Confinement in Einstein's unified field theory

S. Antoci[1], D.-E. Liebscher[2] and L. Mihich[1]

[1]Dipartimento di Fisica "A. Volta" and IPCF of CNR, Pavia, Italia
[2]Astrophysikalisches Institut Potsdam, Potsdam, Deutschland

ABSTRACT. By relying on a thread of essential insights and achievements by Schrödinger, Kurşunoğlu, Lichnerowicz, Hély and Borchsenius, sources are appended to all the field equations of Einstein's Hermitian theory of relativity. From the equations and from the contracted Bianchi identities a sort of gravoelectrodynamics then appears, whose constitutive equation has a much wider scope than the one prevailing in Einstein-Maxwell theory. Now the electric and a magnetic four-current are no longer a physically wrong replica of each other, like it occurs in Maxwell's vacuum.

Particular static exact solutions show that, while electric charges with a pole structure behave according to Coulomb's law, magnetic charges with a pole structure interact with forces not depending on their mutual distance. The latter behaviour is confining for charges of unlike sign, and was already discovered by Treder in 1957 with an approximate calculation, while looking for ordinary electromagnetism in the theory. The exact solutions confirm this finding, already interpreted in 1980 by Treder in a chromodynamic sense.

RÉSUMÉ. Quand on s'attache au fil des résultats atteints par Schrödinger, Kurşunoğlu, Lichnerowicz, Hély et Borchsenius, des sources sont attachées a toutes les équations de champ de la théorie Hermitien de relativité d'Einstein. Faisant usage des indentités Bianchi contractées on trouve une sorte de dynamique gravo-électrique, dont les équations fondamentales ont une domaine plus étendu que celles de la pure théorie Einstein-Maxwell. Maintenant, les courants électriques et magnétiques ne sont plus (physiquement faux) répliques mutuelles, comme on le trouve dans le vacuum de Maxwell.

Des solutions statiques particulières démontrent que les charges électriques à structure polaire se comportent selon la loi de Coulomb. Cependant, les charges magnétiques à structure polaire interagissent *à forces ne pas dépendant de leur distance mutuelle. Ce comportement conduit à un confinement des charges de signe différente, ce que était trouvé en 1957 par Treder au moyen d'un calcul approximé, quand il essayait de trouver une électrodynamique ordinaire. Notres solutions exactes supportent ses résultats, qui sont déjà interpretées selon la terminologie chromodynamique en 1980 par Treder.*

P.A.C.S.: 04.50+h; 04.20.Cv; 04.20.Jb; 12.10.-g; 12.39.Ki

1 Introduction

With their theory of the nonsymmetric field, either in the metric-affine [1, 2, 3, 4] or in the purely affine version [5, 6, 7, 8], while providing a last demonstration of their mathematical insight, Einstein and Schrödinger left as heritage to the future generations the heavy task of trying to attribute a physical interpretation to the very similar field equations that, by proceeding from different startpoints, both of them eventually arrived at. We shall consider here for definiteness the theory proposed by Einstein, in its complex, Hermitian version [3]. In this theory, defined on a real, four-dimensional manifold, one avails, as independent fundamental quantities, of the Hermitian tensor $g_{ik} = g_{(ik)} + g_{[ik]}$ and of the Hermitian affine connection $\Gamma^i_{kl} = \Gamma^i_{(kl)} + \Gamma^i_{[kl]}$. From g_{ik} one builds the Hermitian contravariant tensor g^{ik} such that

$$g^{il} g_{kl} = g^{li} g_{lk} = \delta^i_k, \tag{1}$$

and, since $g \equiv \det(g_{ik})$ is a real quantity, the Hermitian tensor density

$$\mathbf{g}^{ik} = \sqrt{-g} g^{ik}. \tag{2}$$

In Einstein's Hermitian theory, under quite general conditions [9], the Hermitian affine connection Γ^i_{kl} is uniquely defined by the tensor g_{ik} through the transposition invariant equation

$$g_{ik,l} - g_{nk}\Gamma^n_{il} - g_{in}\Gamma^n_{lk} = 0. \tag{3}$$

Let the further field equation

$$\mathbf{g}^{[is]}_{\ \ ,s} = 0 \tag{4}$$

be satisfied. From (3) one gets [10] that (4) is equivalent to the injunction

$$\Gamma_i \equiv \Gamma^l_{[il]} = 0 \tag{5}$$

on the skew part of the affine connection. From (3) alone it stems further:

$$\Gamma^{a}_{(ia),k} = \Gamma^{a}_{(ka),i}. \tag{6}$$

The fulfillment of both (3) and (4) is crucial for the properties of the two generally nonvanishing contractions $R^{i}{}_{kli}$ and $R^{i}{}_{ilm}$ of the Riemann curvature tensor

$$R^{i}{}_{klm}(\Gamma) = \Gamma^{i}_{kl,m} - \Gamma^{i}_{km,l} - \Gamma^{i}_{al}\Gamma^{a}_{km} + \Gamma^{i}_{am}\Gamma^{a}_{kl}. \tag{7}$$

The second contraction reads in general:

$$R^{i}{}_{ilm} = \Gamma^{i}_{il,m} - \Gamma^{i}_{im,l}. \tag{8}$$

When both (3) and (4) are satisfied, this second contraction just vanishes due to (5) and (6); hence, like it occurs with the symmetric theory of 1915, also the problem of choosing which combination of the contractions one should introduce in the field equations simply disappears. Under the same circumstances, the first contraction

$$R_{kl}(\Gamma) = \Gamma^{i}_{kl,i} - \Gamma^{i}_{ki,l} - \Gamma^{a}_{ki}\Gamma^{i}_{al} + \Gamma^{a}_{kl}\Gamma^{i}_{ai}, \tag{9}$$

i.e. the Ricci tensor, happens to be Hermitian. Einstein proposed that its symmetric and skew parts should fulfill the field equations

$$R_{(ik)}(\Gamma) = 0 \tag{10}$$

and

$$R_{[ik],l}(\Gamma) + R_{[kl],i}(\Gamma) + R_{[li],k}(\Gamma) = 0 \tag{11}$$

respectively. The field equations (3), (4) and (10), (11) of what Einstein called the Hermitian generalization of the theory of gravitation can be deduced from a variational principle, e.g. in the manner shown by Einstein in [3], or in the more transparent way, that avails of the "starred affinity", envisaged [6] by Schrödinger.

We have indulged, with these introductory remarks, in expounding the mathematical structure of Einstein's Hermitian theory, since the knowledge of the latter is by no means widespread, while it seems essential for properly understanding what sort of hopes sustained both Einstein and Schrödinger in their decade-long effort, and what means they believed to be the most appropriate for trying to fulfill such hopes.

In the many technical papers written in the decade 1945-1955 on the subject of the "generalized theory of gravitation", Einstein spent very few words on the possible physical content of the theory. In his "Autobiographisches" [11] he was very clear about the reasons for believing that the future progress of physical theory could not be based on quantum theory, due to the statistical character of the latter, and to its allowance for the superposition principle; to him, any real progress could only be achieved by starting from the general theory of relativity, since in Einstein's opinion, "its equations are more likely to assert anything *precise* than all the other equations of physics". From the discovery of general relativity he had also learned that no collection of empirical facts, however extensive, could have been of help in building equations of such intricacy: equations of such complication can only be retrieved when one has found a logically simple mathematical condition that determines the equations in a complete or nearly complete way. Hermitian symmetry or, more generally, invariance under transposition, that both represent a natural mathematical extension of the symmetry properties of the general relativity of 1915, could be sufficiently strong formal conditions, upon which one might attempt a generalization of the previous theory, based on real symmetric quantities.

At variance with the buoyant optimism permeating his first attempt on the subject [1], in his later work Einstein, while sometimes asserting that, since (4) had to hold everywhere, $g_{[[ik],l]}$ might have to assume the rôle of electric four-current [3, 12], became cautious in foretelling what the possible physical content of his new theory might result to be. In the autobiographical notes he limited himself to remark that, in his opinion, equations (3), (4), (10), (11) constituted the most natural generalization of the equations of gravitation, just adding, in a footnote, that in his opinion the theory had a fair likelihood of proving correct, provided that the way to a satisfactory representation of the physical reality on the basis of the continuum will turn out to be feasible in general. He also believed that, since these equations constituted the natural completion of the equations of 1915, no source terms should be appended at their right-hand sides. His "Autobiographisches" therefore ends with a question mark, left like a legacy to the posterity: what happens with the solutions of these equations that are free from singularities in the whole space?

On the possible physical content of the theory, Schrödinger was more explicit already in his first paper [5], where he clearly shows to have perceived the complete novelty of a fundamental feature of the theory, that had to become a crucial issue in the years to come, and eventually led to the abandonment of the efforts aimed at the understanding of the theory, since it constitutes too large a departure from the way we are used to think about the electromagnetic interaction.

2 Interpreting the theory along a path made possible by Schrödinger, Kursunoglu, Lichnerowicz, Hély and Borchsenius

In [5] Schrödinger deals with his own purely affine theory, whose field equations, if considered from a trivially pragmatic standpoint, differ from the ones reported in Section 1 only due to the presence of the "cosmological terms" $\lambda g_{(ik)}$ and $\lambda g_{[[ik],l]}$ at the right-hand sides of equations (10) and (11) respectively. His remarks about the possible electromagnetic meaning of his theory can be extended to the case when $\lambda = 0$, and mean that equations (4) and (11) should be interpreted like a sort of (modified) Maxwell equations, with $\mathbf{g}^{[ik]}$ and $R_{[ik]}$ in the rôles of "contravariant density" and "covariant field tensor" respectively. Needless to say, such an interpretation entails a total departure from the behaviour that one might expect from the acquaintance with Maxwell's equations in vacuo, where the two quantities previously mentioned within quotation marks are mutually related by a simple constitutive equation, that only entails the metric in the usual tasks of raising indices and forming densities from tensors. $\mathbf{g}^{[ik]}$ and $R_{[ik]}$ can play in (4) and (11) the rôles envisaged by Schrödinger only if the constitutive equation of this "electromagnetism" is of a kind never heard of before, namely, a highly involved differential relation, whose content is by no means surveyable in its explicit form, since its determination requires first solving (3) for the affine connection, and then substituting the resulting expressions $\Gamma^i_{kl} = \Gamma^i_{kl}(g_{pq}, g_{pq,r})$ in $R_{[ik]}(\Gamma)$. It is well known [9] that already the first step does not yield in general a surveyable outcome, hence no hint can be drawn a priori about the relation between inductions and fields dictated by the Hermitian theory.

However, despite the total ignorance about its physical meaning, there is one thing that can be subjected to a close scrutiny in this sort of electromagnetism. In keeping with Schrödinger's and Einstein's conviction that the theory did constitute the completion of the theory of 1915, no sources are to be allowed at the right-hand sides of all its field equations. This holds in particular for (4) and (11): as Schrödinger [5] notes with some regret, these equations of unmistakable electromagnetic form are "used up"; their left-hand sides cannot be availed of for defining, like it could have been possible in principle, two conserved four-currents associated with the skew fields. Therefore, and again at variance with what occurs in Maxwell's electromagnetism, we have to look elsewhere for the definition of, say, the electric four-current. Such a further departure from the known patterns could be welcome and sought for, because, as complained by Einstein, "Das Elektron ist ein Fremder in die Elektrodynamik". An electric four-current whose continuous distribution were dictated by the field equations themselves would represent the solution of many problems that plague theoretical physics. This is why Einstein suggested that $g_{[[ik],l]}$ might have to assume the rôle of electric four-current [3, 12]; in [5] Schrödinger added three more candidates to such a high task. But (4) and (11) are just the electromagnetic equations that one would write *in the absence of charges and currents* for some continuum endowed with a very strange constitutive equation, and the Hermitian theory of relativity is a natural generalization of an eminently successful predecessor, whose success was however only possible through the addition, as source, of the phenomenological energy tensor. Therefore the shadow of doubt remained, that the new theory might need phenomenological sources too.

Such a doubt was strengthened by the study of the contracted Bianchi identities. One may find the derivation of these identities e.g. in [7], where Schrödinger, in keeping with his conviction that the theory allowed for a merging of gravitational and nongravitational fields in a total entity, did not split their expression by separating the terms where only symmetric quantities appear from the terms where only skew quantities occur, like e.g. Kurşunoğlu did a few years later [13, 14]. When the field equations (3), (4) hold, the contracted Bianchi identities found by Schrödinger can be written as

$$\left[\sqrt{-g}\left(g^{ik}R_{il}+g^{ki}R_{li}\right)\right]_{,k}=\sqrt{-g}g^{ik}R_{ik,l}. \tag{12}$$

Through the above mentioned splitting, the same identities come to read

$$\begin{aligned}&\left(2\sqrt{-g}g^{(ik)}R_{(il)}\right)_{,k}-\sqrt{-g}g^{(ik)}R_{(ik),l} \\ &=\sqrt{-g}g^{[ik]}\left(R_{[ik],l}+R_{[kl],i}+R_{[li],k}\right).\end{aligned} \tag{13}$$

But in [14] Kurşunoğlu provided an even more allusive writing. He noticed that, if one introduces a symmetric tensor s^{ik} such that

$$\sqrt{-s}s^{ik}=\sqrt{-g}g^{(ik)}, \tag{14}$$

where s is the determinant of the tensor s_{ik}, and $s^{ik}s_{il} = \delta^k_l$, the left-hand side of (13) can be rewritten as follows:

$$\left(2\sqrt{-g}g^{(ik)}R_{(il)}\right)_{,k} - \sqrt{-g}g^{(ik)}R_{(ik),l} \tag{15}$$
$$= \left(2\sqrt{-s}s^{ik}R_{(il)}\right)_{;k} - \left(\sqrt{-s}s^{ik}R_{(ik)}\right)_{;l} .$$

Remarkably enough, the semicolon stands for the covariant differentiation with respect to the Christoffel symbols built with s_{ik}. Hence the contracted Bianchi identities of Einstein's nonRiemannian extension of the vacuum general relativity of 1915 admit a sort of Riemannian rewriting that avails of the tensor s_{ik}:

$$\left(s^{ik}R_{(il)} - \frac{1}{2}\delta^k_l s^{pq}R_{(pq)}\right)_{;k} \tag{16}$$
$$= \frac{1}{2}\sqrt{\frac{g}{s}}g^{[ik]}\left(R_{[ik],l} + R_{[kl],i} + R_{[li],k}\right),$$

provided, of course, that equations (3) and (4) are satisfied. The same form of the weak identities was arrived at later [15] by Hély, who was even more prepared to appreciate the suggestions coming from Kurşunoğlu's way of expression, thanks to a precious result [16, 17] found in the meantime: through his study of the Cauchy problem in Einstein's new theory, Lichnerowicz had concluded that the metric l^{ik} appearing in the eikonal equation

$$l^{ik}\partial_i f \partial_k f = 0 \tag{17}$$

for the wave surfaces of the theory had to be

$$l^{ik} = g^{(ik)}, \tag{18}$$

or, one must add, any metric conformally related to $g^{(ik)}$. Since s_{ik}, defined by (14), just belonged to this class of metrics, Hély had one more reason for critically investigating how the expression (16) might assume a physical meaning, like it occurs in the theory of 1915, where the contracted Bianchi identities just say that the covariant divergence of the energy tensor is vanishing.

When confronted with the weak identity (16), the sort of regret felt by Schrödinger on noticing that the left-hand sides of (4) and (11) were "used up" for expressing the vanishing of two four-currents cannot help becoming a serious concern. One has to withstand one further disappointment: by adhering to the tenet endorsed both by Einstein and by Schrödinger, according to which no source terms should be appended at the right-hand sides of their equations, both sides of (16) simply vanish. Are we not missing in this way an occasion offered by the theory? The very finding of (16) led Kurşunoğlu to modify [14] Einstein's field equations in order to provide the weak identities with physical meaning in a field theoretical way. In a less daring mood, Hély appended [18] phenomenological sources at the right-hand sides of both (10) and (11), with the tentative physical meaning of energy tensor for matter and of electric current respectively. In such a way, (16) comes to assert that the nonvanishing of the covariant divergence of the energy tensor density of charged matter is due to the Lorentz coupling of its electric four-current with the electromagnetic field density $\mathbf{g}^{[ik]}$.

In the same mood, one may well ask what hinders appending phenomenological sources to all the field equations. The question is even more justified, since a class of exact solutions to the equations of the Hermitian theory has been found [19], that intrinsically depend on three coordinates. Solutions belonging to this class appear endowed with physical meaning when sources are appended at the right-hand sides of both (11) and (4).

There is indeed one hindrance, because, as shown in Section 1, the satisfaction of (4) is just one of the necessary conditions for getting a Hermitian Ricci tensor. The remedy was found [20] by Borchsenius; one needs substituting the symmetrized Ricci tensor

$$\bar{R}_{kl}(\Gamma) = \Gamma^{i}_{kl,i} - \frac{1}{2}\left(\Gamma^{i}_{ki,l} + \Gamma^{i}_{li,k}\right) - \Gamma^{a}_{ki}\Gamma^{i}_{al} + \Gamma^{a}_{kl}\Gamma^{i}_{ai}, \tag{19}$$

for the plain Ricci tensor (9). The substitution does not affect the original field equations of Einstein and Schrödinger in vacuo, since there the modified Ricci tensor of Borchsenius is equal to the true Ricci tensor, but is effective in obtaining a set of equations with sources that is always Hermitian. When s_{ik} is adopted as metric, in the footsteps of Hély, this set, whose derivation is reported e.g. in [21], reads:

$$\mathbf{g}^{qr}{}_{,p} + \mathbf{g}^{sr}\Gamma^{q}_{sp} + \mathbf{g}^{qs}\Gamma^{r}_{ps} - \mathbf{g}^{qr}\Gamma^{t}_{(pt)} = \frac{4\pi}{3}(\mathbf{j}^{q}\delta^{r}_{p} - \mathbf{j}^{r}\delta^{q}_{p}), \tag{20}$$

$$\mathbf{g}^{[is]}{}_{,s} = 4\pi\mathbf{j}^{i}, \tag{21}$$

$$\bar{R}_{(ik)}(\Gamma) = 8\pi(T_{ik} - \frac{1}{2}s_{ik}s^{pq}T_{pq}), \tag{22}$$

$$\bar{R}_{[[ik],l]} = 8\pi K_{ikl}. \tag{23}$$

In this way the two conserved four-currents $\mathbf{j}^i$ and K_{ikl}, and the symmetric energy tensor T_{ik} are appended to the original equations in a manner that does not spoil their Hermitian character, and uniquely defines the phenomenological sources in terms of their geometric counterparts. The relevant contracted Bianchi identities are [21] in this case

$$\begin{aligned} -2(\mathbf{g}^{(is)}\bar{R}_{(ik)}(\Gamma))_{,s} + \mathbf{g}^{(pq)}\bar{R}_{(pq),k}(\Gamma) \\ = 2\mathbf{g}^{[is]}_{\ ,s}\bar{R}_{[ik]}(\Gamma) + \mathbf{g}^{[is]}\bar{R}_{[[ik],s]}(\Gamma). \end{aligned} \tag{24}$$

By substituting here the material sources defined above, and by defining the contravariant energy tensor density as

$$\mathbf{T}^{ik} = \sqrt{-s}s^{ip}s^{kq}T_{pq}, \tag{25}$$

one eventually extends Hély's result [18] to the form

$$\mathbf{T}^{ls}_{;s} = \frac{1}{2}s^{lk}\left(\mathbf{j}^i\bar{R}_{[ki]}(\Gamma) + K_{iks}\mathbf{g}^{[si]}\right), \tag{26}$$

where the semicolon again indicates the covariant derivative done with respect to the Christoffel connection built with s_{ik}. By completing Hély's proposal, this equation asserts that the covariant divergence of $\mathbf{T}^{ik}$ does not vanish in general because of the Lorentz coupling of the conserved current K_{iks} with $\mathbf{g}^{[si]}$, and also because of the Lorentz coupling of the conserved current density $\mathbf{j}^i$ with the field $\bar{R}_{[ki]}$. But, since the constitutive equation of this sort of electromagnetism represents a total departure from the one prevailing in the vacuum of Maxwell's electromagnetism, we shall not fear that the duality present in the latter shall lead to a duplicate representation of the same physical behaviour, with electric and magnetic four-currents both producing the same phenomena under a duality transformation. In Maxwell's electromagnetism this occurrence is avoided by imposing, in keeping with experience, that magnetic four-currents do not exist. In Einstein's Hermitian theory this injunction is neither required, nor helpful. The exact solutions show in fact that the two four-currents give rise to completely different interactions, both seemingly needed for the description of nature.

3 The electrostatics of Einstein's Hermitian theory

The simple form of equation (26) should deceive nobody: it is evident that the "particle in field" imagery, already misleading in Maxwell's electrodynamics, is totally out of place both in the general relativity of 1915 and in its Hermitian extension. From such nonlinear theories, both in exact and in approximate solutions, as well exhibited [22] in the work of Einstein and Infeld, one must expect a much subtler link between structure and motion of the field singularities that one uses for representing masses and charges. A particular example of this occurrence is evident [23] in a solution of the Hermitian theory, that one cannot help calling electrostatic in the sense of Coulomb. It can be built by the method reported in [19]; if referred to the coordinates $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = t$, its fundamental tensor g_{ik} reads:

$$g_{ik} = \begin{pmatrix} -1 & 0 & 0 & a \\ 0 & -1 & 0 & b \\ 0 & 0 & -1 & c \\ -a & -b & -c & d \end{pmatrix}, \tag{27}$$

where

$$d = 1 + a^2 + b^2 + c^2, \tag{28}$$

and

$$a = i\chi_{,x}, \; b = i\chi_{,y}, \; c = i\chi_{,z}, \quad i = \sqrt{-1}, \quad \chi_{,xx} + \chi_{,yy} + \chi_{,zz} = 0. \tag{29}$$

The solution is static, and its metric s_{ik} can be written as

$$s_{ik} = \sqrt{d} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{1}{\sqrt{d}} \begin{pmatrix} \chi_{,x}\chi_{,x} & \chi_{,x}\chi_{,y} & \chi_{,x}\chi_{,z} & 0 \\ \chi_{,x}\chi_{,y} & \chi_{,y}\chi_{,y} & \chi_{,y}\chi_{,z} & 0 \\ \chi_{,x}\chi_{,z} & \chi_{,y}\chi_{,z} & \chi_{,z}\chi_{,z} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{30}$$

hence the square of the line element, in the adopted coordinates, reads

$$\mathrm{d}s^2 = s_{ik}\mathrm{d}x^i\mathrm{d}x^k = -\sqrt{d}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}t^2\right) - \frac{1}{\sqrt{d}}(\mathrm{d}\chi)^2. \tag{31}$$

The solution always fulfils the equation $g_{[[ik],l]} = 0$, and one feels entitled to call it electrostatic in the sense of Coulomb. The reason is simple, and geometric in character. It is discussed in detail in [23], to which the reader is referred. Here we recall it briefly. If one allows for sources at the right-hand side of the Laplacian occurring in (29), one notices that the admission of such sources in the representative space corresponds to introducing a true charge density at the right-hand side of (21). Imagine now trying to build localized true charges by starting from localized, disjoint sources in the "Bildraum". One finds that, when the charges are very far apart from each other, they will be both pointlike and spherically symmetric, with all the accuracy needed to account for the empirical constraints, only provided that the charges occupy, in the space whose metric is s_{ik}, just the positions dictated by Coulomb's law of electrostatic equilibrium [23].

One might object that naming "electrostatic" the charges associated with $\mathbf{j}^i$ is wholly premature, since we have not yet explored what happens when net charges are built from K_{ikl}. But an exact solution allowing for such charges dispels the objection because, like one might well have expected, the "magnetostatics" exhibited by such a solution has nothing to do with Maxwell's electromagnetism.

4 In Einstein's Hermitian theory the magnetic charges are confined entities

One solution of this kind is easily found by the method given in [19]; when referred to polar cylindrical coordinates $x^1 = r$, $x^2 = z$, $x^3 = \varphi$, $x^4 = t$, its fundamental tensor g_{ik} reads:

$$g_{ik} = \begin{pmatrix} -1 & 0 & \delta & 0 \\ 0 & -1 & \varepsilon & 0 \\ -\delta & -\varepsilon & \zeta & \tau \\ 0 & 0 & -\tau & 1 \end{pmatrix}, \tag{32}$$

with

$$\zeta = -r^2 + \delta^2 + \varepsilon^2 - \tau^2, \tag{33}$$

and

$$\delta = ir^2\psi_{,r},\ \varepsilon = ir^2\psi_{,z},\ \tau = -ir^2\psi_{,t},\ \ \psi_{,rr} + \frac{\psi_{,r}}{r} + \psi_{,zz} - \psi_{,tt} = 0. \tag{34}$$

Its metric s_{ik} can be written as

$$s_{ik} = \frac{\sqrt{-\zeta}}{r}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \frac{r^3}{\sqrt{-\zeta}}\begin{pmatrix} \psi_{,r}\psi_{,r} & \psi_{,r}\psi_{,z} & 0 & \psi_{,r}\psi_{,t} \\ \psi_{,r}\psi_{,z} & \psi_{,z}\psi_{,z} & 0 & \psi_{,z}\psi_{,t} \\ 0 & 0 & 0 & 0 \\ \psi_{,r}\psi_{,t} & \psi_{,z}\psi_{,t} & 0 & \psi_{,t}\psi_{,t} \end{pmatrix}, \tag{35}$$

hence the square of the line element, in the adopted coordinates, reads

$$\mathrm{d}s^2 = s_{ik}\mathrm{d}x^i\mathrm{d}x^k = \frac{\sqrt{-\zeta}}{r}\left(-\mathrm{d}r^2 - \mathrm{d}z^2 - r^2\mathrm{d}\varphi^2 + \mathrm{d}t^2\right) + \frac{r^3}{\sqrt{-\zeta}}(\mathrm{d}\psi)^2. \tag{36}$$

Let us consider the particular, static solution for which

$$\psi = -\sum_{q=1}^{n} K_q \ln\frac{p_q + z - z_q}{r}, \tag{37}$$

where

$$p_q = [r^2 + (z - z_q)^2]^{1/2}; \tag{38}$$

K_q and z_q are constants. One obtains

$$\delta = i\sum_{q=1}^{n}\frac{K_q r(z - z_q)}{p_q}, \quad \varepsilon = -i\sum_{q=1}^{n}\frac{K_q r^2}{p_q}, \quad \tau = 0, \tag{39}$$

and

$$\zeta = -r^2(1 + F), \tag{40}$$

with

$$F = \sum_{q=1}^{n} K_q^2 + r^2\sum_{q=1}^{n}\sum_{q'=1}^{n(q'\neq q)}\frac{K_q K_{q'}}{p_q p_{q'}} + \sum_{q=1}^{n}\frac{K_q(z - z_q)}{p_q}\sum_{q'=1}^{n(q'\neq q)}\frac{K_{q'}(z - z_{q'})}{p_{q'}}. \tag{41}$$

Let $n = 1$, $z_1 = 0$. Then

$$\delta = i\frac{Krz}{(r^2+z^2)^{1/2}}, \quad \varepsilon = -i\frac{Kr^2}{(r^2+z^2)^{1/2}}, \quad \zeta = -r^2(1+K^2), \tag{42}$$

and the interval reads

$$\mathrm{d}s^2 = \sqrt{1+K^2}\left(-\mathrm{d}r^2 - \mathrm{d}z^2 - r^2\mathrm{d}\varphi^2 + \mathrm{d}t^2\right) + \frac{K^2}{\sqrt{1+K^2}}\frac{(z\mathrm{d}r - r\mathrm{d}z)^2}{r^2+z^2}. \tag{43}$$

It is easy to ascertain that this interval displays a constant deviation from elementary flatness along the z-axis. The length $\mathrm{d}l$ of an infinitesimal vector $\mathrm{d}x^i$, lying in a meridian plane, orthogonal to the z-axis, and drawn from a point for which $r = 0$, $z =$ const. , reads

$$\mathrm{d}l = \left(-s_{11} + \frac{(s_{12})^2}{s_{22}}\right)^{1/2}\mathrm{d}x^1, \tag{44}$$

while the length of the infinitesimal circle drawn by the tip of the vector $\mathrm{d}x^i$, when it is so rotated around the z-axis that φ grows by the amount 2π, is

$$\Delta l = 2\pi\sqrt{-s_{33}}. \tag{45}$$

Since for the circle drawn in this way $r = \mathrm{d}x^1$, the value of the ratio $\mathcal{R}$ between length and radius of the elementary circle turns out to be

$$\mathcal{R} = 2\pi\sqrt{1 - \frac{\delta^2}{r^2}}, \tag{46}$$

hence, for the present particular case with $n = 1$, one obtains

$$\mathcal{R} = 2\pi\sqrt{1+K^2}. \tag{47}$$

But, in an axially symmetric solution, a constant deviation from elementary flatness along the symmetry axis can be removed by simply modifying the definition of the manifold, since nothing enforces the original, tentative choice $0 < \varphi \leq 2\pi$ for the coordinate φ.

Let us first rewrite the interval (43) in spherical polar coordinates R, ϑ, φ, t, obtained by performing, in the meridian planes, the coordinate transformation

$$r = R\sin\vartheta, \quad z = R\cos\vartheta. \tag{48}$$

Then (43) comes to read

$$\mathrm{d}s^2 = \sqrt{1+K^2}\left[-\mathrm{d}R^2 - R^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) + \mathrm{d}t^2\right] + \frac{K^2}{\sqrt{1+K^2}}R^2\mathrm{d}\vartheta^2. \tag{49}$$

By the coordinate transformation and fixation of the manifold

$$\varphi' = \sqrt{1+K^2}\varphi, \quad 0 < \varphi' \leq 2\pi, \tag{50}$$

the interval becomes

$$\mathrm{d}s^2 = \sqrt{1+K^2}\left(-\mathrm{d}R^2 + \mathrm{d}t^2\right) - \frac{R^2}{\sqrt{1+K^2}}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi'^2\right). \tag{51}$$

This manifold, besides displaying elementary flatness everywhere, with the exception of $R = 0$, is spherically symmetric too. One recognizes, in the g_{ik} associated with it, one particular case of the spherically symmetric solutions [24] found by Papapetrou. For this particular solution $\mathbf{j}^i$ is everywhere vanishing, while this is not the case for K_{ikl}. In fact, let us consider in this manifold a closed spatial two-surface Σ, and define the invariant integral

$$I = -\frac{1}{8\pi i}\int_\Sigma \bar{R}_{[ik]}\mathrm{d}f^{ik}, \tag{52}$$

where $\mathrm{d}f^{ik}$ is a surface element of Σ. The integral is always vanishing if Σ does not surround, say, the origin $R = 0$ of the spatial coordinates R, ϑ, φ'. In the opposite case one finds

$$I = \frac{K}{\sqrt{1+K^2}}, \tag{53}$$

i.e. K_{ikl} exhibits a pole of magnetic charge located at $R = 0$ in the representative space, which, according to (51), is a point charge in the metric sense too.

When $n = 2$, the solution defined by (32)-(41) cannot describe the field of two isolated poles of magnetic charge, lying on the z axis, whatever the choice of K_1, K_2 and of z_1, z_2 may be. This negative outcome happens despite the fact that the integral (52) is nonvanishing when it is extended to a closed spatial two-surface Σ surrounding either one or the other of the above mentioned positions, and otherwise arbitrary, thereby proving the existence of net charges built with K_{ikl} both at $r = 0$, $z = z_1$ and at $r = 0$, $z = z_2$ respectively.

In fact, at variance with what happens when $n = 1$, the ratio (46) shows that the deviation from the elementary flatness occurring on the z-axis is only piecewise constant, hence it can not be made to disappear by an appropriate choice of the manifold. Therefore, when $n = 2$, the solution can not be considered as representing the field of two isolated bodies, just like it happens, in the general relativity of 1915, with the Weyl-Levi Civita field for two masses at rest [25, 26, 27].

The $n = 3$ case is more fruitful, for, if we choose

$$K_1 = K_3 = K, \; K_2 = -K, \quad z_1 < z_2 < z_3, \tag{54}$$

we find that

$$\lim_{r \to 0} F = K^2 \tag{55}$$

along the whole z-axis. Therefore the ratio $\mathcal{R}$, defined by (46), says that the deviation from elementary flatness, just like in the case $n = 1$, can be eliminated through the appropriate definition of the manifold, by suitably choosing the range of φ.

Let us remind that, in the electrostatic case [23], we have found that the electric charges did occupy the positions of equilibrium dictated by Coulomb's law, provided that the charges built with $\mathbf{j}^i$ were pointlike in the metric sense, and that the metric s_{ik} happened to be spherically symmetric in an infinitesimal neighbourhood of each charge, with all the accuracy needed to meet with the empirical facts. Let us study under what conditions the three aligned magnetic charges happen to enjoy the same geometric properties.

An inspection of the metric (35) for this solution shows that pointlike charges in the representative space are always pointlike in the metric sense too. To check for the spherical symmetry in an infinitesimal neighbourhood of each charge, we need evaluating the interval $\mathrm{d}s$, expressed by (36), in an infinitesimal neighbourhood of each of the points located at $r = 0$, $z = z_i$, $i = 1, 2, 3$. One finds that, in the close proximity to all the points of the z-axis, the interval (36) can be approximated as

$$\mathrm{d}s^2 = \sqrt{1 + K^2}\left(-\mathrm{d}r^2 - \mathrm{d}z^2 - r^2\mathrm{d}\varphi^2 + \mathrm{d}t^2\right) + \frac{1}{\sqrt{1 + K^2}}(r\mathrm{d}\psi)^2. \tag{56}$$

In the close proximity of the three points mentioned above one can use the further approximation

$$\frac{1}{\sqrt{1+K^2}}(r\mathrm{d}\psi)^2 = \frac{K^2}{\sqrt{1+K^2}}\frac{\left[(z-z_i)\mathrm{d}r - r\mathrm{d}z\right]^2}{r^2+(z-z_i)^2}. \tag{57}$$

Therefore, by performing severally, in the meridian planes, the coordinate transformations

$$r = R\sin\vartheta,\ \ z - z_i = R\cos\vartheta, \tag{58}$$

for $i = 1, 2$ and 3, one will find that in each infinitesimal neighbourhood the interval will always take the same form, given by (49), i.e. the very form that holds in the whole space for the solution with $n = 1$. As a consequence, if one performs the transformation and fixation of the manifold (50) also in this case with $n = 3$, defined by (54), one finds that the interval is spherically symmetric in the infinitesimal neighbourhood of each of the pointlike magnetic charges.

The geometrical conditions on the metric field surrounding the charges, whose fulfillment[1], in the electrostatic solution of Section 3, ensures that Coulomb's law is an outcome of the theory, in the particular solution considered here are always satisfied exactly, whatever the mutual positions of the three magnetic charges may be, provided that the order $z_1 < z_2 < z_3$ is respected. One therefore draws the physical conclusion that these aligned magnetic charges by no means behave like magnetic monopoles would do, if they were allowed for, in Maxwell's electromagnetism. The indifferent equilibrium of the three charges exhibited by this magnetostatic solution of the Hermitian theory is only possible if the interaction of the charges is independent of their mutual distances.

One can object to this conclusion, because the fact that the charges are both pointlike in the metrical sense, and each endowed with a spherically symmetric infinitesimal neighbourhood for whatever choice of $z_1 < z_2 < z_3$, might well mean that these charges are not interacting at all. But, as soon as the conditions (54) for K_i are not respected, a deviation from elementary flatness appears on stretches of the z-axis, that can not be made to disappear through the choice of the manifold, just like it occurs in the solution with $n = 2$, and also in the two-body,

[1]although with the approximation expounded in [23].

static solutions of the general relativity of 1915. Moreover, approximate calculations done by Treder already [28] in 1957 both by the EIH method [29, 22] and by the test-particle method [30] of Papapetrou revealed the existence, in this gravito-electromagnetism, of a central force between the poles built with K_{ikl}, that does not depend on their mutual distance, and that, in the Hermitian theory, is attractive when the poles have charges of opposite sign.

The same conclusion can be drawn also with an argument that relies on another exact solution [32] belonging to the class described in [19]. The solution is a Hermitian generalization of the Curzon metric [33]. In the cylindrical coordinates of its representative space two Curzon masses, located at $r = 0$, $z = z_1$ and $r = 0$, $z = z_2$ respectively, are endowed with point magnetic charges. For fixed z_1 and z_2, by choosing appropriately the values of the constants associated with both the masses and the charges, one succeeds in obtaining that no deviation from elementary flatness occur along the whole z-axis. One interprets this circumstance as showing that the gravitational force between the masses is balanced by the force that the magnetic charges exert on each other. From the weak field limit of this exact solution, when the gravitational pull reduces to the Newtonian behaviour, one concludes too that the force between the magnetic charges is attractive when the charges have opposite sign, and that it does not depend on their mutual distance[2]. In 1980 Treder interpreted [31] his findings of 1957 in a chromodynamic sense.

5 Conclusion

Talking of conclusions, here and now, sounds ironically premature. We are still at the very beginnings, since the theory represents such a total departure from the known paths. Considering $\mathbf{g}^{[ik]}$ and $R_{[ik]}$ as electromagnetic inductions and fields, like Schrödinger first [5] envisaged sixty years ago, leads to a gravito-electromagnetism endowed with a range of possibilities so wide and unexplored, thanks to the intricate differential constitutive relation linking these quantities, that one might well despair that its content will ever be unraveled, and proved to be physically meaningful or not. And yet, thanks to approximate and to exact findings, some glimpses about the possible content of the theory have appeared during the lapse of the decades. Besides, of course, Ein-

[2]In the mentioned paper [32], the deviation from the elementary flatness was calculated by availing of $g_{(ik)}$ as metric. The calculation was repeated with the right metric s_{ik}, and has provided just the same result.

stein's gravitation of 1915, the theory appears to contain, according to particular exact solutions, electric charges that behave as prescribed by Coulomb's electrostatics [23], as well as magnetic poles that interact with forces not depending on their mutual distance. When confronted with such outcomes, one can not help remembering the hopes expressed by Schrödinger in the paper quoted above:

> "We may, I think, hold out the prospect, that those skew fields together, whatever may emerge as the appropriate interpretation, embrace both the electromagnetic and the nuclear field and their interplay with each other and with gravitation."

and dare suggesting, on the basis of the admittedly scant, but unambiguous evidence gathered until now, that the work on this theory, abandoned so many decades ago, be resumed in the years to come.

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(Manuscrit reçu le 28 mars 2007)