Neutral Particles in 4-Space Dirac Theory

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ABSTRACT. Four-space Dirac theory permits bispinor wave functions describing neutral spin-$\frac{1}{2}$ particles of generally indeterminate proper mass. Though the 4-current vector is always null, the 4-momentum carried by these particles may be timelike, and in this case the wave function oscillates.

1 Introduction

Evidence of neutrino mass has invalidated the Weyl description, which requires the proper mass to be zero. A formulation of neutrino flavours as superpositions of mass eigenstates has been put forward to explain the observed oscillations, but debate continues, and Lim has suggested [1] a pseudo-Dirac model incorporating Majorana masses. This may give better agreement with experiment, though at the expense of extra parameters. If neutrinos are indeed massive particles, one would expect them to be subluminal, and an unequivocal, independent demonstration of this would help to clarify their status, since empirically it still seems possible at present that they travel at the speed of light. But if neutrino speeds cannot be clearly distinguished experimentally from that of light (*in vacuo*), the revised neutrino theory may itself be questioned, leaving the Standard Model open to challenge by one or more of the 4-space theories, sometimes called parametrized relativistic quantum theories [2, 3, 4].

This paper shows that neutral particles can be described within a 4-space Dirac theory as a special class of solutions satisfying conditions for the absence of charge. In the picture outlined below, a neutral particle always has a null current, but it may, nonetheless, carry timelike 4-momentum - this possibility arising because the 4-space theory allows states that are not eigenstates of the proper mass, m_0. If it is sharp, then m_0 must be precisely zero (this case corresponds closely to the Weyl neutrino), but in general m_0 is indeterminate, and the expected proper mass, $\langle m_0 \rangle$, lies within a non-negative range that depends on the particle's 4-momentum. Even a null 4-momentum allows $\langle m_0 \rangle$ to be positive. Of particular interest are states with timelike 4-momentum and indeterminate proper mass - in these the wave function oscillates.

The 4-space Dirac formulation used here requires a non-standard interpretation of the wave function, with both negative and positive charge contributions, as follows. The 4-component spinor ψ is a function of the spacetime coordinates $X^\lambda = (x^k, ct)$ and the invariant parameter τ, which corresponds either to the proper time of classical theory, or to the affine parameter of a null geodesic. (We use the same symbol τ in all cases.) The conventions are generally those of [5, 6, 7]; in particular, the Lorentz metric tensor is $\eta^{\alpha\beta} = \mathrm{diag}(1\ ,1\ ,\ 1\ \ ,\ \ -1)$, and the Dirac matrices γ^λ have a chiral representation. In [5], m was employed for proper mass, but m_0 is used here. We begin below with an approach that applies to all free spin-$\frac{1}{2}$ particles, but later specialize to neutral particles.

There is an invariant expected net charge density $F(\mathbf{X}, \tau) \equiv \psi^\dagger(i\gamma^4)\psi$ in space-time, and an expected particle 4-current density $\mathbf{J}(\mathbf{X}, \tau) \equiv -\psi^\dagger\gamma^4\boldsymbol{\gamma}\psi$, so that $J^4 = \psi^\dagger\psi \geq 0$ implies a flow in the positive time direction. In general, $F = F_1 - F_2$, where F_1 and F_2 are expected densities of negative and positive charge, and $\mathbf{J}$ is the sum of the expected particle and antiparticle currents, which in most cases have a common 4-velocity $\mathbf{U}$ given by $c\mathbf{J} = (F_1 + F_2)\mathbf{U}$. There is also a second invariant, $Q \equiv \psi^\dagger\gamma^0\psi$, where $\gamma^0 \equiv -i\gamma^1\gamma^2\gamma^3$, which is related to F and $\mathbf{J}$ by $\mathbf{J} \cdot \mathbf{J} = -(F^2 + Q^2)$. As a consequence, $F_1 = (\sqrt{F^2 + Q^2} + F)/2$ and $F_2 = (\sqrt{F^2 + Q^2} - F)/2$: note that $F_1 \geq 0$, $F_2 \geq 0$, and $F_1 = F_2 = 0 \Leftrightarrow F = Q = 0$. These points are outlined in [5], and illustrated in [6], where the 4-space approach is applied to Klein's paradox. The interpretation of the model, especially as regards probabilities, is given more fully in [7].

However, these earlier papers are concerned only with charged particles (electrons and positrons): they do not cover the case with which we are mainly concerned here, viz. $F_1 = F_2 = 0$. The relation $c\mathbf{J} = (F_1 + F_2)\mathbf{U}$ is then no longer directly applicable - though we note that it does not imply $\mathbf{J} = \mathbf{0}$. Just as the more familiar relation $\mathbf{P} = m_0\mathbf{V}$ of relativistic particle mechanics does not imply $\mathbf{P} = \mathbf{0}$ when $m_0 = 0$, but rather that $\mathbf{P}$ is a null vector ($\mathbf{P} \cdot \mathbf{P} = 0$), in the present context we find that $\mathbf{J}$ is a null vector, because of an identity mentioned above: $\mathbf{J} \cdot \mathbf{J} = -(F^2 + Q^2)$ (see Appendix B of [5]). We continue to suppose that the charge densities F_1 and F_2 are given as above by the invariants F and Q, so that $F = Q = 0$.

2 Free-Particle Solutions

For the present purpose, we wish to solve the free-field equation

$$\boldsymbol{\gamma}\cdot\boldsymbol{\partial}\psi = \left(\frac{i}{c}\right)\frac{\partial\psi}{\partial\tau}. \tag{1}$$

There is a class of solutions of the form

$$\psi = \exp(\mathbf{P}\cdot\,[i\mathbf{X}I + c\tau\boldsymbol{\gamma}]/\hbar)\;\zeta, \tag{2}$$

where ζ is any constant bispinor. These have sharp 4-momentum $\mathbf{P}$, but are not eigenstates of proper mass. Moreover, for timelike $\mathbf{P}$ they are oscillating solutions, as we see on rewriting (2) in the form

$$\psi = e^{i\mathbf{P}\cdot\mathbf{X}/\hbar}\;\{I\cos(cP\tau/\hbar) + \frac{\mathbf{P}\cdot\boldsymbol{\gamma}}{P}\sin(cP\tau/\hbar)\}\;\zeta. \tag{3}$$

Here $P = \sqrt{-\mathbf{P}\cdot\mathbf{P}}$, but since the proper mass m_0 is not sharp, we cannot use $P = m_0 c$. The case $P = 0$ (i.e. $\mathbf{P}$ a null vector) can be obtained by letting $P \to 0$ in (3):

$$\psi = e^{i\mathbf{P}\cdot\mathbf{X}/\hbar}\;\{I + c\tau\,\mathbf{P}\cdot\boldsymbol{\gamma}/\hbar)\}\;\zeta. \tag{4}$$

So far, with m_0 indeterminate, ψ does not represent the usual kind of plane wave. However, in the 4-space theory there is a proper mass operator $\hat{m}_0 = (-i\hbar/c^2)\partial/\partial\tau$, and imposing the condition $\hat{m}_0\psi = m_0\psi$ on (2) gives

$$(\mathbf{P}\cdot\boldsymbol{\gamma} - im_0cI)\;\zeta = 0. \tag{5}$$

Applying (5) to (2), we obtain a plane wave closer to the conventional form:

$$\psi = \exp(i[\mathbf{P}\cdot\mathbf{X} + E_0\tau]/\hbar)\;\zeta. \tag{6}$$

(The usual Dirac plane wave lacks the term $E_0\tau \equiv m_0c^2\tau$ in the exponential.) As in standard Dirac theory, the constant bispinor ζ is now an eigenvector defined by (5): for the chiral representation we find, provided $m_0 \neq 0$, that

$$\zeta = \begin{bmatrix} i\alpha m_0 c \\ i\beta m_0 c \\ \alpha(P_3 - P_4) + \beta(P_1 - iP_2) \\ \alpha(P_1 + iP_2) - \beta(P_3 + P_4) \end{bmatrix}, \tag{7}$$

where α and β are arbitrary complex constants. The existence of ζ requires the mass-shell condition

$$\mathbf{P} \cdot \mathbf{P} = -m_0^2 c^2. \tag{8}$$

There is an obvious formal similarity between these 4-space plane waves and their conventional analogues. Wave packets with sharp proper mass, though with τ (not t) as the evolution parameter, can be generated from them by superposition in the usual way. And the charge density is of only one sign ($F_1 > 0, F_2 = 0$) - this is easily verified by choosing a frame in which the spatial momentum components are zero, when we find

$$F = 2m_0^2c^2(|\alpha|^2 + |\beta|^2); Q = 0. \tag{9}$$

It follows that a neutral particle with sharp proper mass must have $m_0 = 0$, and this is a separate case, covered below. We note in passing that linear superposition of wave functions for massive plane waves at rest (and hence for those with a common 4-velocity) cannot give neutral particles: we find, as in (9), that F is proportional to $|\psi|^2$.

In the case of a massive particle, the phase velocity of a 4-space plane wave contrasts with that of conventional Dirac theory. The 4-velocity of the wave (6) is just $\mathbf{P}/m_0$, as for the corresponding classical particle, whereas the usual Dirac plane wave has a phase 3-velocity of magnitude $E/|\mathbf{p}| > c$, where $\mathbf{p}$ is the 3-momentum. The phase 4-velocity $\mathbf{P}/m_0$ of (6) coincides here also with the 4-velocity $\mathbf{U}$ of the particle current (Section 1). The requirement for a neutral particle to give a null current is consistent with this, because a neutral plane wave is massless.

We noted earlier that (7) is valid only if $m_0 \neq 0$; i.e. when $\mathbf{P}$ is not null. If $m_0 = 0$, (5) becomes $(\mathbf{P}\cdot \boldsymbol{\gamma})\zeta = 0$, and the consequent condition

$det(\mathbf{P}\cdot\boldsymbol{\gamma}) = 0$, required for a non-trivial solution, reduces as expected to $\mathbf{P} \cdot \mathbf{P} = 0$. However, we do not recover the associated eigenvectors by setting $m_0 = 0$ in (7): the result for the chiral representation when $m_0 = 0$ is

$$\zeta = \begin{bmatrix} \alpha(P_3 + P_4) \\ \alpha(P_1 + iP_2) \\ \beta(P_3 - P_4) \\ \beta(P_1 + iP_2) \end{bmatrix} . \tag{10}$$

We need not pursue this in detail, because it is equivalent to the two-component Weyl description, as is suggested by the form of (10). One can readily check that (10) implies $F = Q = 0$ (and hence $F_1 = F_2 = 0$): both F and Q contain a factor $\mathbf{P} \cdot \mathbf{P} = 0$. In particular, eigenstates of helicity are all equivalent to the Weyl neutrino (or antineutrino), with their distinct and invariant chiralities.

3 Neutral Particles with Indeterminate Mass

Though reassuringly familiar, plane wave solutions with sharp proper mass are not especially interesting. The type that we first looked at (equations (2)-(4)), in which m_0 is not sharp, offers more possibilities. Solutions with $P = 0$ (see (3), (4)) are now obtainable from the general case, in contrast to the plane wave solutions that are eigenstates of m_0 (see (7), (10)). We find (with the aid of the identities $\boldsymbol{\gamma}^\dagger\gamma^4 = -\gamma^4\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}^\dagger\gamma^0 = \gamma^0\boldsymbol{\gamma}$) that the invariants F and Q determining the expected charge densities are (if $P \neq 0$)

$$F = \zeta^\dagger(i\gamma^4)\zeta; \tag{11}$$

$$Q = \zeta^\dagger\gamma^0\zeta\cos(2cP\tau/\hbar) + (\mathbf{P}/P)\cdot(\zeta^\dagger\gamma^0\boldsymbol{\gamma}\zeta)\sin(2cP\tau/\hbar). \tag{12}$$

If $P = 0$ (see (4)), we can get the correct Q by letting $P \to 0$ in (12):

$$Q = \zeta^\dagger\gamma^0\zeta + (2c\tau\mathbf{P}/\hbar)\cdot(\zeta^\dagger\gamma^0\boldsymbol{\gamma}\zeta). \tag{13}$$

In all cases the expected charge densities F_1, F_2, are zero if and only if

$$\zeta^\dagger\gamma^4\zeta = \zeta^\dagger\gamma^0\zeta = \mathbf{P}\cdot(\zeta^\dagger\gamma^0\boldsymbol{\gamma}\zeta) = 0. \tag{14}$$

Here we have just three conditions on the eight real components of ζ.

In general, the proper mass operator $\hat{m}_0 = (-i\hbar/c^2)\partial/\partial\tau$ is defined so that an integral over spacetime gives $\langle m_0 \rangle$. In the present case we have an eigenstate of 4-momentum, so that ψ is not localized in space-time, and the usual integral is therefore undefined. However, in cases like this we can regard the integrand, $\psi^\dagger(i\gamma^4)\hat{m}_0\psi$, as an expected proper mass. (A factor involving γ^4 is required for Lorentz invariance of inner products [6].) We now find that the expected proper mass is the invariant (and constant) expression

$$\langle m_0 \rangle = (\mathbf{P}/c) \cdot (\zeta^\dagger \gamma^4 \boldsymbol{\gamma} \zeta) = (-1/c)\,\mathbf{P} \ \cdot \ \mathbf{J}. \tag{15}$$

It is helpful here to write ζ in terms of two 2-component spinors, ξ and η:

$$\zeta = \begin{bmatrix} \xi \\ \eta \end{bmatrix}. \tag{16}$$

Introducing the 3-momentum vector $\mathbf{p}$, so that (in contravariant form) $\mathbf{P} = (\mathbf{p}, E/c)$, from (15) we obtain

$$(c^2/2)\langle m_0 \rangle = E\,\xi^\dagger\xi + c\,\mathbf{p} \cdot (\xi^\dagger \boldsymbol{\sigma} \xi) \tag{17}$$

and

$$(c^2/2)\langle m_0 \rangle = E\,\eta^\dagger\eta - c\,\mathbf{p} \cdot (\eta^\dagger \boldsymbol{\sigma} \eta), \tag{18}$$

where $\boldsymbol{\sigma}$ represents the "3-vector" of Pauli matrices. Assuming that $E > 0$, and using the fact that $|\xi^\dagger \boldsymbol{\sigma} \xi| = \xi^\dagger \xi$, we can write (from (17))

$$(E - cp)|\xi|^2 \leq (c^2/2)\langle m_0 \rangle \leq (E + cp)|\xi|^2, \tag{19}$$

where $p \equiv |\mathbf{p}|$. Adding this and the corresponding result from (18), we get

$$(E - cp)|\zeta|^2 \leq c^2\langle m_0 \rangle \leq (E + cp)|\zeta|^2. \tag{20}$$

If, as seems plausible, we can suppose that ζ is normalized, then the classical value of $m_0 c^2$ is the geometric mean of the bounds in (20). If $\mathbf{P}$ is null, (20) simplifies to

$$0 \leq c^2\langle m_0 \rangle \leq 2E|\zeta|^2. \tag{21}$$

4 Conclusions and Comments

We have looked at spin-$\frac{1}{2}$ particles within the framework of a 4-space Dirac theory, in which both charged and neutral particles are accommodated in a single unified scheme. This is more restrictive than the usual approach, since it requires neutrinos to satisfy zero-charge conditions, and does not introduce new mass parameters, yet it allows a wide range of behaviour.

Solutions corresponding to the massless Weyl neutrino are still found, and if we suppose that the proper mass is sharp, there are no other cases: the proper mass must be zero, and the momentum (if also sharp) must be null. These solutions do not contain the invariant evolution parameter τ: their only variable factor is the scalar function $e^{i\mathbf{P}\cdot\mathbf{X}/\hbar}$.

But when we allow the proper mass to be indeterminate, as the 4-space picture implies we can, interesting new possibilities arise. The 4-momentum $\mathbf{P}$ (still assumed sharp) need not be null, and if $\mathbf{P}$ is time-like, ψ contains an oscillating function of τ, as in eq. (3). It seems clear that this function is capable of causing an oscillating probability of interaction. Although its sine and cosine factors must both be present, because $\mathbf{P}$ would be null if we had $(\mathbf{P}\cdot\boldsymbol{\gamma})\zeta = 0$, it is possible that $(\mathbf{P}\cdot\boldsymbol{\gamma})\zeta$ is relatively small, so that the cosine term dominates.

In all cases the particle current $\mathbf{J}$ is a null 4-vector: these neutral particles always propagate at the speed of light. In this respect, the present model is closer to the Weyl formulation than are the recent massive-neutrino alternatives, and the characteristic helicities of the Weyl neutrino and antineutrino are maintained.

Finally, we should note that the conditions for the absence of charge were imposed only after we had found oscillating solutions, which therefore are not confined to neutral particles. For charged particles, however, solutions of the oscillating type considered here require a constant potential, which suggests that interactions caused by charges may suppress the oscillating states.

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