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On the electromagnetism's invariance.

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ABSTRACT. We recall how the quantum theory explains the relativistic invariance of the Dirac theory, introducing $SL(2,\mathbb{C})$ which is a subset of the Pauli algebra. We study that algebra, which is also the Clifford algebra of the physical space. There is an homomorphism from the group Cl_3^* of the invertible elements of the algebra, into the group of the Lorentz dilations. The kernel of the homomorphism is the chiral gauge group of the G. Lochak's monopole theory. The Dirac equation, and all the electromagnetism, even with magnetic monopoles, is invariant under Cl_3^*. We propose a second gauge invariance for the homogenous wave equation. The extended invariance is compatible with an oriented time and an oriented space.

1 - Relativistic invariance of the Dirac equation.

We start from the Dirac equation

$$[\gamma^\mu(\partial_\mu + iqA_\mu) + im]\psi = 0 \ ; \quad q := \frac{e}{\hbar c} \ ; \quad m := \frac{m_0 c}{\hbar} \tag{1.1}$$

where e is the negative electron's charge, m_0 the proper mass and A_μ are the covariant components of the electromagnetic vector. The study of the relativistic invariance is generally made in the Weyl's representation, with

$$\gamma^0 = \gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \ ; \ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ ; \ \psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \ ; \ \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \ ; \ \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$
$$\gamma^j = -\gamma_j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix} \ ; \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ ; \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ ; \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.2}$$

Using the shorter notation $\vec{A} = \sigma_1 A^1 + \sigma_2 A^2 + \sigma_3 A^3$ instead of $\vec{\sigma} \cdot \vec{A}$ and $\vec{\partial} = \sigma_1 \partial_1 + \sigma_2 \partial_2 + \sigma_3 \partial_3$ instead of $\vec{\sigma} \cdot \vec{\nabla}$, the Dirac equation is equivalent to the system

$$\begin{aligned}
&(\partial_0 + \vec{\partial})\xi + iq(A_0 - \vec{A})\xi + im\eta = 0 \\
&(\partial_0 - \vec{\partial})\eta + iq(A_0 + \vec{A})\eta + im\xi = 0
\end{aligned} \tag{1.3}$$

To get the relativistic invariance of (1.1), it is necessary to consider the set $SL(2, \mathbb{C})$ of the complex 2×2 matrices

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \; ; \quad 1 = \det(M) = \alpha\delta - \beta\gamma. \tag{1.4}$$

It is also necessary to associate to each event, with coordinates $(x^0,\ x^1,\ x^2,\ x^3)$, $x^0 = ct$, the matrix

$$x = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}. \tag{1.5}$$

The transformation R defined by

$$R : x \mapsto x' := M x M^\dagger \tag{1.6}$$

verifies

$$\begin{aligned}
\det(x') = ({x'}^0)^2 - ({x'}^1)^2 - ({x'}^2)^2 - ({x'}^3)^2 &= \det(M x M^\dagger) \\
= \det(M)\det(x)\det(M^\dagger) &= |\det(M)|^2 \det(x) = \det(x) \\
= (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 &
\end{aligned} \tag{1.7}$$

since $\det(M) = 1$. So R is a Lorentz transformation. It is well known [1] that R conserves the time and space orientation, and is an element of the restricted Lorentz group $\mathcal{L}_+^\uparrow$. If we consider

$$f : M \mapsto R \tag{1.8}$$

f is an homomorphism from $SL(2, \mathbb{C})$ into $\mathcal{L}_+^\uparrow$ and the kernel of f is $\{\pm I\}$. With

$$\begin{aligned}
&{x'}^\mu = R^\mu_\nu x^\nu \; ; \quad {\partial'}_\mu := \frac{\partial}{\partial {x'}^\mu} \; ; \quad \partial_\nu = R^\mu_\nu {\partial'}_\mu \\
&\widehat{M} := \begin{pmatrix} \delta^* & -\gamma^* \\ -\beta^* & \alpha^* \end{pmatrix} \; ; \; N := \begin{pmatrix} M & 0 \\ 0 & \widehat{M} \end{pmatrix}
\end{aligned} \tag{1.9}$$

we get the general and non trivial relation

$$R^\mu_\nu \gamma^\nu = N^{-1}\gamma^\mu N. \tag{1.10}$$

The form invariance of the Dirac equation comes from

$$\psi' = N\psi \ ; \quad \psi = N^{-1}\psi' \tag{1.11}$$

that gives

$$\begin{aligned}
0 &= [\gamma^\nu(\partial_\nu + iqA_\nu) + im]\psi \\
&= [\gamma^\nu R^\mu_\nu(\partial'_\mu + iqA'_\mu) + im]N^{-1}\psi' \\
&= [N^{-1}\gamma^\mu N(\partial'_\mu + iqA'_\mu) + im]N^{-1}\psi' \\
&= N^{-1}[\gamma^\mu(\partial'_\mu + iqA'_\mu) + im]\psi'
\end{aligned} \tag{1.12}$$

We can notice that with (1.2), (1.9) and (1.11) we get, for the Weyl's spinors ξ and η

$$\xi' = M\xi \ ; \quad \eta' = \widehat{M}\eta. \tag{1.13}$$

2 - Space algebra

When we use the 2×2 complex matrices in (1.5), we actually work with the Clifford algebra Cl_3 of the physical space. Since that real algebra can't be ignored in the quantum theory, there are only advantages to understand and to use that tool.

The general element of the space algebra Cl_3 reads

$$u = s + \vec{v} + i\vec{w} + ip \tag{2.1}$$

where s is a scalar (real number), $\vec{v}$ is a vector, with three real components. $i\vec{w}$ is a pseudo-vector, $\vec{w}$ is an axial vector, and ip is a pseudo-scalar. As $i^2 = -1$, Cl_3 is a generalization of the complex field. If $(\sigma_1, \sigma_2, \sigma_3)$ is an orthonormal basis of the physical space, that is

$$\sigma_j \cdot \sigma_k = 0 \ , \ j \neq k \ ; \quad \sigma_j^2 = 1 \tag{2.2}$$

we can write any vector $\vec{v}$ as

$$\vec{v} = v^1\sigma_1 + v^2\sigma_2 + v^3\sigma_3 \tag{2.3}$$

If we use the Pauli representation (1.2) for the σ_j, and if we identify scalars and scalar matrices, the sum and the product of two terms in the space algebra is exactly the sum and the matrix product. This product has a symmetric and an anti-symmetric part, for instance

$$\vec{v}\vec{w} = \vec{v} \cdot \vec{w} + i\ \vec{v} \times \vec{w} \tag{2.4}$$

where $\vec{v} \cdot \vec{w}$ is the scalar product and $\vec{v} \times \vec{w}$ is the vectorial product. The space algebra uses the differential operator

$$\vec{\partial} := \sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3\ ;\ \partial_j := \frac{\partial}{\partial x^j} \tag{2.5}$$

which is known as the gradient when applied to a scalar

$$\text{grad}\quad s = \vec{\partial}s \tag{2.6}$$

and gives the divergence and the curl when applied to a vector

$$\vec{\partial}\vec{v} = \vec{\partial} \cdot \vec{v} + i\ \vec{\partial} \times \vec{v}\ ;\ \vec{\partial} \cdot \vec{v} = \ \text{div}\ \ \vec{v}\ ;\ \vec{\partial} \times \vec{v} = \ \text{curl}\ \ \vec{v}. \tag{2.7}$$

The matrix x of (1.5) was named paravector by Baylis [2]. It is the sum of the scalar x^0 and the vector $\vec{x} = x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3$,

$$x = x^0 + \vec{x} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}. \tag{2.8}$$

And we recover (1.5). With the Pauli representation (1.2) the differential operator $\vec{\partial}$ reads

$$\vec{\partial} = \begin{pmatrix} \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & -\partial_3 \end{pmatrix}. \tag{2.9}$$

The space algebra Cl_3 is greater than the complex field and we use here three conjugations $u^\dagger$, $\widehat{u}$, $\overline{u}$:

$$u^\dagger = s + \vec{v} - i\vec{w} - ip \tag{2.10}$$

$$\widehat{u} = s - \vec{v} + i\vec{w} - ip \tag{2.11}$$

$$\overline{u} = s - \vec{v} - i\vec{w} + ip \tag{2.12}$$

and we get

$$(uv)^\dagger = v^\dagger u^\dagger\ ;\ \widehat{uv} = \widehat{u}\widehat{v}\ ;\ \overline{uv} = \overline{v}\ \overline{u}$$

$$\overline{u} = \widehat{u}^\dagger = \widehat{u^\dagger}\ ;\quad \widehat{u} = \overline{u}^\dagger = \overline{u^\dagger}\ ;\quad u^\dagger = \widehat{\overline{u}} = \overline{\widehat{u}} \tag{2.13}$$

With the Pauli representation the conjugation $\widehat{\ }$ verifies also (1.9) and $u^\dagger$ is the adjoint matrix. If v is a space-time vector verifying

$$v = \begin{pmatrix} v^0 + v^3 & v^1 - iv^2 \\ v^1 + iv^2 & v^0 - v^3 \end{pmatrix} = v^0 + \vec{v}\ ; \quad \vec{v} = v^j \sigma_j \tag{2.14}$$

and if we identify scalars and scalar matrices, we get

$$v\overline{v} = \det(v) = v \cdot v = (v^0)^2 - (v^1)^2 - (v^2)^2 - (v^3)^2 \tag{2.15}$$

3 - Invariance under Cl_3^*

When we study the relativistic invariance of the Dirac equation we use a 2×2 matrix M with $\det(M) = 1$. But we can also use any 2×2 matrix M and we define again the R transformation by

$$R : x \mapsto x' = MxM^\dagger\ ; \quad \det(M) = re^{i\theta} \tag{3.1}$$

Then we get

$$\begin{aligned} \det(x') &= (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2 = \det(MxM^\dagger) \\ &= \det(M)\det(x)\det(M^\dagger) = |\det(M)|^2 \det(x) = r^2 \det(x) \\ &= r^2[(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2] \end{aligned} \tag{3.2}$$

R is now a transformation which multiplies each scalar product by r^2 and each length by r. So we call R a "Lorentz dilation" and r is called the ratio of that dilation. If R^μ_ν is the 4×4 real matrix of the Lorentz dilation R, that is

$$x'^\mu = R^\mu_\nu x^\nu \tag{3.3}$$

we get, for any $M \neq 0$, $R^0_0 > 0$. Consequently R conserves the time's arrow. We get also, for any M,

$$\det(R^\mu_\nu) = |\det(M)|^4 = r^4. \tag{3.4}$$

And if $r \neq 0$ we get $\det(R^\mu_\nu) > 0$: R is invertible and conserves the space-time orientation. Consequently R conserves the time and space orientation. Now we consider

$$f : M \mapsto R \tag{3.5}$$

f is an homomorphism from the space algebra Cl_3 (which is also the Pauli algebra) into the set of the Lorentz dilation : If $R = f(M)$ and $R' = f(M')$ with

$$R : x \mapsto x' = MxM^\dagger \ ; \quad R' : x' \mapsto x'' = M'x'M'^\dagger \tag{3.6}$$

we get

$$\begin{aligned} R' \circ R : x \mapsto x'' = M'(MxM^\dagger)M'^\dagger = (M'M)x(M'M)^\dagger \\ f(M') \circ f(M) = R' \circ R = f(M'M). \end{aligned} \tag{3.7}$$

The restriction of f to the set Cl_3^* of the invertible M, with $r > 0$, is a group homomorphism from $(Cl_3^*, \times)$ into the group $(D^*, \circ)$, where D^* is the set of the dilations with a ratio $r > 0$. Moreover the kernel of the homomorphism is the set

$$\mathrm{Ker(f) = \{M \ / \ M = e^{i\frac{\theta}{2}}I\}} \tag{3.8}$$

That kernel is reduced to $\pm I$ if we reduce f to the case where $\det(M) = 1$. The two-valued representations of the quantum theory are a particular case, and if we don't restrict M, each Lorentz dilation $R = f(M)$ verifies also $R = f(e^{i\frac{\theta}{2}}M)$. That kernel is a $U(1)$ group. It is exactly the gauge group used by G. Lochak for the magnetic monopole's theory, and to explain that fact later, it will be necessary to use the Pauli algebra that is Cl_3.

The first equation (1.3) is equivalent to

$$(\partial_0 + \vec{\partial}^*)\xi^* - iq(A_0 - \vec{A}^*)\xi^* - im\eta^* = 0. \tag{3.9}$$

Multiplying by $-i\sigma_2$ by the left, and using

$$-i\sigma_2\vec{\partial}^* = \vec{\partial}i\sigma_2 \ ; \quad -i\sigma_2\vec{A}^* = \vec{A}i\sigma_2 \tag{3.10}$$

we get

$$(\partial_0 - \vec{\partial})(-i\sigma_2\xi^*) + iq(A_0 + \vec{A})(i\sigma_2\xi^*) + im(i\sigma_2\eta^*) = 0. \tag{3.11}$$

The two column matrices ξ and $-i\sigma_2\eta^*$ give one 2×2 matrix : We use

$$\phi = \sqrt{2}(\xi \quad -i\sigma_2\eta^*) \tag{3.12}$$

which gives

$$\widehat{\phi} = \sqrt{2}(\eta \quad -i\sigma_2\xi^*). \tag{3.13}$$

Consequently the Dirac equation (1.1) or the system (1.3) are equivalent to

$$(\partial_0 - \vec{\partial})\widehat{\phi} + iq(A_0 + \vec{A})\widehat{\phi}\sigma_3 + im\phi\sigma_3 = 0. \tag{3.14}$$

And with

$$\nabla := \partial_0 - \vec{\partial}\ ; \quad A := A^0 + \vec{A}\ ; \quad \sigma_{12} := \sigma_1\sigma_2 = i\sigma_3, \tag{3.15}$$

we get the Dirac equation, written in Cl_3 :

$$\nabla\widehat{\phi} + qA\widehat{\phi}\sigma_{12} + m\phi\sigma_{12} = 0 \tag{3.16}$$

The transformation law (1.13) for the spinors ξ and η, with (3.12), implies that, for a Lorentz rotation defined by a M matrix, we must take

$$\phi' = M\phi\ ; \quad \widehat{\phi}' = \widehat{M}\widehat{\phi}. \tag{3.17}$$

As a Lorentz rotation is a particular case of a Lorentz dilation, we must set the preceding relations for any M. With

$$\nabla = \sigma^\mu\partial_\mu\ ; \ \nabla' = \sigma^\mu\partial'_\mu\ ; \ \sigma_0 = \sigma^0 := I\ ; \ \sigma^j = -\sigma_j\ ; \ j = 1,2,3. \tag{3.18}$$

and with (3.1), we get, for any M, the relation

$$\nabla = \overline{M}\nabla'\widehat{M}. \tag{3.19}$$

The Dirac equation is invariant under the Lorentz dilation defined by (3.1) if we get

$$\nabla'\widehat{\phi}' + qA'\widehat{\phi}'\sigma_{12} + m'\phi'\sigma_{12} = 0 \tag{3.20}$$

The gauge invariance of the Dirac equation implies that A transforms as ∇ :

$$A = \overline{M}A'\widehat{M}. \tag{3.21}$$

Consequently (3.16) gives

$$\begin{aligned} 0 &= (\overline{M}\nabla'\widehat{M})\overline{\phi} + q(\overline{M}A'\widehat{M})\widehat{\phi}\sigma_{12} + m\phi\sigma_{12} \\ &= \overline{M}(\nabla'\widehat{\phi}' + qA'\widehat{\phi}'\sigma_{12}) + m\phi\sigma_{12} \\ &= \overline{M}(-m'\phi'\sigma_{12}) + m\phi\sigma_{12} \\ &= (-m'\overline{M}M + m)\phi\sigma_{12}. \end{aligned} \tag{3.22}$$

But $\overline{M}M = \det(M) = re^{i\theta}$, so the Dirac equation is invariant under a Lorentz dilation if and only if

$$m = re^{i\theta}m'. \tag{3.23}$$

As a Lorentz dilation R multiplies each space-time length by r, and as a proper mass is, with the Planck constant, the inverse of a space-time length, we can understand the relation $m = rm'$. The factor $e^{i\theta}$ is unexpected, but necessary to get the form invariance of the linear Dirac equation. That factor may be avoided if we use the non-linear equation [3]

$$\nabla\widehat{\phi} + qA\widehat{\phi}\sigma_{12} + me^{-i\beta}\phi\sigma_{12} = 0 \tag{3.24}$$

where β [4] is the Yvon-Takabayasi angle, verifying

$$\det(\phi) = \Omega_1 + i\Omega_2 = \rho e^{i\beta} \tag{3.25}$$

So we can say that the wave equation (3.24) is invariant under any Lorentz dilation coming from the M matrix such as

$$\begin{aligned} &x' = R(x) = MxM^\dagger \ ; \quad x'^\mu = R^\mu_\nu x^\nu \\ &\nabla = \sigma^\mu\partial_\mu = \overline{M}\nabla'\widehat{M} \ ; \quad \nabla' = \sigma^\mu\partial'_\mu \\ &\phi' = M\phi \ ; \quad \det(M) = re^{i\theta} \ ; \quad m = rm' \end{aligned} \tag{3.26}$$

We get also

$$\begin{aligned} \rho' e^{i\beta'} &= \det(\phi') = \det(M\phi) = \det(M)\det(\phi) = re^{i\theta}\rho e^{i\beta} = r\rho e^{i(\theta+\beta)} \\ \rho' &= r\rho \ ; \quad \beta' = \theta + \beta. \end{aligned} \tag{3.27}$$

which gives

$$m\rho = (rm')\rho = m'(r\rho) = m'\rho' \tag{3.28}$$

Consequently that is the product $m\rho$ which is invariant under the general transformation (3.26), not m and ρ separately.

There is no difference between the 2×2 matrix M giving a Lorentz dilation R and the 2×2 matrix ϕ of the electron's wave. ϕ is a function from space and time with value into the algebra Cl_3. Moreover there is no difference between the product $M'M$ in (3.7), which induces the product of two dilations, and the product $M\phi$ in (3.17), which gives the transformation of the ϕ wave under a Lorentz dilation. Therefore we can associate to ϕ the dilation $f(\phi)$, in each point of the space-time. $f(\phi)$

applies the local tangent space-time into the observer's space-time. If we call y the general element of the tangent space-time and $D = f(\phi)$:

$$y = y^\mu \sigma_\mu \ ; \quad x = x^\mu \sigma_\mu = D(y) = \phi y \phi^\dagger \tag{3.29}$$

we get also

$$x' = M x M^\dagger = M(\phi y \phi^\dagger) M^\dagger = (M\phi) y (M\phi)^\dagger = \phi' y {\phi'}^\dagger \tag{3.30}$$

Therefore we get $x = D(y)$ and $x' = D'(y)$ with $D' = R \circ D$ and the same y. The tangent space-time is intrinsic to the wave and independent from the observer.

4 - Invariance of the electromagnetism

The laws got by Louis de Broglie [5] for the electromagnetism with a photon are

$$\begin{aligned}
-\frac{1}{c}\frac{\partial \vec{H}}{\partial t} &= \text{ curl } \vec{E} \ ; \quad \text{div } \vec{H} = 0 \ ; \quad \vec{H} = \text{ curl } \vec{A} \\
\frac{1}{c}\frac{\partial \vec{E}}{\partial t} &= \text{ curl } \vec{H} + k_0^2 \vec{A} \ ; \quad \text{div } \vec{E} = -k_0^2 V \ ; \quad \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \text{ grad } V \\
\frac{1}{c}\frac{\partial V}{\partial t} &+ \text{ div } \vec{A} = 0
\end{aligned} \tag{4.1}$$

where $k_0 = \frac{m_0 c}{\hbar}$, m_0 being the very small proper mass of the photon. With

$$x^0 := ct \ ; \quad A^0 := V \ ; \quad A := A^0 + \vec{A} \ ; \quad F := \vec{E} + i\vec{H} \ ; \quad \widehat{\nabla} := \partial_0 + \vec{\partial} \tag{4.2}$$

the seven equations (4.1) are equivalent to

$$F = \nabla \widehat{A} \ ; \quad \widehat{\nabla} F = -k_0^2 \widehat{A} \tag{4.3}$$

These equations are invariant under the dilation $R = f(M)$ if we get

$$F' = \nabla' \widehat{A}' \ ; \quad \widehat{\nabla}' F' = -{k_0'}^2 \widehat{A}' \tag{4.4}$$

We know, after (3.21) and (3.26) that

$$\nabla = \overline{M} \nabla' \widehat{M} \ ; \quad A = \overline{M} A' \widehat{M} \ ; \quad k_0 = r k_0' \tag{4.5}$$

Consequently (4.3) gives

$$\begin{aligned}F &= \nabla\widehat{A} = (\overline{M}\nabla'\widehat{M})\widehat{\overline{M}A'\widehat{M}} = \overline{M}\nabla'\widehat{M\overline{M}}\widehat{A}'M\\ &= re^{-i\theta}\overline{M}(\nabla'\widehat{A}')M = re^{-i\theta}\overline{M}F'M\\ F &= re^{-i\theta}\overline{M}F'M\end{aligned} \tag{4.6}$$

which gives

$$\begin{aligned}\widehat{\nabla}F &= (M^\dagger\widehat{\nabla}'M)(re^{-i\theta}\overline{M}F'M) = M^\dagger\widehat{\nabla}'r^2F'M\\ &= r^2M^\dagger(\widehat{\nabla}'F')M = r^2M^\dagger(-{k_0'}^2\widehat{A}')M\\ &= -(rk_0')^2\widehat{\overline{M}A'\widehat{M}} = -k_0^2\widehat{A}\end{aligned} \tag{4.7}$$

We can say that the laws of the electromagnetism with photon are invariant under Cl_3^*, a greater group than the relativistic invariance group, if and only if the electromagnetic field transforms as

$$F = re^{-i\theta}\overline{M}F'M \tag{4.8}$$

We get a r factor, because the electromagnetic true field coming from the de Broglie's theory is not F but a tensor $\mathbf{F}$ verifying

$$F = k_0\mathbf{F}\ ;\quad \nabla\widehat{A} = k_0\mathbf{F}\ ;\quad \widehat{\nabla}\mathbf{F} = -k_0\widehat{A} \tag{4.9}$$

So the laws are invariant under Cl_3^* if

$$\mathbf{F} = e^{-i\theta}\overline{M}\mathbf{F}'M \tag{4.10}$$

The presence of the $e^{-i\theta}$ factor implies that $\mathbf{F}$ is invariant under the kernel of f, that is the set of the $M = e^{i\frac{\theta}{2}}$ which gives

$$\mathbf{F} = e^{-i\theta}e^{i\frac{\theta}{2}}\mathbf{F}'e^{i\frac{\theta}{2}} = \mathbf{F}' \tag{4.11}$$

The laws of the electromagnetism with electric and magnetic charges are [6]

$$\begin{aligned}&\vec{E} = -\text{ grad } V - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} + \text{ curl }\vec{B}\ ;\ \vec{H} = \text{ curl }\vec{A} + \text{ grad } W + \frac{1}{c}\frac{\partial\vec{B}}{\partial t}\\ &0 = \partial_\mu A^\mu = \frac{1}{c}\frac{\partial V}{\partial t} + \vec{\partial}\cdot\vec{A}\ ;\quad 0 = \partial_\mu B^\mu = \frac{1}{c}\frac{\partial W}{\partial t} + \vec{\partial}\cdot\vec{B}\\ &\text{curl }\vec{H} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j}\ ;\quad \text{div }\vec{E} = 4\pi\rho\\ &\text{curl }\vec{E} + \frac{1}{c}\frac{\partial\vec{H}}{\partial t} = \frac{4\pi}{c}\vec{k}\ ;\quad \text{div }\vec{H} = -4\pi\mu\end{aligned} \tag{4.12}$$

where $\vec{j}$ is the density of electric current, ρ is the electric charge density, $\vec{k}$ is the density of magnetic current, $\mu = k^0$ is the magnetic charge density, $W = B^0$ and $\vec{B}$ are the potential terms of Cabibbo-Ferrari (first found by Louis de Broglie). [1] The eight equations are resumed, in the space algebra, and with $B = B^0 + \vec{B}$, $k = k^0 + \vec{k}$, into

$$F = \nabla(\widehat{A + iB}) \; ; \quad \widehat{\nabla} F = \frac{4\pi}{c}(\widehat{j + ik}) \tag{4.13}$$

The existence of the magnetic monopoles simply corresponds to the replacement of A by $A + iB$ and the replacement of j by $j + ik$. As B is linked to A we will set, under a Lorentz dilation

$$B = \overline{M} B' M \tag{4.14}$$

and we get in (4.13)

$$\begin{aligned}
\frac{4\pi}{c}\widehat{j + ik} &= \widehat{\nabla} F = (M^\dagger \widehat{\nabla}' M)(re^{-i\theta}\overline{M} F' M) = M^\dagger \widehat{\nabla}' r^2 F' M \\
&= r^2 M^\dagger (\widehat{\nabla}' F') M = r^2 M^\dagger \frac{4\pi}{c}\widehat{j' + ik'} M \\
j &= r^2 \overline{M} j' \widehat{M} \; ; \quad k = r^2 \overline{M} k' \widehat{M}
\end{aligned} \tag{4.15}$$

There are r^2 terms which can be avoided if we use

$$F = k_0 \mathbf{F} \; ; \quad j = k_0^2 \mathbf{j} \; ; \quad k = k_0^2 \mathbf{k}. \tag{4.16}$$

The electromagnetic laws with electric and magnetic charges and currents read then

$$\nabla \widehat{A + iB} = k_0 \mathbf{F} \; ; \quad \widehat{\nabla} \mathbf{F} = \frac{4\pi k_0}{c} \widehat{\mathbf{j} + i\mathbf{k}} \tag{4.17}$$

and are invariant under Cl_3^* with (4.10) and if A, B, $\mathbf{j}$ and $\mathbf{k}$ transforms as ∇.

So it is possible that the electromagnetic laws are invariant under a greater group than the Lorentz group, with new involvements : on the true tensors of the theory, on the presence of mass terms in the laws. And we must distinguish the "contravariants vectors" transforming as x, and the "covariant vectors" transforming as ∇.

[1] We use a sign for the magnetic charge different from the G. Lochak's monopole, to avoid a minus sign in (4.13).

5 - Wave equation and invariance of the magnetic monopole

We start with the linear wave equation without mass term of G. Lochak [6] :

$$\begin{aligned}&[\frac{1}{c}\frac{\partial}{\partial t}-\vec{\sigma}\cdot\vec{\nabla}-i\frac{g}{\hbar c}(W+\vec{\sigma}\cdot\vec{B})]\eta=0\\&[\frac{1}{c}\frac{\partial}{\partial t}+\vec{\sigma}\cdot\vec{\nabla}+i\frac{g}{\hbar c}(W-\vec{\sigma}\cdot\vec{B})]\xi=0\end{aligned}\tag{5.1}$$

that are, with the preceding notations [2]

$$[\partial_0-\vec{\partial}-i\frac{g}{\hbar c}(B^0+\vec{B})]\eta=0\tag{5.2}$$

$$[\partial_0+\vec{\partial}+i\frac{g}{\hbar c}(B^0-\vec{B})]\xi=0\tag{5.3}$$

Conjugating (5.3) and multiplying by $-i\sigma_2$ by the left, we get

$$\begin{aligned}&-i\sigma_2[\partial_0+\vec{\partial}^*-i\frac{g}{\hbar c}(B^0-\vec{B}^*)]\xi^*=0\\&[\partial_0-\vec{\partial}-i\frac{g}{\hbar c}(B^0+\vec{B})](-i\sigma_2\xi^*)=0\end{aligned}\tag{5.4}$$

(5.2) and (5.4) are equivalent to one equation with ϕ :

$$(\nabla-\frac{ig}{\hbar c}B)\widehat{\phi}=0\tag{5.5}$$

The chiral gauge of G. Lochak reads here

$$B\mapsto B'=B-\nabla\varphi\ ;\quad \phi\mapsto\phi'=e^{i\frac{g}{\hbar c}\varphi}\phi\tag{5.6}$$

So the $i=\sigma_1\sigma_2\sigma_3$ that we get in the pseudo-scalar ip, or in $\vec{E}+i\vec{H}$, or in $e^{i\theta}$, or in $e^{i\beta}$, is the generator or the chiral gauge. And the kernel of f is actually the chiral gauge group.

The non-linear wave equation of G. Lochak :

$$\begin{aligned}&[\frac{1}{c}\frac{\partial}{\partial t}-\vec{\sigma}\cdot\vec{\nabla}-i\frac{g}{\hbar c}(W+\vec{\sigma}\cdot\vec{B})]\eta+i\frac{c}{\hbar}m(4|\eta^\dagger\xi|^2)(\xi^\dagger\eta)\xi=0\\&[\frac{1}{c}\frac{\partial}{\partial t}+\vec{\sigma}\cdot\vec{\nabla}+i\frac{g}{\hbar c}(W-\vec{\sigma}\cdot\vec{B})]\xi+i\frac{c}{\hbar}m(4|\eta^\dagger\xi|^2)(\eta^\dagger\xi)\eta=0\end{aligned}\tag{5.7}$$

[2] We have changed the sign of the magnetic charge, as in the preceding paragraph. And to get the usual Dirac equation (1.1), we have also exchanged ξ and η. Those differences are the only ones with G. Lochak's equations.

is equivalent to

$$(\nabla - \frac{ig}{\hbar c}B)\widehat{\phi} + \frac{c}{\hbar}m(\rho^2)\frac{\rho}{2}e^{-i\beta}\phi\sigma_{12} = 0 \tag{5.8}$$

That equation is homogenous if $m(\rho^2)\frac{\rho}{2} = m_0$, giving

$$(\nabla - \frac{ig}{\hbar c}B)\widehat{\phi} + me^{-i\beta}\phi\sigma_{12} = 0 \tag{5.9}$$

(5.9) is similar to (3.24) because we used (5.9) to build (3.24) [7]. We can notice that (5.5) is a particular case of (5.9) with $m = 0$. (5.9) is invariant under Cl_3^* with

$$\begin{aligned} &x \mapsto x' = MxM^\dagger \ ; \quad \det(M) = re^{i\theta} \ ; \quad m = rm' \\ &\phi \mapsto \phi' = M\phi \ ; \quad \nabla = \overline{M}\nabla'\widehat{M} \ ; \quad B = \overline{M}B'\widehat{M} \end{aligned} \tag{5.10}$$

which gives for any M :

$$(\nabla - \frac{ig}{\hbar c}B)\widehat{\phi} + me^{-i\beta}\phi\sigma_{12} = \overline{M}[(\nabla' - \frac{ig}{\hbar c}B')\widehat{\phi'} + m'e^{-i\beta'}\phi'\sigma_{12}]. \tag{5.11}$$

6 - A second gauge for the monopole's equation

If we compare (5.9) to the electromagnetic laws with magnetic monopoles, we see that (4.13) has two potential terms, A and B, whereas (5.9) has only a B term and the Dirac equation has only a A term. It is not possible to add a B term to the Dirac equation, but it is possible to add a A term to (5.9), giving

$$[\nabla - \frac{g}{\hbar c}(A + iB)]\widehat{\phi} + me^{-i\beta}\phi\sigma_{12} = 0 \tag{6.1}$$

which is invariant under the gauge transformation (5.6) and under

$$A \mapsto A' = A + \nabla\varphi \ ; \quad \phi \mapsto \phi' = e^{\frac{g}{\hbar c}\varphi}\phi \tag{6.2}$$

That equation is compatible with the law (4.17) of the electromagnetism with magnetic charges and currents. A wave following that equation will see both the potential created by an electric charge and the potential created by a magnetic monopole. But the study of that non-linear equation is not simple. The angular momentum of the wave is not trivial, because the contravariant vectors J and K of the Dirac theory, $J = \phi\phi^\dagger$ and $K = \phi\sigma_3\phi^\dagger$, verify

$$\partial_\mu J^\mu = -\frac{2g}{\hbar c}A_\mu J^\mu \ ; \quad \partial_\mu K^\mu = -\frac{2g}{\hbar c}A_\mu K^\mu \tag{6.3}$$

So (6.1) cannot be got from a Lagrangian density.

7 - Concluding remarks.

Too often, quantum books forget the $\{\pm I\}$ kernel of f and identify the complex matrix M and the Lorentz transformation $R = f(M)$. Here it is not possible to identify the Cl_3^* group, which is a 8-dimensionnal Lie group, and the group of the Lorentz dilations, which is a 7-dimensionnal Lie group.

$f(M)$ is invertible only if $r \neq 0$. Physically the main set is not Cl_3^*, it is the full algebra Cl_3, because ϕ may be not everywhere invertible. For instance the Darwin solutions for the hydrogen atom have places where $\det(\phi) = 0$ ($\Omega_1 = \Omega_2 = 0$), and at those places any observer is on the light cone of the tangent space-time.

Even in that case $R_0^0 > 0$ and the dilation R conserves the time's arrow. The Cl_3^* invariance is compatible with an oriented time and an oriented space, with the same orientation in each tangent space-time. That is physically very important, because the time is not invertible, and because the weak interactions distinguish a left and a right orientation of the physical space.

To use the space algebra instead of the Dirac matrices is useful : we see that ρ is a dilation ratio. We may also see the geometrical meaning of the de Broglie's wave : there are two space-time varieties, the relative space-time, which is flat here, but we know that the gravitation curves that space-time, and the intrinsic space-time, linked to the wave, with a torsion $m\rho$. And the de Broglie's wave is the link between those two space-time, by $x = \phi y \phi^\dagger$.

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