Electromagnetic knots and the magnetic flux in superconductors

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1 Introduction

This contribution summarizes briefly a topological model of electromagnetism proposed by Antonio F. Rañada in 1992 [1], which is locally equivalent to Maxwell's standard theory but implies some topological quantization conditions with physical meaning [2, 3, 4]. The model makes use of two fundamental complex scalar fields, the level curves of which are the magnetic and electric lines, respectively.

In Section 2, we will see how from the study of the magnetic lines an electromagnetic field appears that satisfies two of the Maxwell equations, and how topology can be included easily in the game. In Section 3, magnetic knots are analyzed, including the topological meaning of the magnetic helicity. Section 4 is devoted to the definition and main properties of electromagnetic knots in vacuum. The topological quantization of the electromagnetic helicity in the model of electromagnetic knots is studied in Section 5. The topological quantization of the electric charge is summarized in Section 6. Section 7 is devoted to the quantization of the magnetic flux in an infinite solenoid. In Section 8 some conclusions are drawn.

2 Magnetic lines

Consider a magnetic field $\mathbf{B}(\mathbf{r}, t)$ defined in every point $\mathbf{r}$ of the three–dimensional space at every time t. Such magnetic field can be given in terms of a vector potential but, alternatively, it can be defined through the magnetic lines at every time. These lines are the solutions of the dynamical system given by the equations

$$\mathbf{B}\left(\mathbf{r}(\tau)\right) = \frac{d\mathbf{r}(\tau)}{d\tau}, \tag{1}$$

at every time. In this expression, τ is the arc-length of the curve. If the dynamical system (1) is integrable, there should be two conserved quantities, namely $\phi_1(\mathbf{r},t)$ and $\phi_2(\mathbf{r},t)$. Then $\mathbf{B}$ can be written as

$$\mathbf{B}(\mathbf{r},t) = f(\mathbf{r},t)\,\nabla\phi_1 \times \nabla\phi_2, \tag{2}$$

f being any function of space and time. This magnetic field has to satisfy the condition $\nabla\cdot\mathbf{B} = 0$. It imposes constraints in the form of the function f. In particular, f has to be function of ϕ_1 and ϕ_2 only,

$$\mathbf{B}(\mathbf{r},t) = f(\phi_1,\phi_2)\,\nabla\phi_1 \times \nabla\phi_2. \tag{3}$$

Defining a complex scalar field ϕ as

$$\phi = \phi_1 + i\,\phi_2, \tag{4}$$

equation (3) transforms into

$$\mathbf{B}(\mathbf{r},t) = f(\phi,\bar{\phi})\,\nabla\bar{\phi} \times \nabla\phi. \tag{5}$$

We will use natural units, in which

$$\hbar = c = \varepsilon_0 = \mu_0 = 1. \tag{6}$$

Covariance imposes that the electric field associated to the magnetic field (5) can be written, in these units, as

$$\mathbf{E}(\mathbf{r},t) = -f(\phi,\bar{\phi})\left(\frac{\partial\bar{\phi}}{\partial t}\nabla\phi - \frac{\partial\phi}{\partial t}\nabla\bar{\phi}\right). \tag{7}$$

Consequently, from the idea of magnetic lines an electromagnetic field has appeared naturally, given by the expressions (5) and (7). This electromagnetic field has some interesting properties:

1. The magnetic lines are intersections of the surfaces $\mathrm{Re}\,\phi = \mathrm{Re}\,\phi_0$ and $\mathrm{Im}\,\phi = \mathrm{Im}\,\phi_0$, where ϕ_0 is a complex number that labels the magnetic surfaces. Alternatively, the magnetic lines coincide with the level curves of the complex scalar field ϕ.

2. The magnetic and electric fields are mutually orthogonal. These kind of fields are called radiation or singular fields.

3. The electromagnetic field defined through (5) and (7) automatically satisfy two of Maxwell equations, namely

$$\nabla \cdot \mathbf{B} = 0, \tag{8}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \tag{9}$$

At this point, two things lack in order to obtain solutions of the Maxwell equations. First, to impose contour conditions on the scalar field ϕ. These conditions will allow us to study the topology of the magnetic and electric lines. Second, to satisfy the other two Maxwell equations (both in vacuum or in a space with charges and currents).

Topology is included into the set of electromagnetic fields given by equations (5) and (7) through contour conditions:

1. To impose the electric and magnetic fields to be zero at infinity, we take the scalar field ϕ to have only one value at infinity (the space R^3 is compactified).

2. We assume that the image of the scalar field is the compactified complex plane.

With these conditions, the scalar field ϕ can be interpreted, at every time, as a map

$$\phi : S^3 \to S^2. \tag{10}$$

These maps can be classified into homotopy classes, each one characterized by a topological integer number: the Hopf index (equal to the linking number of any two level curves of the map). The Hopf index of ϕ can be written as the integral

$$H(\phi) = \int_{S^3} \omega \wedge d\omega = \text{integer}, \tag{11}$$

where $d\omega$ is the pull-back by the map ϕ of the area two-form in S^2. The Hopf index will allow us to select the function f in such way that the topological properties of the electromagnetic field given by (5) and (7) will become completely clear.

3 Magnetic knots

The Hopf index (11) gives a tool to study the topology of the magnetic lines. So the function f in (5) and (7) is chosen as

$$f = -\frac{\sqrt{a}}{2\pi i(1+\bar{\phi}\phi)^2}, \tag{12}$$

a being a constant with dimensions of action times velocity, and a pure number in natural units. With this function f, the magnetic and electric fields (5) and (7) take the form

$$\mathbf{B}(\mathbf{r},t) = \frac{\sqrt{a}}{2\pi i(1+\bar{\phi}\phi)^2}\,\nabla\phi\times\nabla\bar{\phi}, \tag{13}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{\sqrt{a}}{2\pi i(1+\bar{\phi}\phi)^2}\,\left(\frac{\partial\bar{\phi}}{\partial t}\nabla\phi - \frac{\partial\phi}{\partial t}\nabla\bar{\phi}\right). \tag{14}$$

With these expressions, the magnetic helicity h_m of this field satisfy that it is equal to the constant a times the Hopf index (11) of the map ϕ,

$$h_m = \int_{R^3} \mathbf{A}\cdot\mathbf{B}\, d^3r = a\, H(\phi), \tag{15}$$

where $\mathbf{A}$ is a vector potential for the magnetic field, i. e. $\mathbf{B} = \nabla\times\mathbf{A}$.

The fields (13) and (14) are called magnetic knots. In these knots, the level curves of ϕ coincide with the magnetic lines, and the magnetic helicity is topologically quantized,

$$h_m = a\, n, \tag{16}$$

where n is the linking number of any two magnetic lines. Magnetic knots have to satisfy two of the Maxwell equations, leading to applications in plasma physics, models of ball lightning, etc. The magnetic helicity in these applications will be a constant of motion, and will have the topological meaning of linking number of magnetic lines, as long as the condition of orthogonality between the electric and the magnetic field is satisfied at every time. They can be used to describe locally every situation in which the magnetic helicity is conserved.

4 Electromagnetic knots

In vacuum, where no charges or currents are present, the electric field is solenoidal. We can then describe the electric lines as the level curves of another complex scalar field $\theta : S^3 \to S^2$. With both maps ϕ and θ, we can define an electromagnetic field in vacuum as

$$\begin{aligned}\mathbf{B}(\mathbf{r},t) &= \frac{\sqrt{a}}{2\pi i(1+\bar{\phi}\phi)^2}\,\nabla\phi\times\nabla\bar{\phi} \\ &= \frac{\sqrt{a}}{2\pi i(1+\bar{\theta}\theta)^2}\,\left(\frac{\partial\bar{\theta}}{\partial t}\nabla\theta - \frac{\partial\theta}{\partial t}\nabla\bar{\theta}\right),\end{aligned} \tag{17}$$

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \frac{\sqrt{a}}{2\pi i(1+\bar{\phi}\phi)^2}\,\left(\frac{\partial\bar{\phi}}{\partial t}\nabla\phi - \frac{\partial\phi}{\partial t}\nabla\bar{\phi}\right) \\ &= \frac{\sqrt{a}}{2\pi i(1+\bar{\theta}\theta)^2}\,\nabla\bar{\theta}\times\nabla\theta.\end{aligned} \tag{18}$$

Following the same principles as in the previous section, the magnetic and electric helicities are then conserved, being equal to the constant a times the Hopf indices of the maps ϕ and θ respectively,

$$h_m = \int_{R^3} \mathbf{A}\cdot\mathbf{B}\, d^3r = a\, H(\phi), \tag{19}$$

$$h_e = \int_{R^3} \mathbf{C}\cdot\mathbf{E}\, d^3r = a\, H(\theta), \tag{20}$$

where $\mathbf{A}$ and $\mathbf{C}$ are magnetic and electric vector potentials in vacuum, that satisfy

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{\partial\mathbf{C}}{\partial t}, \tag{21}$$

$$\mathbf{E} = \nabla\times\mathbf{C} = -\frac{\partial\mathbf{A}}{\partial t}, \tag{22}$$

Electromagnetic fields in vacuum defined through equations (17) and (18) are called electromagnetic knots. The following properties are satisfied by these fields:

1. They are solutions of Maxwell equations in vacuum in which the magnetic lines are the level curves of a map ϕ between the three-sphere and the two-sphere, and the electric lines are the level curves of another map θ.

2. They can be written as in (17)–(18). If these equations hold, they satisfy automatically all the Maxwell equations.

3. The electric and magnetic helicities of an electromagnetic knot are conserved quantities and they are proportional to the linking numbers of the magnetic and the electric lines, respectively.

4. The evolution in time of an electromagnetic knot imposes that the magnetic helicity is always equal to the electric helicity

$$h_m = h_e = a\, n. \tag{23}$$

5. Every solution of Maxwell equations in vacuum can be written locally as the sum of two electromagnetic knots. This means that a topological model of electromagnetism based on electromagnetic knots can be constructed, completely equivalent to Maxwell standard theory from the local point of view, but with topological quantization conditions.

In the following, we will study some of these topological quantization conditions.

5 Topological quantization of the electromagnetic helicity

The electromagnetic helicity is the sum of the magnetic and electric helicities

$$h = h_m + h_e. \tag{24}$$

Writing the electromagnetic field in terms of Fourier components, and comparing the result with the quantum expressions for the particle numbers, it can be proved that the electromagnetic helicity is proportional to the classical limit of the difference between right- and left-handed photons. Concretely, we have

$$h = 2\hbar c \left(N_R - N_L \right). \tag{25}$$

By the other way, for the electromagnetic knots, since the magnetic and electric helicities have to be equal, equation (23) follows in

$$h = 2an. \tag{26}$$

Comparing expressions (25) and (26), we arrive at the topological quantization of the electromagnetic helicity in the model of electromagnetic knots: the classical limit of the difference between right- and left-handed photons is equal to the linking number of any two magnetic lines (or electric lines),

$$(N_R - N_L) = n, \tag{27}$$

if the constant a of the model is set as

$$a = \hbar c, \tag{28}$$

or $a = 1$ in natural units. Note that equation (27) links two meanings of the word helicity: the particle meaning $N_R - N_L$ and the topological meaning n.

6 Topological quantization of the electric charge

Experimentally, it is found that all the isolated charges satisfy

$$Q = n\, e, \quad e = 0.3028\ldots \tag{29}$$

where e is written in natural units.

For the electromagnetic knots, a point charge is a point in which the electric (Coulomb) field is not well defined (it is infinite). This means that the associated complex scalar field can be interpreted, at every time, as a map

$$\theta : S^2 \times R \to S^2. \tag{30}$$

The degree of a regular map $\theta : S^2 \to S^2$ is an integer number n that can be written as

$$\deg(\theta) = \int_{S^2} \frac{d\bar{\theta} \wedge d\theta}{2\pi i (1 + \bar{\theta}\theta)^2} = n, \tag{31}$$

By the other way, if an electric field $\mathbf{E}$ is created by an isolated point charge Q at rest, the Gauss law for this field, written in terms of the complex scalar field θ, gives

$$\int_S \mathbf{E} \cdot d\mathbf{a} = \sqrt{a} \int_S \frac{\nabla\bar{\theta} \times \nabla\theta}{2\pi i (1 + \bar{\theta}\theta)^2} \cdot d\mathbf{a} = Q. \tag{32}$$

As can be seen, there is a close similarity between the degree of a map θ in S^2 and the Gauss law for a Coulomb field in the model of electromagnetic knots. Comparing equations (31) and (32), we can conclude that, for the model of electromagnetic knots,

$$Q = \sqrt{a}\, n, \tag{33}$$

so we obtain a topological quantization of the electric (or magnetic) charge in the model of electromagnetic knots: all the isolated charges in this model satisfy

$$Q = nq_0, \tag{34}$$

in which the fundamental charge q_0 is given by $q_0 = \sqrt{a}$, or $q_0 = 1$ in natural units.

7 The magnetic flux of a superconducting ring

It is found that the magnetic flux of a superconducting ring satisfies, in the experiments, the condition

$$\Phi_m = \frac{n}{2}\, g_0, \tag{35}$$

where n is an integer, and $g_0 = 2\pi/e$ is the numerical value of the Dirac's monopole. We will see that, in the case of the topological model of electromagnetism based on electromagnetic knots, this relation is the same, but the fundamental value of the monopole is different, being equal to $\sqrt{a}$.

We will study the case of an infinite solenoid (the case of a finite one is similar). Let us consider an infinite perfect solenoid around the z-axis (perfect means that no flux escapes through the coils). The electric field is zero. The magnetic field vanishes outside and is constant inside the ring, so that

$$\mathbf{B} = B\, \mathbf{u}_z. \tag{36}$$

In the model of electromagnetic knots, this magnetic field can be written as

$$\mathbf{B} = \frac{\sqrt{a}}{2\pi i (1+\bar{\phi}\phi)^2}\, \nabla\phi \times \nabla\bar{\phi} = \sqrt{a}\, \nabla p \times \nabla q, \tag{37}$$

in which ϕ is a complex scalar field, and p and q are defined as

$$p = \frac{1}{1+\bar{\phi}\phi}\ ,\ q = \frac{1}{2\pi}\, \mathrm{ArcTan}\left(\frac{\mathrm{Im}\phi}{\mathrm{Re}\phi}\right), \tag{38}$$

so that $0 \leq p \leq 1$ and q is a phase function.

In the case of the magnetic field created by the infinite solenoid, given the configuration of the magnetic lines (36), we require ϕ to be regular for the induced map $\phi : S^2 \to S^2$ given by $\phi(x,y) \in C \cup \{\infty\}$. Since the magnetic field vanishes outside the solenoid, p and q have to satisfy one of three possibilities:

1. q is a phase function of p outside the solenoid.
2. q is constant outside the solenoid.
3. p is constant outside the solenoid.

In the first case, where $q = f(p)$ outside the solenoid, we can perform a canonical transformation that does not affect the value of the magnetic field,

$$q \to q - f(p). \tag{39}$$

The new expression of ϕ is real outside the solenoid, but complex inside it. Consequently, the magnetic flux across a section of the solenoid is

$$\Phi_m = \sqrt{a}\,\mathrm{Area}\,\phi(S), \tag{40}$$

where S is any surface that completely cuts the solenoid and is bordered by a circuit outside it. In fact, this area must necessarily be $n/2$, n integer, because any curve contained in a great circle of a sphere encircles a integral multiple of semi-spheres. Or, considering the stereographic projection, if the curve is contained in the real axis it encircles an integral number of semi-planes. Consequently, in the first case, the magnetic flux is

$$\Phi_m = \sqrt{a}\,\frac{n}{2}. \tag{41}$$

In the second case, if q is constant outside the solenoid, the approach and situation is similar to that of the first case. It gives the same flux quantization (41).

In the third case, $p = p_0$ outside the solenoid, where p_0 is a constant, let us take, for the scalar field outside the solenoid,

$$\phi = \sqrt{\frac{1-p_0}{p_0}}\,e^{2\pi i q(r,\alpha)}, \tag{42}$$

in which $r = \sqrt{x^2+y^2}$ and $\alpha = \mathrm{ArcTan}\,(y/x)$. Moreover, since q is a phase,

$$\int_0^{2\pi} \frac{\partial q}{\partial \alpha}\,d\alpha = \text{integer}. \tag{43}$$

In order for ϕ to be a regular map of the plane xy on the complete complex plane with q a non-constant function of α, either $p = 0$ outside the solenoid or $p = 1$ there. Otherwise ϕ would not be well defined at infinity in this plane. In both cases, equation (41) holds. Consequently, in the topological model of electromagnetism there is a topological quantization of the magnetic flux of a superconductor, given by the expression (41).

8 Conclusions

Electromagnetic knots are solutions of Maxwell equations in vacuum such that their magnetic lines and their electric lines are the level curves of two complex scalar fields. The linking numbers of any two magnetic lines and of any two electric lines are the same in an electromagnetic knot, and they are Hopf indices of the scalar fields. A model of electromagnetism based on electromagnetic knots is locally equivalent to Maxwell theory. However, the model of electromagnetic knots includes classical quantization conditions. Three of them are (in natural units)

$$N_R - N_L = \text{integer}, \tag{44}$$

$$Q = \text{integer}, \tag{45}$$

$$\Phi_m(\text{Superconductor}) = \frac{\text{integer}}{2}, \tag{46}$$

in which the integers have topological meaning.

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