An illustration of the quantitative problem of the many worlds interpretation of quantum mechanics and the motivation for outcome counting

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ABSTRACT. The "quantitative problem" of the many worlds interpretation (MWI) of quantum mechanics is to justify the interpretation of the Born rule measure $|a_n|^2$ – the squared norm of the amplitude associated with the n^{th} out of N possible results – as a probability. The essential difficulty is that the basic framework of the MWI would seem to suggest an alternative probability rule, outcome counting, which is that each separate outcome should be equally likely. The purpose of this paper is to illustrate the aesthetic advantages of outcome counting over the Born rule, and to argue that outcome counting is the most "natural" definition of probability in the MWI.

RÉSUMÉ. Le le problème quantitatif de l'interprétation des mondes multiples sera obligé à justifier l'interprétation de la mesure de règle de Born $|a_n|^2$ *– la norme carrée de l'amplitude associée avec le* n^{th} *de* N *les résultats possibles – comme une probabilité. La difficulté essentielle est que le cadre fondamental de l'interprétation des mondes multiples semblerait suggérer une règle de probabilité alternative, le calcul d'issue, qui est que chaque issue séparée doit être également probable. Le but de ce papier sera obligé à illustrer les avantages esthétiques de calcul d'issue par-dessus la règle de Born, et disputer ce calcul d'issue est la définition la plus naturelle de probabilité dans l'interprétation des mondes multiples.*

P.A.C.S.: 01.70.+w; 02.50.Cw; 03.65.Ta

1 Introduction

In the many worlds interpretation (MWI) of quantum mechanics (originally put forth by Everett as the "relative state formulation"[4]), the probability of obtaining a particular experimental outcome is prescribed by the Born rule. According to the Born rule, a probability measure $m_n = |a_n|^2$ is assigned to each of the N experimental results associated with a measurement, with a_n being the complex coefficient of the n^{th} result as calculated by the Schrödinger equation. From an empiric perspective, the success of the Born rule is undisputed.

From an ontologic perspective, the Born rule is not so successful. Perhaps its greatest ontological weakness is what has been called the "quantitative problem"[12]. Graham, many years ago, expressed the essential difficulty: "it is extremely difficult to see what significance such a measure [the Born rule measure] can have when its implications are completely contradicted by a simple count of the worlds involved, worlds that Everett's own work assures us must all be on the same footing" [5]. From this perspective, the essential ontology of the MWI would seem to suggest an alternative probability rule, which is that each possible experimental outcome should be equally likely. This alternative probability rule has gone by various names, including "outcome counting" [13] (which will be the term used in this paper), "egalitarianism" [6], or the "alternate projection postulate" [11]. The difficulty with outcome counting, of course, is immediately obvious: if one simply replaces the Born rule with outcome counting, the predictions are *wrong*. Any attempt to fix the model through additional modification, so that predictions are correct[1], would likely – or so the argument goes – be so difficult, or ugly, or otherwise unappealing, that there is no use even making the attempt. Better it would be not to worry so much about ontology.

Nevertheless, for those of us who worry about ontology, this situation is unsatisfying. It feels as if there is a form of double-speak behind the interpretation of the Born rule, whereby each world is "equally real" – except that some worlds are "more real" (more probable) than others.

The purpose of this paper is to *illustrate* why it is that this state of affairs is unsatisfactory, the purpose being to motivate the construction of a model based on outcome counting. This will be performed through the analysis of an experiment commonly encountered in quantum mechanics: a spin measurement. This analysis will define and make use of

[1] In such a model, we would expect outcome counting, as the fundamental probability rule, to operate at the fine-grained level, and the Born rule to operate at a coarser-grained level. The Born rule would not be *wrong*, per se; but its implementation would be such that it no longer plays the role that it does in our understanding of "what probability really is."

an axiom labeled here the "probability criterion," which – loosely (and somewhat facetiously) stated – is that it is better to be more frequently right than more frequently wrong. The argument recapitulates but also expands upon a similar argument by Graham [5]. For a more abstract philosophic analysis, the reader is referred to a growing literature[12] [6] [9] [10] [1].

2 A simple experiment

Consider an experiment in which M identically prepared spin-1/2 particles are prepared, with their spins being measured sequentially. Using the Born rule, each particle is predicted to be observed to be spin up or down with probabilities m_{up} or m_{down}, respectively. Since each individual spin measurement produces two separate branches (worlds), there will be a total of 2^M worlds at the end of the M measurements. In each of these worlds, upon the completion of the M measurements, the observer is imagined to calculate the frequency p_{up} with which the particles were observed to be in the spin up state. In other words, m_{up} is what the theory predicts, and p_{up} is what is actually observed (with m_{up} being calculated once prior to experiment, and p_{up} being calculated separately in each of the 2^M worlds, after completion of the experiment). From a practical perspective, the prediction is tested by comparing the prediction m_{up} with the observation p_{up} (and likewise for spin down), using as large a value for M as is practically feasible. In particular, it is hoped that as $M \to \infty$, the physical measure of the number of worlds in which the Born rule appears to be false – that is, in which p_{up} deviates from m_{up} by an arbitrarily chosen (small) number δ – approaches zero.

The notion of testing a "probability rule" by comparing the predicted frequency to the observed frequency may be stated more generally for any arbitrary quantum mechanical experiment, using the following definitions:

Definition 1 *Experiments M. M is the number of times that the experiment is run.*

Definition 2 *Outcomes N. N is the number of mutually exclusive possible outcomes for one experimental trial.*

Note the assumption that the spectrum of experimental outcomes is *discrete*. This assumption may be made without loss of generality, and is necessary for the purpose of defining a *measure* over the number of worlds in which a particular outcome is observed. A quantum mechanical experiment with a continuous spectrum of outcomes, such as a position measurement, can be conceptualized as a theoretical (perhaps unattainable) limit as the number of discrete elements of the position measurement apparatus approaches infinity[2].

Definition 3 *Observed probability measure p_n. For any individual world (of which there are N^M), $P_n \in [0, M]$ is the number times that the n^{th} outcome was observed, with $p_n = P_n/M \in [0, 1]$ being the frequency of this outcome among the M measurements.*

Definition 4 *Predicted probability measure m_n. m_n is the predicted probability associated with the n^{th} outcome, $n \in [1, \ldots, N]$.*

Note that p_n is an attribute of an individual world, whereas m_n is a predicted quantity that is independent of any individual experimental result. The expression for m_n will depend on which probability rule we choose; e.g., the Born rule $m_n = m_n^{Born} = |a_n|^2$, outcome counting $m_n = m_n^{OC} = 1/N$, or some other rule.

Definition 5 *Error ϵ_n. The difference between the predicted probability measure m_n and the observed quantity p_n will be referred to as the "error" ϵ_n, that is, $\epsilon_n = |p_n - m_n|$.*

Definition 6 *Measure of belief F_n. $F_n \in [0, 1]$ is the proportion of the N^M worlds in which the observer concludes that the probability measure is valid, as determined by the error being less than or equal to an arbitrarily chosen cutoff, $\epsilon_n \leq \delta$.*

Given these definitions, the probability criterion may be stated in the following manner:

Axiom 1 *Probability criterion. For any arbitrary δ, $\lim_{M\to\infty} F_n = 1$.*

[2]For example, the photographic plate used in a two slit experiment could be modeled as a two dimensional array of detector elements. Mathematically, we can imagine the number of detectors to approach infinity; practically, it is difficult to see how the number of detectors could exceed, say, the number of atoms in the plate.

The probability criterion is essentially a mathematical statement of the notion that *"most" of the worlds will produce an observer who concludes that the theoretical prediction is correct*, to within some arbitrarily chosen cutoff. In a sense, the probability criterion may be interpreted as a *definition* of the very notion of probability: it is a requirement that must be met by the probability measure m_n. It therefore may come as a surprise that the probability criterion is not generally met if m_n is calculated by the Born rule, $m_n = m_n^{Born} = |a_n|^2$. That is to say that in *most* of the N^M worlds, the observer will conclude that the Born rule is false! The probability criterion *is* met, however, if m_n is calculated by outcome counting, $m_n = m_n^{OC} = 1/N$.

By way of illustration, suppose $m_{up} = 0.9$, $m_{down} = 0.1$, $\delta = 0.1$ and $M = 100$. These 100 measurements result in 2^{100} worlds, and it is asked: in what proportion of these worlds does the observer find that p_{up} falls within the interval $m_{up}^{Born} \pm \delta$? A quick calculation (using the formulae derived in the next section) shows that the answer is a miniscule $5.58 * 10^{-8}$ percent. In contrast, the probability criterion *is* met if the probability measure is calculated using outcome counting, $m_{up}^{OC} = 1/N$. In this case, $m_{up}^{OC} = 1/2$, and the proportion of worlds in which the observer finds that p_{up} falls within the interval $m_{up}^{OC} \pm \delta$ is a much larger 96.4 percent.

In summary: in the spin measurement described above, the proportion of worlds in which the observer finds that *the Born rule is wrong* is a rather large 99.9999999442 percent. Conversely, the proportion of worlds in which the observer finds that *the outcome counting rule is right* is 96.4 percent. And for any fixed δ, these percentages get closer and closer to one hundred percent as M approaches ∞. Our experience, of course, is the opposite: our experiments tell us that the Born rule is right, and outcome counting is wrong. Therefore, according to current formulation, the probability criterion is not met, and *we are forced to conclude that the vast majority of Everettian worlds, somehow, just don't matter.* This is the essence of the quantitative problem.

3 Generalization

It is not so difficult to see that outcome counting is a general solution to the probability criterion, and so is *generally* immune to the above difficulty. Assume the definitions above. It is further assumed that m_n can be expressed as a function of N. (Compare this to Everett's assumption [4], discussed in [11], that m_n can be expressed as a function of a_n.) The goal is to find a general expression that tells us whether a given m_n does or does not satisfy the probability criterion.

The number of ways to make P_n observations of the n^{th} outcome is calculated to be $\frac{(N-1)^{(M-P_n)}M!}{P_n!(M-P_n)!}$, as follows. There is only one way to get the n^{th} result on P_n out of P_n trials. Next, the number of ways to get anything other than the n^{th} result on $M-P_n$ out of $M-P_n$ trials is $(N-1)^{M-P_n}$. Next, the number of ways to mix an ordered sequence with P_n elements and an ordered sequence of $M-P_n$ elements, i.e. the number of ways of distributing P_n elements over M elements is M choose P_n, i.e. $\frac{M!}{P_n!(M-P_n)!}$. Multiplying these expressions yields: $\frac{(N-1)^{(M-P_n)}M!}{P_n!(M-P_n)!}$. Dividing this expression by the total number of worlds N^M yields the proportion $f(p_n) \in [0,1]$ of such worlds: $f(p_n) = \frac{(N-1)^{(M-P_n)}M!}{P_n!(M-P_n)!N^M}$. The integrated proportion $F(m_n, \delta) \in [0,1]$ of worlds in which the observed frequency p_n is close to the predicted frequency m_n, i.e. falls anywhere within the closed interval $m_n \pm \delta$, i.e. falls within $[m_n - \delta, m_n + \delta]$, is calculated as the sum: $F(m_n, \delta) = \sum_{p_n=m_n-\delta}^{p_n=m_n+\delta} f(p_n)$. The probability criterion states that for any δ, $\lim_{M\to\infty} F(m_n, \delta) = 1$. By setting $m_n = 1/N$ in the above expression, it is readily seen that the outcome counting rule does in fact satisfy this equation. Computerized numeric calculations confirms this solution. Therefore, it can be concluded that *outcome counting is a general solution to the probability criterion.*

The essential similarity between outcome counting and the probability criterion is that in both cases, each of the N possible "branches" associated with a single measurement – or equivalently, each of the N^M distinct worlds resulting from M measurements – is considered to be, ontologically speaking, on an "equal footing." The Born rule, on the other hand, does not seem to offer a clear ontological picture of the "reality" of alternate worlds. The fact that outcome counting, but not the Born rule, is a solution to the probability criterion is precisely what makes it the more natural choice for a probability rule.

4 Discussion

The purpose of this paper has been to present the aesthetic appeal of outcome counting as an alternative to the Born rule for the calculation of probabilities in the MWI. We have seen that if a theory is based on outcome counting, then the measure of worlds in which prediction matches observation approaches unity, so the probability criterion is met. On the other hand, if a theory is based on the Born rule, then the measure of worlds in which prediction does *not* match observation approaches unity, so the probability criterion is *not* met. This state of affairs *is* the quantitative problem, and it forces us to reject the majority of Everettian worlds as nonexistent or (perhaps more charitably?) irrelevant.

Whether or not this state of affairs is troubling is, perhaps, no more than a matter of taste. As it now stands, many people, including Everett himself, have dedicated much effort towards the demonstration that the Born rule is a natural consequence of the essential structure of the MWI – thus rendering outcome counting not even an option, and hence "solving" the quantitative problem. Hartle has contributed to these attempts [8]. More recently, Deutsch-Wallace decision theory [12] [3] has been proposed to demonstrate the inevitability of the Born rule, essentially by deriving it *a priori*. Others, however, have criticized derivations of these types on the grounds that they are based on circular reasoning, because they contain hidden probabilistic assumptions that effectively "sneak" the Born rule into the formalism, so that they in fact assume what they have set out to prove [11] [1] [2]. The ontological status of the probability interpretation of the MWI, therefore, continues to be a subject of controversy.

There is another option: to accept outcome counting as fundamental, and around this to build a *reformulation* of quantum mechanics so that it makes correct predictions. The reason that a reformulation is necessary is that a simple replacement of the Born rule with outcome counting yields a theory that makes predictions that are inconsistent with experiment. The goal of reformulation is therefore to identify some *additional* modification (besides replacing the Born rule with outcome counting) so that the observed quantum statistics may be restored to the overall programme. If the resulting scheme were to make *exactly* the same predictions as standard quantum mechanics, then its (proposed) justification would be nothing beyond the philosophic: that it solves the problem of probability. One might imagine, however, the existence of an outcome counting-based reformulation that is consistent with all currently known experimental data, yet still makes novel predictions (in situations that have not yet been tested). Such a theory, in principle, would be testable against the standard formulations, in the same sense that general relativity was testable against Newtonian mechanics. In this hypothetical scenario, it is at least conceivable that outcome counting could play the role of a symmetry principle, analogous to that of the principle of relativity.

Although the construction of such a formulation is well beyond the scope of the present paper, there are several schemes that may be mentioned in brief. The first scheme is trivial, and is based upon a simple redefinition of the term "outcome." According to this scheme, what is typically thought of as a *single outcome* – say, the n^{th} out of N possible results, with associated probability $|a_n|^2$ – is redefined so that it corresponds to a *multitude of distinct outcomes*, the number of which is proportional to $|a_n|^2$. The difficulty with this scheme – the reason it is trivial – is that there is nothing in the theory to distinguish any one of the $|a_n|^2$ outcomes from another; nothing to justify the existence of multiple copies of the same outcome. Ideally, such a scheme would define "distinct" as *physically distinct*, would *describe* this distinction in some detail, would tell us how to enumerate them, and would do so in independent fashion; that is, without simply setting the count equal to $|a_n|^2$ by fiat. Without an independent justification, this scheme simply uses circular reasoning to solve the quantitative problem.

There are several non-trivial counting schemes that have been put forth in the literature.[3] One is the "mangled worlds" model [7], according to which the memory of observers in "small" worlds is destroyed or somehow converted to remember events from "large" worlds. (This gives us a reason for the elimination of the 99.9999999442 percent of worlds, discussed above, in which the Born rule appears to be false.) A second proposal [13] assumes nonlinearity in the time-dependent evolution of the observer state, as well as an exponential time dependence for the nonlinear effects, in a fashion that effectively generates $|a_n|^2$ Everettian branches for each experimental outcome. A third model (the work of the author, to be published separately) is similar to the scheme of the preceeding paragraph, except that setting the count equal to $|a_n|^2$ turns out to be an approximation of a completely different scheme that hopefully meets the above criteria to be nontrivial.

The main point of this paper is to suggest that these efforts to base a theory on outcome counting have been too few and far between. Instead of taking an empirically successful rule – the Born rule – and trying to demonstrate that "it could be no other way," we should at least consider a different approach: to assume outcome counting as fundamental, to do whatever reformulation is necessary to make the scheme empirically successful, and see where it leads us. It could just be that this is – or

[3] Graham [5] introduced his own model; unfortunately, I have failed to understand how it works.

will help to identify – one small, yet essential ingredient to a theory of quantum gravity.

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References

[1] David Baker. Measurement outcomes and probability in everettian quantum mechanics. *Studies in History and Philosophy of Modern Physics*, 38:153–169, 2007.

[2] H. Barnum, C. M. Caves, J. Finkelstein, C. A. Fuchs, and R. Schack. Quantum probability from decision theory? *Proc. Roy. Soc. Lond. A*, 456:1175–1182, 2000.

[3] David Deutsch. Quantum theory of probability and decisions. *Proc. R. Soc. Lond. A*, 455:3129–3137, 1999.

[4] Hugh Everett. Relative state formulation of quantum mechanics. *Reviews of Modern Physics*, 29:454–462, 1957.

[5] Neill Graham. The measurement of relative frequency. *The Many-Worlds Interpretation of Quantum Mechanics*, pages 229–252, 1973.

[6] Hilary Greaves. Understanding deutsch's probability in a deterministic multiverse. *Studies in the History and Philosophy of Modern Physics*, 35:423–456, 2004.

[7] Robin D. Hanson. When worlds collide: Quantum probability from observer selection? *Foundations of Physics*, 33:1129 – 1150, 2003.

[8] JB Hartle. Quantum mechanics of individual systems. *American Journal of Physics*, 36:704–712, 1968.

[9] Simon Saunders. Time, quantum mechanics, and probability. *Synthese*, 114:373–404, 1998.

[10] Simon Saunders. What is probability? *Quo Vadis Quantum Mechanics?*, A. Elitzur, S. Dolev, and N. Kolenda:Springer–Verlag, Berlin, 2005.

[11] Patrick Van Esch. On the born rule and the everett programme. *Annales de la Fondation Louis de Broglie*, 32:51–60, 2007.

[12] David Wallace. Quantum probability and decision theory, revisited. *Studies in the History and Philosophy of Modern Physics*, 34:415–438, 2003.

[13] M. B. Weissman. Emergent measure-dependent probabilities from modified quantum dynamnics without state-vector reduction. *Foundations of Physics Letters*, 12:407–426, 1999.

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