.
\end{aligned} \tag{194}
$$

Collecting the above formulas we end with

$$
D_{\alpha}{}^{\star}T^{\mathbf{a}\alpha\nu} + \frac{1}{2}(T^{\alpha}_{\mu\alpha}{}^{\star}T^{\mathbf{a}\mu\nu} - T^{\mu}_{\mu\rho}{}^{\star}T^{\mathbf{a}\rho\nu} + T^{\nu}_{\mu\rho}{}^{\star}T^{\mathbf{a}\mu\rho}) = (-1)^{n-1}({}^{\star}R^{\mathbf{ca}\ \ \mathbf{d}}_{\ \ \ \mathbf{c}} q^{\nu}_{\mathbf{d}} - \omega^{\mathbf{a}\star}_{\alpha\mathbf{b}} T^{\mathbf{b}\alpha\nu}), \tag{195}
$$

which is another expression for the first Bianchi identity written in terms of duals.

Remark 15 *Consider, e.g., the term* $D_{\alpha}{}^{\star}T^{\mathbf{a}\alpha\nu}$ *in the above equation and write*

$$
D_{\alpha}{}^{\star}T^{\mathbf{a}\alpha\nu} = D_{\alpha}(q^{\mathbf{a}\star}_{\rho} T^{\rho\alpha\nu}). \tag{196}
$$

We now show that

$$
D_{\alpha}(q^{\mathbf{a}\star}_{\rho} T^{\rho\alpha\nu}) \neq q^{\mathbf{a}}_{\rho} D_{\alpha}{}^{\star}T^{\rho\alpha\nu}. \tag{197}
$$

Indeed, recall that we already found that

$$(D_\alpha{}^\star T^{\mathbf{a}\alpha\nu})\theta_\nu = -\delta\star\mathcal{T}^{\mathbf{a}} + \frac{1}{2}(T^{\alpha}_{\mu\alpha}{}^\star T^{\mathbf{a}\mu\nu} - T^{\mu}_{\mu\rho}{}^\star T^{\mathbf{a}\rho\nu} + T^{\nu}_{\mu\rho}{}^\star T^{\mathbf{a}\mu\rho})\theta_\nu, \quad (198)$$

and taking into account the second formula in Eq.(111) *we can write*

$$\theta^\alpha \lrcorner (D_{\partial_\alpha}\star\mathcal{T}^{\mathbf{a}}) = -\delta\star\mathcal{T}^{\mathbf{a}} + \frac{1}{2}(T^{\alpha}_{\mu\alpha}{}^\star T^{\mathbf{a}\mu\nu} - T^{\mu}_{\mu\rho}{}^\star T^{\mathbf{a}\rho\nu} + T^{\nu}_{\mu\rho}{}^\star T^{\mathbf{a}\mu\rho})\theta_\nu. \quad (199)$$

Now, writing $\star\mathcal{T}^{\mathbf{a}} = \frac{1}{2}q^{\mathbf{a}}_\rho\ {}^\star T^{\rho\mu\nu}\theta_\mu \wedge \theta_\nu$ *and get*

$$\begin{aligned}
&\theta^\alpha \lrcorner (D_{\partial_\alpha} \star \mathcal{T}^{\mathbf{a}}) \\
&= \frac{1}{2}\theta^\alpha \lrcorner [D_{\partial_\alpha}(q^{\mathbf{a}\star}_\rho T^{\rho\mu\nu}\theta_\mu \wedge \theta_\nu)] \\
&= \frac{1}{2}\vartheta^\alpha \lrcorner [\partial_\alpha(q^{\mathbf{a}\star}_\rho T^{\rho\mu\nu})\theta_\mu \wedge \theta_\nu + q^{\mathbf{a}\star}_\rho T^{\rho\mu\nu} D_{\partial_\alpha}(\theta_\mu \wedge \theta_\nu)] \\
&= \frac{1}{2}\vartheta^\alpha \lrcorner [(\partial_\alpha q^{\mathbf{a}}_\rho)^\star T^{\rho\mu\nu}\theta_\mu \wedge \theta_\nu + q^{\mathbf{a}}_\rho \partial_\alpha({}^\star T^{\rho\mu\nu})\theta_\mu \wedge \theta_\nu + q^{\mathbf{a}\star}_\rho T^{\rho\mu\nu} D_{\partial_\alpha}(\theta_\mu \wedge \theta_\nu)] \\
&= \frac{1}{2}\vartheta^\alpha \lrcorner [(\partial_\alpha q^{\mathbf{a}}_\rho)^\star T^{\rho\mu\nu}\theta_\mu \wedge \theta_\nu + q^{\mathbf{a}}_\rho D_\alpha({}^\star T^{\rho\mu\nu})\theta_\mu \wedge \theta_\nu] \\
&= (\partial_\alpha q^{\mathbf{a}}_\rho)^\star T^{\rho\mu\nu}\delta^\alpha_\mu\theta_\nu + q^{\mathbf{a}}_\rho D_\alpha({}^\star T^{\rho\mu\nu})\delta^\alpha_\mu\theta_\nu. \qquad (200)
\end{aligned}$$

Comparing the Eq.(198) *with* Eq.(199) *using* Eq.(200) *we get*

$$D_\alpha{}^\star T^{\mathbf{a}\alpha\nu}\theta_\nu = D_\alpha(q^{\mathbf{a}\star}_\rho T^{\mathbf{a}\alpha\nu}\theta_\nu) = (\partial_\alpha q^{\mathbf{a}}_\rho)^\star T^{\rho\mu\nu} + q^{\mathbf{a}}_\rho D_\alpha({}^\star T^{\rho\mu\nu}), \quad (201)$$

thus proving our statement and showing the danger of applying a so called "tetrad postulate" asserting without due care on the meaning of the symbols that " the covariant derivative of the tetrad is zero, and thus using "$D_\alpha q^{\mathbf{a}}_\rho = 0$*"."*

Exercise 16 *Show that the coordinate expression of the second Bianchi identity* $\mathbf{D}\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} = 0$ *is*

$$\sum_{(\mu\nu\rho)} D_\mu R_{\beta\ \nu\rho}^{\ \alpha} = \sum_{(\mu\nu\rho)} T^{\alpha}_{\nu\mu} R_{\beta\ \alpha\rho}^{\ \alpha}. \quad (202)$$

Exercise 17 *Calculate* $\star\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}}$ *in an orthonormal basis.*

Solution: First we recall the $\star\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}} = \theta^{\mathbf{b}} \wedge \star\mathcal{R}^{\mathbf{a}}_{\mathbf{b}}$ and then use the formula in the third line of Eq.(35) to write:

$$\begin{aligned}\theta^{\mathbf{b}} \wedge \star\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} &= -\star(\theta^{\mathbf{b}} \lrcorner \mathcal{R}^{\mathbf{a}}_{\mathbf{b}}) \\ &= -\star\left[\frac{1}{2}\theta^{\mathbf{b}} \lrcorner (R_{\mathbf{b}\ \mathbf{cd}}^{\ \mathbf{a}}\theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}})\right] \\ &= -\star[R_{\mathbf{b}\ \mathbf{cd}}^{\ \mathbf{a}}\eta^{\mathbf{bc}}\theta^{\mathbf{d}}] \\ &= -\star[R^{\mathbf{ca}}_{\ \ \mathbf{cd}}\theta^{\mathbf{d}}] = -\star[R^{\mathbf{ac}}_{\ \ \mathbf{dc}}\theta^{\mathbf{d}}] \\ &= -\star[R^{\mathbf{a}}_{\mathbf{d}}\theta^{\mathbf{d}}] = -\star\mathcal{R}^{\mathbf{a}} \end{aligned} \tag{203}$$

Of course, if the connection is the Levi-Civita one we get

$$\theta^{\mathbf{b}} \wedge \star\mathring{\mathcal{R}}^{\mathbf{a}}_{\mathbf{b}} = -\star(\theta^{\mathbf{b}} \lrcorner \mathring{\mathcal{R}}^{\mathbf{a}}_{\mathbf{b}}) = -\mathring{R}^{\mathbf{a}}_{\mathbf{b}}\theta^{\mathbf{b}} = -\star\mathring{\mathcal{R}}^{\mathbf{a}}. \tag{204}$$

13 A Remark on Evans 101^{th} Paper on "ECE Theory"

Eq. (195) or its equivalent Eq.(201) is to be compared with a wrong one derived by Evans from where he now claims that the Einstein-Hilbert (gravitational) theory which uses in its formulation the Levi-Civita connection $\mathring{D}$ is incompatible with the first Bianchi identity. Evans conclusion follows because he thinks to have derived "from first principles" that

$$\mathbf{D}\star\mathcal{T}^{\mathbf{a}} = \star\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}}, \tag{205}$$

an equation that *if* true implies as we just see from Eq.(203) that for the Levi-Civita connection for which $\mathcal{T}^{\mathbf{a}} = 0$ the Ricci tensor of the connection $\mathring{D}$ is null.

We show below that Eq.(205) is a false one in two different ways, firstly by deriving the correct equation for $\mathbf{D}\star\mathcal{T}^{\mathbf{a}}$ and secondly by showing explicit counterexamples for some trivial structures.

Before doing that let us show that we can derive from the first Bianchi identity that

$$\mathring{R}_{\mathbf{a}\ \mathbf{cd}}^{\ \mathbf{a}} = 0, \tag{206}$$

an equation that eventually may lead Evans in believing that for a Levi-Civita connection the first Bianchi identity implies that the Ricci tensor is null. As we know, for a Levi-Civita connection the first Bianchi identity gives (with $\mathcal{R}^{\mathbf{a}}_{\mathbf{b}} \longmapsto \mathring{\mathcal{R}}^{\mathbf{a}}_{\mathbf{b}}$):

$$\mathring{\mathcal{R}}^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}} = 0. \tag{207}$$

Contracting this equation with $\theta_{\mathbf{a}}$ we get

$$\begin{aligned}\theta_{\mathbf{a}}\lrcorner(\mathring{\mathcal{R}}_{\mathbf{b}}^{\mathbf{a}}\wedge\theta^{\mathbf{b}}) &= \theta_{\mathbf{a}}\lrcorner(\theta^{\mathbf{b}}\wedge\mathring{\mathcal{R}}_{\mathbf{b}}^{\mathbf{a}})\\ &= \delta_{\mathbf{a}}^{\mathbf{b}}\mathring{\mathcal{R}}_{\mathbf{b}}^{\mathbf{a}}-\theta^{\mathbf{b}}\wedge(\theta_{\mathbf{a}}\lrcorner\mathring{\mathcal{R}}_{\mathbf{b}}^{\mathbf{a}})\\ &= \mathring{\mathcal{R}}_{\mathbf{a}}^{\mathbf{a}}-\frac{1}{2}\theta^{\mathbf{b}}\wedge[\theta_{\mathbf{a}}\lrcorner(\mathring{R}_{\mathbf{b}\ \mathbf{cd}}^{\ \mathbf{a}}\theta^{\mathbf{c}}\wedge\theta^{\mathbf{d}})]\\ &= \mathring{\mathcal{R}}_{\mathbf{a}}^{\mathbf{a}}-\mathring{R}_{\mathbf{b}\ \mathbf{ad}}^{\ \mathbf{a}}\theta^{\mathbf{b}}\wedge\theta^{\mathbf{d}}\end{aligned}$$

Now, the second term in this last equation is null because according to the Eq.(106), $-R_{\mathbf{b}\ \mathbf{ad}}^{\ \mathbf{a}}=R_{\mathbf{b}\ \mathbf{da}}^{\ \mathbf{a}}=R_{\mathbf{bd}}$ are the components of the Ricci tensor, which is a symmetric tensor for the Levi-Civita connection. For the first term we get

$$\mathring{R}_{\mathbf{a}\ \mathbf{cd}}^{\ \mathbf{a}}\theta^{\mathbf{c}}\wedge\theta^{\mathbf{d}}=0, \tag{208}$$

which implies that as we stated above that

$$\mathring{R}_{\mathbf{a}\ \mathbf{cd}}^{\ \mathbf{a}}=0. \tag{209}$$

But according to Eq.(106) the $\mathring{R}_{\mathbf{a}\ \mathbf{cd}}^{\ \mathbf{a}}$ are not the components of the Ricci tensor, and so there is not any contradiction. As an additional verification recall that the standard form of the first Bianchi identity in Riemannian geometry is

$$\mathring{R}_{\mathbf{b}}{}^{\mathbf{a}}{}_{\mathbf{cd}}+\mathring{R}_{\mathbf{c}}{}^{\mathbf{a}}{}_{\mathbf{db}}+\mathring{R}_{\mathbf{d}}{}^{\mathbf{a}}{}_{\mathbf{bc}}.=0 \tag{210}$$

Making $\mathbf{b}=\mathbf{a}$ we get

$$\begin{aligned}&\mathring{R}_{\mathbf{a}}{}^{\mathbf{a}}{}_{\mathbf{cd}}+\mathring{R}_{\mathbf{c}}{}^{\mathbf{a}}{}_{\mathbf{da}}+\mathring{R}_{\mathbf{d}}{}^{\mathbf{a}}{}_{\mathbf{ac}}\\ &=\mathring{R}_{\mathbf{a}}{}^{\mathbf{a}}{}_{\mathbf{cd}}-\mathring{R}_{\mathbf{c}}{}^{\mathbf{a}}{}_{\mathbf{ad}}+\mathring{R}_{\mathbf{d}}{}^{\mathbf{a}}{}_{\mathbf{ac}}\\ &=\mathring{R}_{\mathbf{a}}{}^{\mathbf{a}}{}_{\mathbf{cd}}+\mathring{R}_{\mathbf{cd}}-\mathring{R}_{\mathbf{dc}}\\ &=\mathring{R}_{\mathbf{a}}{}^{\mathbf{a}}{}_{\mathbf{cd}}=0.\end{aligned} \tag{211}$$

14 Direct Calculation of $\mathbf{D}\star\mathcal{T}^{\mathbf{a}}$

We now present using results of Clifford bundle formalism, recalled above (for details, see, e.g., [22]) a calculation of $\mathbf{D}\star\mathcal{T}^{\mathbf{a}}$.

We start from Cartan first structure equation

$$\mathcal{T}^{\mathbf{a}}=d\theta^{\mathbf{a}}+\omega_{\mathbf{b}}^{\mathbf{a}}\wedge\theta^{\mathbf{b}}. \tag{212}$$

By definition

$$\mathbf{D}\star\mathcal{T}^{\mathbf{a}}=d\star\mathcal{T}^{\mathbf{a}}+\omega_{\mathbf{b}}^{\mathbf{a}}\wedge\star\mathcal{T}^{\mathbf{b}}. \tag{213}$$

Now, if we recall Eq.(39), since the $\mathcal{T}^{\mathbf{a}}\in\sec\bigwedge^{2}T^{*}M\hookrightarrow\sec\mathcal{C}\ell(M,\mathtt{g})$ we can write

$$d\star\mathcal{T}^{\mathbf{a}}=\star\delta\mathcal{T}^{\mathbf{a}}. \tag{214}$$

We next calculate $\delta\mathcal{T}^{\mathbf{a}}$. We have:

$$\begin{aligned}\delta\mathcal{T}^{\mathbf{a}}&=\delta\left(d\theta^{\mathbf{a}}+\omega_{\mathbf{b}}^{\mathbf{a}}\wedge\theta^{\mathbf{b}}\right)\\&=\delta d\theta^{\mathbf{a}}+d\delta\theta^{\mathbf{a}}-d\delta\theta^{\mathbf{a}}+\delta(\omega_{\mathbf{b}}^{\mathbf{a}}\wedge\theta^{\mathbf{b}})\ .\end{aligned} \tag{215}$$

Next we recall the definition of the Hodge D'Alembertian which, recalling Eq.(112) permit us to write the first two terms in Eq.(215) as the negative of the square of the standard Dirac operator (associated with the Levi-Civita connection)[19]. We then get:

$$\begin{aligned}\delta\mathcal{T}^{\mathbf{a}}&=-\boldsymbol{\partial}^{2}\theta^{\mathbf{a}}-d\delta\theta^{\mathbf{a}}+\delta(\omega_{\mathbf{b}}^{\mathbf{a}}\wedge\theta^{\mathbf{b}})\\&\overset{\text{Eq.(115)}}{=}-\mathring{\square}\theta^{\mathbf{a}}-(\boldsymbol{\partial}\wedge\boldsymbol{\partial})\theta^{\mathbf{a}}-d\delta\theta^{\mathbf{a}}+\delta(\omega_{\mathbf{b}}^{\mathbf{a}}\wedge\theta^{\mathbf{b}})\\&\overset{\text{Eq.(127)}}{=}-\mathring{\square}\theta^{\mathbf{a}}-\mathring{\mathcal{R}}^{\mathbf{a}}-d\delta\theta^{\mathbf{a}}+\delta(\omega_{\mathbf{b}}^{\mathbf{a}}\wedge\theta^{\mathbf{b}})\\&=-\mathring{\square}\theta^{\mathbf{a}}-\mathcal{R}^{\mathbf{a}}+\mathcal{J}^{\mathbf{a}}-d\delta\theta^{\mathbf{a}}+\delta(\omega_{\mathbf{b}}^{\mathbf{a}}\wedge\theta^{\mathbf{b}})\end{aligned} \tag{216}$$

where we have used Eq(107) to write

$$\mathcal{R}^{\mathbf{a}}=R_{\mathbf{b}}^{\mathbf{a}}\theta^{\mathbf{b}}=(\mathring{R}_{\mathbf{b}}^{\mathbf{a}}+J_{\mathbf{b}}^{\mathbf{a}})\theta^{\mathbf{b}}. \tag{217}$$

So, we have

$$d\star\mathcal{T}^{\mathbf{a}}=-\star\mathring{\square}\theta^{\mathbf{a}}-\star\mathcal{R}^{\mathbf{a}}+\star\mathcal{J}^{\mathbf{a}}-\star d\delta\theta^{\mathbf{a}}+\star\delta(\omega_{\mathbf{b}}^{\mathbf{a}}\wedge\theta^{\mathbf{b}})$$

and finally

$$D\star\mathcal{T}^{\mathbf{a}}=-\star\mathring{\square}\theta^{\mathbf{a}}-\star\mathcal{R}^{\mathbf{a}}+\star\mathcal{J}^{\mathbf{a}}-\star d\delta\theta^{\mathbf{a}}+\star\delta(\omega_{\mathbf{b}}^{\mathbf{a}}\wedge\theta^{\mathbf{b}})+\omega_{\mathbf{b}}^{\mathbf{a}}\wedge\star\mathcal{T}^{\mathbf{b}} \tag{218}$$

or equivalently recalling Eq.(35)

$$D\star\mathcal{T}^{\mathbf{a}}=-\star\mathring{\square}\theta^{\mathbf{a}}-\star\mathcal{R}^{\mathbf{a}}+\star\mathcal{J}^{\mathbf{a}}-\star d\delta\theta^{\mathbf{a}}+\star\delta(\omega_{\mathbf{b}}^{\mathbf{a}}\wedge\theta^{\mathbf{b}})-\star(\omega_{\mathbf{b}}^{\mathbf{a}}\lrcorner\mathcal{T}^{\mathbf{b}}) \tag{219}$$

[19] Be patient, the Riemann-Cartan connection will appear in due time.

Remark 18 Eq.(219) *does not implies that* $D\star\mathcal{T}^{\mathbf{a}} = \star\mathcal{R}^{\mathbf{a}}_{\mathbf{b}}\wedge\theta^{\mathbf{b}}$ *because taking into account* Eq.(203)

$$\star\mathcal{R}^{\mathbf{a}}_{\mathbf{b}}\wedge\theta^{\mathbf{b}} = -\star\mathcal{R}^{\mathbf{a}} \neq D\star\mathcal{T}^{\mathbf{a}} = -\star\mathring{\square}\theta^{\mathbf{a}} - \star\mathcal{R}^{\mathbf{a}} + \star\mathcal{J}^{\mathbf{a}} - \star d\delta\theta^{\mathbf{a}} + \star\delta(\omega^{\mathbf{a}}_{\mathbf{b}}\wedge\theta^{\mathbf{b}}) - \star(\omega^{\mathbf{a}}_{\mathbf{b}}\lrcorner\mathcal{T}^{\mathbf{b}}) \tag{221}$$

in general.

So, for a Levi-Civita connection we have that $D\star\mathcal{T}^{\mathbf{a}} = 0$ and then Eq.(218) implies

$$D\star\mathcal{T}^{\mathbf{a}} = 0 \Leftrightarrow -\mathring{\square}\theta^{\mathbf{a}} - \mathring{\mathcal{R}}^{\mathbf{a}} - d\delta\theta^{\mathbf{a}} + \delta(\mathring{\omega}^{\mathbf{a}}_{\mathbf{b}}\wedge\theta^{\mathbf{b}}) = 0 \tag{220}$$

or since $\mathring{\omega}^{\mathbf{a}}_{\mathbf{b}}\wedge\theta^{\mathbf{b}} = -d\theta^{\mathbf{b}}$ for a Levi-Civita connection,

$$D\star\mathcal{T}^{\mathbf{a}} = 0 \Leftrightarrow -\mathring{\square}\theta^{\mathbf{a}} - \mathring{\mathcal{R}}^{\mathbf{a}} - d\delta\theta^{\mathbf{a}} - \delta d\theta^{\mathbf{a}} = 0 \tag{221}$$

or yet

$$-\mathring{\square}\theta^{\mathbf{a}} - \mathring{\mathcal{R}}^{\mathbf{a}} = -\mathring{\partial}^2\,\theta^{\mathbf{a}} = d\delta\theta^{\mathbf{a}} + \delta d\theta^{\mathbf{a}}, \tag{222}$$

an identity that we already mentioned above (Eq.(113)).

14.1 Einstein Equations

The reader can easily verify that Einstein equations in the Clifford bundle formalism is written as:

$$\mathring{\mathcal{R}}^{\mathbf{a}} - \frac{1}{2}\mathring{R}\theta^{\mathbf{a}} = \boldsymbol{T}^{\mathbf{a}}, \tag{223}$$

where $\mathring{R}$ is the scalar curvature and $\boldsymbol{T}^{\mathbf{a}} = -T^{\mathbf{a}}_{\mathbf{b}}\theta^{\mathbf{b}}$ are the energy-momentum 1-form fields. Comparing Eq.(221) with Eq.(223). We immediately get the "wave equation" for the cotetrad fields:

$$\boldsymbol{T}^{\mathbf{a}} = -\frac{1}{2}\mathring{R}\theta^{\mathbf{a}} - \mathring{\square}\theta^{\mathbf{a}} - d\delta\theta^{\mathbf{a}} - \delta d\theta^{\mathbf{a}}, \tag{224}$$

which does not implies that the Ricci tensor is null.

Remark 19 *We see from Eq.(224) that a Ricci flat spacetime is characterized by the equality of the Hodge and covariant D' Alembertians acting on the coterad fields, i.e.,*

$$\mathring{\square}\theta^{\mathbf{a}} = \Diamond\theta^{\mathbf{a}}, \tag{225}$$

a non trivial result.

Exercise 20 *Using* Eq.(120) *and* Eq.(121) write $\mathring{\square}\theta^{\mathbf{a}}$ *in terms of the connection coefficients of the Riemann-Cartan connection.*

15 Two Counterexamples to Evans (Wrong) Equation "$\mathbf{D}\star\mathcal{T}^{\mathbf{a}}=\star\mathcal{R}_{\mathbf{b}}^{\mathbf{a}}\wedge\theta^{\mathbf{b}}$"

15.1 The Riemannian Geometry of S^2

Consider the well known Riemannian structure on the unit radius sphere [12] $\{S^2, \boldsymbol{g}, \mathring{D}\}$. Let $\{x^i\}$, $x^1 = \vartheta$, $x^2 = \varphi$, $0 < \vartheta < \pi$, $0 < \varphi < 2\pi$, be spherical coordinates covering $U = \{S^2 - l\}$, where l is the curve joining the north and south poles.

A coordinate basis for TU is then $\{\boldsymbol{\partial}_\mu\}$ and its dual basis is $\{\theta^\mu = dx^\mu\}$. The Riemannian metric $\boldsymbol{g} \in \sec T_0^2 M$ is given by

$$\boldsymbol{g} = d\vartheta \otimes d\vartheta + \sin^2\vartheta d\varphi \otimes d\varphi \tag{226}$$

and the metric $\mathtt{g}\in \sec T_2^0 M$ of the cotangent space is

$$\mathtt{g} = \boldsymbol{\partial}_1 \otimes \boldsymbol{\partial}_1 + \frac{1}{\sin^2\vartheta}\boldsymbol{\partial}_2 \otimes \boldsymbol{\partial}_2. \tag{227}$$

An orthonormal basis for TU is then $\{\mathbf{e_a}\}$ with

$$\mathbf{e_1} = \boldsymbol{\partial}_1, \mathbf{e_2} = \frac{1}{\sin\vartheta}\boldsymbol{\partial}_2, \tag{228}$$

with dual basis $\{\theta^{\mathbf{a}}\}$ given by

$$\theta^{\mathbf{1}} = d\vartheta, \theta^{\mathbf{2}} = \sin\vartheta d\varphi. \tag{229}$$

The structure coefficients of the orthonormal basis are

$$[\mathbf{e_i}, \mathbf{e_j}] = c_{\mathbf{ij}}^{\mathbf{k}}\mathbf{e_k} \tag{230}$$

and can be evaluated, e.g., by calculating $d\theta^{\mathbf{i}} = -\frac{1}{2}c_{\mathbf{jk}}^{\mathbf{i}}\theta^{\mathbf{j}} \wedge \theta^{\mathbf{k}}$. We get immediately that the only non null coefficients are

$$c_{\mathbf{12}}^{\mathbf{2}} = -c_{\mathbf{21}}^{\mathbf{2}} = -\cot\theta. \tag{231}$$

To calculate the connection 1-form $\omega_{\mathbf{d}}^{\mathbf{c}}$ we use Eq.(92), i.e.,

$$\omega^{\mathbf{cd}} = \frac{1}{2}(-c_{\mathbf{jk}}^{\mathbf{c}}\eta^{\mathbf{dj}} + c_{\mathbf{jk}}^{\mathbf{d}}\eta^{\mathbf{cj}} - \eta^{\mathbf{ca}}\eta_{\mathbf{bk}}\eta^{\mathbf{dj}}c_{\mathbf{ja}}^{\mathbf{b}})\theta^{\mathbf{k}}.$$

Then,

$$\omega^{\mathbf{21}} = \frac{1}{2}(-c^{\mathbf{2}}_{\mathbf{12}}\eta^{\mathbf{11}} - \eta^{\mathbf{22}}\eta_{\mathbf{22}}\eta^{\mathbf{11}}c^{\mathbf{2}}_{\mathbf{12}})\theta^{\mathbf{2}} = \cot\vartheta\theta^{\mathbf{2}}. \tag{232}$$

Then

$$\begin{aligned}\omega^{\mathbf{21}} &= -\omega^{\mathbf{12}} = \cot\vartheta\theta^{\mathbf{2}},\\ \omega^{\mathbf{2}}_{\mathbf{1}} &= -\omega^{\mathbf{1}}_{\mathbf{2}} = \cot\vartheta\theta^{\mathbf{2}},\end{aligned} \tag{233}$$

$$\mathring{\omega}^{\mathbf{2}}_{\mathbf{21}} = \cot\vartheta\ ,\ \ \mathring{\omega}^{\mathbf{2}}_{\mathbf{11}} = 0. \tag{234}$$

Now, from Cartan' s second structure equation we have

$$\begin{aligned}\mathring{\mathcal{R}}^{\mathbf{1}}_{\mathbf{2}} &= d\mathring{\omega}^{\mathbf{1}}_{\mathbf{2}} + \mathring{\omega}^{\mathbf{1}}_{\mathbf{1}} \wedge \mathring{\omega}^{\mathbf{1}}_{\mathbf{1}} + \mathring{\omega}^{\mathbf{1}}_{2} \wedge \mathring{\omega}^{\mathbf{2}}_{\mathbf{2}} = d\mathring{\omega}^{\mathbf{1}}_{\mathbf{2}}\\ &= \theta^{\mathbf{1}} \wedge \theta^{\mathbf{2}}\end{aligned} \tag{235}$$

and[20]

$$\mathring{R}_{\mathbf{2}}{}^{\mathbf{1}}{}_{\mathbf{12}} = -\mathring{R}_{\mathbf{2}}{}^{\mathbf{1}}{}_{\mathbf{21}} = -\mathring{R}_{\mathbf{1}}{}^{\mathbf{2}}{}_{\mathbf{12}} = \mathring{R}_{\mathbf{1}}{}^{\mathbf{2}}{}_{\mathbf{21}} = \frac{1}{2}. \tag{236}$$

Now, let us calculate $\star\mathcal{R}^{\mathbf{1}}_{\mathbf{2}} \in \sec \bigwedge^0 T^*M$. We have

$$\begin{aligned}\star\mathcal{R}^{\mathbf{1}}_{\mathbf{2}} &= \widetilde{\mathcal{R}^{\mathbf{1}}_{\mathbf{2}}} \lrcorner \tau_{\mathbf{g}} = -(\theta^{\mathbf{1}} \wedge \theta^{\mathbf{2}}) \lrcorner (\theta^{\mathbf{1}} \wedge \theta^{\mathbf{2}}) = -\theta^{\mathbf{1}}\theta^{\mathbf{2}}\theta^{\mathbf{1}}\theta^{\mathbf{2}}\\ &= \left(\theta^{\mathbf{1}}\right)^2 \left(\theta^{\mathbf{2}}\right)^2 = 1\end{aligned} \tag{237}$$

and

$$\star\mathcal{R}^{\mathbf{1}}_{\mathbf{a}} \wedge \theta^{\mathbf{a}} = \mathcal{R}^{\mathbf{1}}_{\mathbf{2}} \wedge \theta^{\mathbf{2}} = \theta^{\mathbf{2}} \neq 0. \tag{238}$$

Now, Evans equation implies that $\star\mathcal{R}^{\mathbf{1}}_{\mathbf{a}} \wedge \theta^{\mathbf{1}} = 0$ for a Levi-Civita connection and thus as promised we exhibit a counterexample to his wrong equation.

Remark 21 *We recall that the first Bianchi identity for* $(S^2, \mathbf{g}, \mathring{D})$, *i.e.,* $\mathbf{D}\mathcal{T}^{\mathbf{a}} = \mathcal{R}^{\mathbf{a}}_{\mathbf{b}} \wedge \theta^{\mathbf{b}} = 0$ *which translate in the orthonormal basis used above in* $\mathring{R}_{\mathbf{b}}{}^{\mathbf{a}}{}_{\mathbf{cd}} + \mathring{R}_{\mathbf{c}}{}^{\mathbf{a}}{}_{\mathbf{db}} + \mathring{R}_{\mathbf{d}}{}^{\mathbf{a}}{}_{\mathbf{bc}}. = 0$ *is rigorously valid. Indeed, we have*

$$\begin{aligned}\mathring{R}_{\mathbf{2}}{}^{\mathbf{1}}{}_{\mathbf{12}} + \mathring{R}_{\mathbf{1}}{}^{\mathbf{1}}{}_{\mathbf{21}} + \mathring{R}_{\mathbf{2}}{}^{\mathbf{1}}{}_{\mathbf{21}} &= \mathring{R}_{\mathbf{2}}{}^{\mathbf{1}}{}_{\mathbf{12}} - \mathring{R}_{\mathbf{2}}{}^{\mathbf{1}}{}_{\mathbf{12}} = 0,\\ \mathring{R}_{\mathbf{1}}{}^{\mathbf{2}}{}_{\mathbf{12}} + \mathring{R}_{\mathbf{1}}{}^{\mathbf{2}}{}_{\mathbf{21}} + \mathring{R}_{\mathbf{2}}{}^{\mathbf{2}}{}_{\mathbf{21}} &= \mathring{R}_{\mathbf{1}}{}^{\mathbf{2}}{}_{\mathbf{12}} - \mathring{R}_{\mathbf{1}}{}^{\mathbf{2}}{}_{\mathbf{12}} = 0.\end{aligned} \tag{239}$$

[20] Observe that with our definition of the Ricci tensor it results that $\mathring{R} = \mathring{R}^{\mathbf{1}}_{\mathbf{1}} + \mathring{R}^{\mathbf{2}}_{\mathbf{2}} = -1$.

15.2 The Teleparallel Geometry of $(\mathring{S}^2, \mathbf{g}, D)$

Consider the manifold $\mathring{S}^2 = \{S^2 \backslash \text{north pole}\} \subset \mathbb{R}^3$, it is an sphere of unitary radius excluding the north pole. Let $\boldsymbol{g} \in \sec T_2^0 \mathring{S}^2$ be the standard Riemann metric field for $\mathring{S}^2$ (Eq.(226)). Now, consider besides the Levi-Civita connection another one, D, here called the Nunes (or navigator [17]) connection[21]. It is defined by the following parallel transport rule: a vector is parallel transported along a curve, if at any $x \in \mathring{S}^2$ the angle between the vector and the vector tangent to the latitude line passing through that point is constant during the transport (see Figure 1)

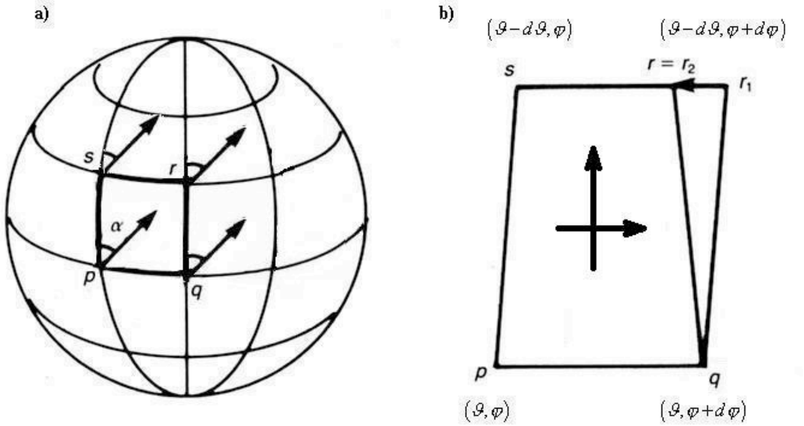

Figure 1: Geometrical Characterization of the Nunes Connection.

As before $(x^1, x^2) = (\vartheta, \varphi)$ $0 < \vartheta < \pi$, $0 < \varphi < 2\pi$, denote the standard spherical coordinates of a $\mathring{S}^2$ of unitary radius, which covers $U = \{\mathring{S}^2 - l\}$, where l is the curve joining the north and south poles.

Now, it is obvious from what has been said above that our connection is characterized by

$$D_{\mathbf{e_j}} \mathbf{e_i} = 0. \tag{240}$$

[21] See some historical details in [22].

Then taking into account the definition of the curvature tensor we have

$$\mathbf{R}(\mathbf{e_k}, \theta^{\mathbf{a}}, \mathbf{e_i}, \mathbf{e_j}) = \theta^{\mathbf{a}} \left(\left[D_{\mathbf{e_i}} D_{\mathbf{e_j}} - D_{\mathbf{e_j}} D_{\mathbf{e_i}} - D^{c}_{[\mathbf{e_i}, \mathbf{e_j}]} \right] \mathbf{e_k} \right) = 0. \quad (241)$$

Also, taking into account the definition of the torsion operation we have

$$\begin{aligned} \tau(\mathbf{e_i}, \mathbf{e_j}) &= T^{\mathbf{k}}_{\mathbf{ij}} \mathbf{e_k} = D_{\mathbf{e_j}} \mathbf{e_i} - D_{\mathbf{e_i}} \mathbf{e_j} - [\mathbf{e_i}, \mathbf{e_j}] \\ &= [\mathbf{e_i}, \mathbf{e_j}] = c^{\mathbf{k}}_{\mathbf{ij}} \mathbf{e_k}, \end{aligned} \quad (242)$$

$$T^{\mathbf{2}}_{\mathbf{21}} = -T^{\mathbf{2}}_{\mathbf{12}} = \cot\vartheta \;, \;\; T^{\mathbf{1}}_{\mathbf{21}} = -T^{\mathbf{1}}_{\mathbf{12}} = 0. \quad (243)$$

It follows that the unique non null torsion 2-form is:

$$\mathcal{T}^{\mathbf{2}} = -\cot\vartheta \theta^{\mathbf{1}} \wedge \theta^{\mathbf{2}}.$$

If you still need more details, concerning this last result, consider Figure 1(b) which shows the standard parametrization of the points p, q, r, s in terms of the spherical coordinates introduced above [17]. According to the geometrical meaning of torsion, we determine its value at a given point by calculating the difference between the (infinitesimal)[22] vectors pr_1 and pr_2 determined as follows. If we transport the vector pq along ps we get the vector $\vec{v} = sr_1$ such that $|\mathbf{g}(\vec{v}, \vec{v})|^{\frac{1}{2}} = \sin\vartheta \triangle\varphi$. On the other hand, if we transport the vector ps along pr we get the vector $qr_2 = qr$. Let $\vec{w} = sr$. Then,

$$|\mathbf{g}(\vec{w}, \vec{w})|^{\frac{1}{2}} = \sin(\vartheta - \triangle\vartheta)\triangle\varphi \simeq \sin\vartheta\triangle\varphi - \cos\vartheta\triangle\vartheta\triangle\varphi, \quad (244)$$

Also,

$$\vec{u} = r_1 r_2 = -u(\frac{1}{\sin\vartheta}\boldsymbol{\partial}_2) \;, \quad u = |\mathbf{g}(\vec{u}, \vec{u})| = \cos\vartheta\triangle\vartheta\triangle\varphi. \quad (245)$$

[22] This wording, of course, means that this vectors are identified as elements of the appropriate tangent spaces.

Then, the connection D of the structure $(\mathring{S}^2, \boldsymbol{g}, D)$ has a non null torsion tensor Θ. Indeed, the component of $\vec{u} = r_1 r_2$ in the direction $\boldsymbol{\partial}_2$ is precisely $T^{\varphi}_{\vartheta\varphi}\triangle\vartheta\triangle\varphi$. So, we get (recalling that $D_{\boldsymbol{\partial}_j}\boldsymbol{\partial}_i = \Gamma^k_{ji}\boldsymbol{\partial}_k$)

$$T^{\varphi}_{\vartheta\varphi} = \left(\Gamma^{\varphi}_{\vartheta\varphi} - \Gamma^{\varphi}_{\varphi\vartheta}\right) = -\cot\vartheta. \tag{246}$$

Exercise 22 *Show that D is metrical compatible, i.e., $D\boldsymbol{g} = 0$.*

Solution:

$$\begin{aligned} 0 = \quad & D_{\mathbf{e_c}}\boldsymbol{g}(\mathbf{e_i}, \mathbf{e_j}) = (D_{\mathbf{e_c}}\boldsymbol{g})(\mathbf{e_i}, \mathbf{e_j}) + \boldsymbol{g}(D_{\mathbf{e_c}}\mathbf{e_i}, \mathbf{e_j}) + \boldsymbol{g}(\mathbf{e_i}, D_{\mathbf{e_c}}\mathbf{e_j}) \\ & = (D_{\mathbf{e_c}}\boldsymbol{g})(\mathbf{e_i}, \mathbf{e_j}) \end{aligned} \tag{247}$$

Remark 23 *Our counterexamples that involve the parallel transport rules defined by a Levi-Civita connection and a teleparallel connection in $\mathring{S}^2$ show clearly that we cannot mislead the Riemann curvature tensor of a connection defined in a given manifold with the fact that the manifold may be bend as a surface in an Euclidean manifold where it is embedded. Neglecting this fact may generate a lot of wishful thinking.*

16 Conclusions

In this paper after recalling the main definitions and a collection of tricks of the trade concerning the calculus of differential forms on the Cartan, Hodge and Clifford bundles over a Riemannian or Riemann-Cartan space or a Lorentzian or Riemann-Cartan spacetime we solved with details several exercises involving different grades of difficult and which we believe, may be of some utility for pedestrians and even for experts on the subject. In particular we found using technology of the Clifford bundle formalism the correct equation for $\mathbf{D}\star\mathcal{T}^{\mathbf{a}}$. We show that the result found by Dr. Evans [10], "$\mathbf{D}\star\mathcal{T}^{\mathbf{a}} = \star\mathcal{R}^{\mathbf{a}}_{\mathbf{b}}\wedge\mathcal{T}^{\mathbf{b}}$" because it contradicts the right formula we found. Besides that, the wrong formula is also contradicted by two simple counterexamples that we exhibited in Section 15 . The last sentence before the conclusions is a crucial remark, which each one seeking truth must always keep in mind: do not confuse the Riemann curvature tensor[23] of a connection defined in a given manifold with the fact that the manifold may be bend as a surface in an Euclidean manifold where it is embedded.

[23] The remark applies also to the torsion of a connection.

We end the paper with a necessary explanation. An attentive reader may ask: Why write a bigger paper as the present one to show wrong a result not yet published in a scientific journal? The justification is that Dr. Evans maintain a site on his (so called) "ECE theory" which is read by thousand of people that thus are being continually mislead, thinking that its author is creating a new Mathematics and a new Physics. Besides that, due to the low Mathematical level of many referees, Dr. Evans from time to time succeed in publishing his papers in SCI journals, as the recent ones., [8, 9]. In the past we already showed that several published papers by Dr. Evans and colleagues contain serious flaws (see, e.g., [5, 21]) and recently some other authors spent time writing papers to correct Mr. Evans claims (see, e.g.,[1, 2, 3, 14, 15, 28]) It is our hope that our effort and of the ones by those authors just quoted serve to counterbalance Dr. Evans influence on a general public[24] which being anxious for novelties may be eventually mislead by people that claim among other things to *know* [6, 7, 8, 9] how to project devices to withdraw energy from the vacuum.

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[24] And we hope also on many scientists, see a partial list in [6, 7]!

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