

## A Quasi-Newtonian Approach to Bohmian Mechanics II: Inherent Quantization

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RÉSUMÉ. Dans le rapport précédent nous avons obtenu une forme fonctionnelle du potentiel quantum, par une image quasi - Newtonienne, et sans avoir recours à la fonction des ondes. Nous avons même, succinctement décrit, les propriétés de l'approche quasi-Newtonienne à la mécanique de Bohme. Dans cet article nous considérons le problème de quantisation et nous montrons que le postulat d'Eigenvalue, est une conséquence naturelle des conditions de la continuité, et il n'est pas nécessaire de postuler que l'éventail de l'énergie et le mouvement angulaire sont d'Eigenvalues, de l'opérateur relié. A vrai dire, mécanique Bohmienne, prédit le postulat d'Eigenvalue.

*ABSTRACT. In a previous paper, we obtained the functional form of quantum potential by a quasi-Newtonian approach and without appealing to the wave function. We also described briefly the characteristics of this approach to the Bohmian mechanics. In this article, we consider the quantization problem and we show that the 'eigenvalue postulate' is a natural consequence of continuity condition and there is no need for postulating that the spectrum of energy and angular momentum are eigenvalues of their relevant operators. In other words, the Bohmian mechanics predicts the 'eigenvalue postulate'.*

Keywords: Bohmian Mechanics, Quantum Potential, Hamilton-Jacobi, Eigenvalue Postulate

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### 1 Introduction

In a previous paper, we considered a quasi-Newtonian approach to the Bohmian mechanics. We could obtain the functional form of the quan-

tum potential without appealing to any wave function. In this article, we consider the method of solving quantum problems in this quasi-Newtonian approach and we show that the 'eigenvalue postulate' for energy and angular momentum are natural consequence of the continuity equation and there is no need for postulating it in the Bohmian mechanics. This fact throws light on the problem of eigenvalues in the quantum theory. There is a hope that one can generalize this statement for physical quantities other than energy and orbital angular momentum, specially for *spin*.

In the previous paper, we also described the main differences between our quasi-Newtonian approach to Bohmian mechanics and the usual one. We mentioned that the imposition of the uniqueness condition on  $S$  and positive-definiteness condition on  $R$  are not necessary for solving quantum problems. In this article, we practically see these facts by solving some specific problems.

## 2 Solving problems in the quasi-Newtonian approach to Bohmian mechanics

### 2.1 About the eigenvalue postulate

Quantum mechanics consists of two basic parts: Schrödinger equation and operators. On the one hand, we assume that the wave function of any physical system satisfies the time-dependent Schrödinger equation, and on the other hand, for any physical quantity like energy, momentum, etc., there exists an operator, the eigenvalues of which constitute all possible values of that quantity. This assumption does not result from the Schrödinger equation, and in a certain sense is one of the important postulates of physics, since Galileo and Newton. The *eigenvalue postulate* raises some questions. For example, all operators do not have common eigenfunctions (like e.g. the components of angular momentum operator). What can one do with physical quantities corresponding to these operators? How can one get their values and how are they defined? If we take this postulate seriously, then one must conclude that all of the components of angular momentum do not exist at the same time. Then the question arises about the physical meaning of the *non-existence* of a component of angular momentum, or how can one imagine a physical quantity without having any definite value? Any alternative theory that tries to explain quantum phenomena, without accepting this postulate, has to explain why the measured values of the physical quantities are simply the eigenvalues of their corresponding operators? In the

Bohmian interpretation of quantum mechanics, it is possible to show how the process of measurement terminates with an eigenvalue of the relevant quantum operator without any collapse of the wave function [1],[2, Chap. 8]. Of course, it seems that the concept of operators and operator relations and their roles in the quantum theory are not made clearer in the ordinary Bohmian interpretation of quantum mechanics.

When we want to find the energy levels of atoms in the Bohmian mechanics, we solve the time-independent Schrödinger equation

$$\hat{H}\psi = E\psi \quad (1)$$

as in the standard quantum mechanics. Also, for finding angular momentum values we solve the equations

$$\hat{L}^2\psi = l(l+1)\hbar^2\psi \quad (2)$$

$$\hat{l}_z\psi = m\hbar\psi. \quad (3)$$

These equations are postulates of the standard quantum mechanics. Why we must use them in the Bohmian mechanics too? In the Bohm's postulates there are no operators and operator relations. Therefore, we must *prove* these relations in the Bohmian mechanics.

In the next sections of this paper, we attempt to show that one can solve quantum mechanical problems without starting from operator methods. Also, one can clarify the meaning of operator methods, specially eigenvalue problems. We show that one can *prove* the validity of eigenvalue equations, and during this proof the meaning of these equations becomes clearer. Consequently, we shall discuss about the situations that we can use or not use these equations.

## 2.2 The stationary states as conservative states

In classical mechanics, whenever classical potential  $V$  does not depend on time, the 'system' is conservative. For example, when you throw a particle with an arbitrary velocity, the energy of system remains invariant ( $dH/dt = \partial V/\partial t = 0$ ). But, in Bohmian mechanics, because of the presence of quantum potential  $Q$ , we have

$$dH/dt = \partial(V + Q)/\partial t = \partial Q/\partial t. \quad (4)$$

Therefore, for the motion to be conservative, we must have

$$\partial Q/\partial t = 0. \quad (5)$$

When  $R$  tends to zero at infinity (or at system boundary), this equation reduces to  $\partial R/\partial t = 0$ , i.e., the state must be stationary. Therefore, in stationary states the motion is conservative, and due to this fact we can call the stationary states as 'conservative states'. Thus, in the Bohmian mechanics, even when the potential  $V$  is classically conservative, the energy is not necessarily conservative. The conservative motions are restricted to the cases that the state is stationary. This is the reason for the significance of stationary states, among arbitrary states.

### 2.3 *Hamilton's canonical equations and quantization problem*

The central concept in the ordinary quantum mechanics, is that of wave function. In the case of stationary states (where  $\partial R/\partial t = 0$  and  $S = W(x) - Et$ ), we look for the eigenfunctions and the eigenvalues of the operators (accurately or approximately). In the scattering problems as well as those problems involving time-dependent potentials, the wave function plays a central role. But a question arises about whether the concept of wave function is necessary for the conceptual structure and the mathematical formulation of the quantum mechanics. Is it really necessary to consider both  $R$  and  $S$  as physical functions which are unique at all points of space (apart from additive constants for  $S$ )? Is it not sufficient to refer quantum phenomena only to  $R$  and regard  $S$  as an auxiliary (multi-valued) function like its classical counterpart, and not as the phase of a wave function? To what extent appealing to wave function and operator-based approaches is necessary for solving quantum problems?

One possibility for finding the answer of these questions is to add the quantum potential to the classical potential and try to solve the problems through Hamilton's canonical equations to describe the quantum phenomena like interference, passage through a potential barrier,  $\dots$ , and specifically the quantization of physical quantities (energy, angular momentum,  $\dots$ ). What that justifies this method is that the quantum potential appears in the Hamiltonian as a function of space. Thus, its role in Hamilton's canonical equations is similar to the role of classical

potential. But, one must note that the mere addition of a function, named quantum potential, to the canonical equations, does not bring about the quantization of the quantities, as it doesn't occur for any classical potential. The cause of quantization lies elsewhere. We shall see how, by adding quantum potential to the mechanics of the particle and regarding the continuity equation, the quantization of quantities (energy and angular momentum, in this article) for bound conserved states *becomes a necessity*. We show that the role of continuity equation in the quantization of quantities is essential. We prove that, the continuity condition is sufficient for understanding the fact that the energy and angular momentum of particle in the stationary states are simply the eigenvalues of the relevant operators and there is no need to take this statement as a postulate. This fact shows a merit of Bohmian views (in the framework of a quasi-Newtonian approach) that reduces the number of postulates we need to explain quantum phenomena.

In order to show that this method works, we shall solve some problems by this method in the next sections and show that in order to solve quantum problems, it does not seem necessary to use the usual *operator-based* methods used in the standard quantum mechanics. We shall show that one can get *quantization without any reference to the wave function and the eigenvalue postulate*.

### 3 The analysis of some problems using canonical equations

#### 3.1 Stationary (conservative) states in one dimension

Consider a one-dimensional single-particle system with the Hamiltonian:

$$H(x, p_x) = \frac{p_x^2}{2m} + V(x) + Q(x) \quad (6)$$

in which the quantum potential  $Q$  is of the form:

$$Q(x) = -\frac{\hbar^2}{2m} \frac{1}{R} \frac{d^2 R}{dx^2}, \quad R = R(x). \quad (7)$$

The canonical equations are

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \\ \dot{p}_x &= -\frac{\partial H}{\partial x} = -\frac{dV}{dx} - \frac{dQ}{dx}. \end{aligned}$$

From these two equations, one concludes that

$$p_x^2 + 2m(V + Q) = \alpha$$

where  $\alpha$  is the constant of integration, and by equating  $H$  with  $E$ , one gets  $\alpha = 2mE$ . Therefore, for a stationary state in one-dimensional potentials, we have

$$\frac{R}{\hbar^2} p_x^2 = \frac{d^2 R}{dx^2} + 2m(E - V(x)) \frac{R}{\hbar^2}. \quad (8)$$

There is another important condition that must be considered which is the continuity equation

$$\frac{d}{dx}(R^2 p_x) = 0. \quad (9)$$

In classical mechanics, one obtains the classical laws of motion, by taking the extremum of an integral of action along the path of the particle, whereas the continuity equation is obtained by identifying the canonical distribution function  $f(x, p_x)$  with  $R^2(x)\delta(p_x - \partial S/\partial x)$  and using Liouville's equation [2, Chap. 2]. Therefore, in classical mechanic the expression  $d^2 R/dx^2$  does not enter into the equation (8), without any need to have  $d^2 R/dx^2 = 0$ .

Now, we pay attention to the condition (9) which implies that  $R^2 p_x = \lambda = \text{const}$ . The constant  $\lambda$  is the same throughout the particle path, being either zero or non-zero. If  $\lambda \neq 0$ , then we should always have  $p_x \neq 0$  because  $R$  as physical quantity can not be infinite, which means that the particle never stops and thus cannot have classical turning points. This state is not a bound one. But, if  $\lambda = 0$ , then because  $R$  is not always zero, we must have  $p_x = 0$ . Thus, the particle would be at rest. Therefore, a bound particle in a stationary state is always at rest. Now, the question arises: whereas the continuity equation is independent of the form of quantum potential  $Q$  as a function of  $R$  and the continuity equation is common between classical and quantum mechanics, why we do have turning points in classical mechanics for bound particles? In response, one may suggest that the classical dynamics must be considered as non-stationary states with  $R = R(x, t)$  and  $S = W(x) - Et$ . Therefore, we must consider

$$\frac{\partial}{\partial t}R^2 + \frac{\partial}{\partial x}\left(R^2\frac{p_x}{m}\right) = 0 \tag{10}$$

instead of equation (9) for classical dynamics. Therefore, because  $R$  in classical mechanics appears only in the continuity equation (10) and not in the energy equation, one can always find a well-behavior function  $R(x, t)$  such that it satisfies the equation (10). Therefore, turning point *becomes possible*.

Thus, in the bound stationary states of quantum mechanics, the particle is at rest and the energy eigenvalues are obtained from

$$\frac{d^2R}{dx^2} + \frac{2m}{\hbar^2}(E - V(x))R = 0. \tag{11}$$

This is the time-independent Schrödinger equation for a real time-independent wave function  $R$ . In the standard quantum mechanics, it is only after the introduction of the time-reversal operator and the imposition of the invariance under this operator that we prove that in the case of one-dimensional bound states, the wave function is real and therefore  $R$  replace  $\psi$  in the eigenvalue relation.

An important point about the equation (11) is that its solutions for  $R$  are not necessarily positive definite functions. This is a fact that we mentioned in the section 1. For example, for the case of a particle in a box, we have

$$V(x) = \begin{cases} 0 & \text{if } 0 < x < a \\ \infty & \text{otherwise.} \end{cases}$$

The solutions are

$$R(x) \propto \sin \frac{n\pi x}{a}$$

in which  $R$  is negative in certain intervals. Thus the  $R$  that appears here is not exactly the same as  $R$  that appears in Bohm's formulation. If we write  $R$  as

$$R = |R|e^{ix}$$

then, for  $R \geq 0$ , we have  $\chi = 0$  and for  $R < 0$ , we have  $\chi = \pi$ . In the usual formulation of Bohmian mechanics,  $\chi$  is a part of the phase of the wave function. Thus the phase of wave function is equal to

$$S'(x, t)/\hbar = \chi - Et/\hbar.$$

We have denoted the phase of wave function by  $S'$ , because in our approach here  $S = -Et/\hbar$ . In the usual Bohmian mechanics, although  $S'$  is not the same at all points, but as it is constant in any interval between the zero points of  $R$ , the particle remains at rest. In this situation, the value of  $S'$  near the points where  $R$  is equal to zero, i.e., at nodes, changes non-continuously. But, from the point of view adopted in this paper,  $S$  is continuous. The function  $S$  is constant throughout the whole box and the value of  $R$  changes continuously between positive and negative values. Therefore, it is more suitable to let  $R$  take negative values too. Thus, its interpretation as the amplitude of a wave is not correct, because the amplitude of a wave cannot be negative. Of course, in the Bohmian mechanics,  $R$  appears in the form of  $R^2$  or  $R^{-1}\nabla^2 R$ . Thus, negative  $R$  is not a problem, and the condition  $R \geq 0$  does not seem to be necessary. This supports our argument that appealing to the wave function is not necessary for quantum theory. As we shall see in the case of central potential in three dimension too, the positive definiteness of  $R$  is superfluous.

### 3.2 Stationary (conservative) states of central potentials

Consider the case of a particle which moves in a three dimensional central problem with a Hamiltonian

$$H(r, \theta, \phi, p_r, p_\theta, p_\phi) = \frac{1}{2m} \left( p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) + V(r) + Q(r, \theta, \phi).$$

Now, considering the spherical symmetry of  $V(r)$ , we write  $R$  as

$$R(r, \theta, \phi) = R_r(r)R_\theta(\theta)R_\phi(\phi). \quad (12)$$

Then the quantum potential takes the form

$$Q(r, \theta, \phi) = Q_r(r) + \frac{Q_\theta(\theta)}{r^2} + \frac{Q_\phi(\phi)}{r^2 \sin^2 \theta} \quad (13)$$



in which

$$Q_r(r) = -\frac{\hbar^2}{2m} \frac{1}{R_r} \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} R_r \right) \quad (14)$$

$$Q_\theta(\theta) = -\frac{\hbar^2}{2m} \frac{1}{R_\theta} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} R_\theta \right) \quad (15)$$

$$Q_\phi(\phi) = -\frac{\hbar^2}{2m} \frac{1}{R_\phi} \frac{\partial^2}{\partial \phi^2} R_\phi. \quad (16)$$

The canonical equations of motion for the  $\phi$  coordinate are

$$\begin{aligned} \dot{\phi} &= \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mr^2 \sin^2 \theta} \\ \dot{p}_\phi &= -\frac{\partial H}{\partial \phi} = -\frac{1}{r^2 \sin^2 \theta} \frac{\partial Q_\phi}{\partial \phi} \end{aligned}$$

which lead to

$$p_\phi^2 + 2mQ_\phi(\phi) = \alpha_\phi^2 \quad (17)$$

in which  $\alpha_\phi^2$  is the constant of integration and thus a constant of motion. Similarly, the canonical equations for the coordinate  $\theta$  and  $r$  lead to

$$p_\theta^2 + \frac{\alpha_\phi^2}{\sin^2 \theta} + 2mQ_\theta(\theta) = \alpha_\theta^2 \quad (18)$$

$$H = \frac{1}{2m} \left( p_r^2 + \frac{\alpha_\theta^2}{r^2} \right) + V(r) + Q_r(r) = E. \quad (19)$$

The equations (17)-(19) can be written in the form:

$$\frac{R_\phi}{\hbar^2} p_\phi^2 = \frac{\partial^2}{\partial \phi^2} R_\phi + \frac{\alpha_\phi^2}{\hbar^2} R_\phi \quad (20)$$

$$\frac{R_\theta}{\hbar^2} p_\theta^2 = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} R_\theta \right) + \left( \alpha_\theta^2 - \frac{\alpha_\phi^2}{\sin^2 \theta} \right) \frac{R_\theta}{\hbar^2} \quad (21)$$

$$\frac{R_r}{\hbar^2} p_r^2 = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial}{\partial r} R_r) + \frac{2m}{\hbar^2} \left[ E - V(r) - \frac{\alpha_\theta^2}{2mr^2} \right] R_r. \quad (22)$$

If the left-hand sides of the equations (21) and (22) are zero, then these two equations reduce to the angular and radial Schrödinger equations. For the case of equation (20) and the quantization of  $\alpha_\phi^2$ , we shall talk about later. We observe that, for  $\alpha_\theta^2$  and  $E$  to be quantized, it is necessary for  $p_\theta$  and  $p_r$  to be zero, and we shall see that the continuity condition compels these quantities to be zero.

In the case of one dimensional problem, we saw that it was more natural to let  $R$  taking negative values as well. Here, as it is clear from the relevant equations, the functions  $R_r$ ,  $R_\theta$  and  $R_\phi$  can take negative values too. Thus, again, the interpretation of  $R$  as the amplitude of the wave function is not suitable.

What is the meaning of the constants  $\alpha_\theta$  and  $\alpha_\phi$ ? In order to see meaning of these constants, it would be better to compare the equations (17) to (19) with their classical counterparts. In the classical case [3, pp. 450-451], we have

$$p_\phi = \alpha_\phi \quad (23)$$

$$p_\theta^2 + \frac{\alpha_\phi^2}{\sin^2 \theta} = \alpha_\theta^2 = l^2 \quad (24)$$

$$p_r^2 + \frac{\alpha_\theta^2}{r^2} + 2mV(r) = 2mE \quad (25)$$

where  $l$  is the magnitude of the classical angular momentum. In the classical case,  $\alpha_\phi$  is the  $z$ -component of angular momentum and  $\alpha_\theta$  is the magnitude of the angular momentum. But, in the quantum case, even though  $\alpha_\theta$  and  $\alpha_\phi$  are constants, they do not have exactly the same meaning. Here the  $z$ -component of angular momentum  $p_\phi$  and the magnitude of the angular momentum vector  $(p_\theta^2 + p_\phi^2/\sin^2 \theta)^{1/2}$  are not necessarily constants of motion. What we measure in the laboratory as the  $z$ -component of angular momentum ( $m\hbar$ ) and the square of the magnitude of the angular momentum vector ( $l(l+1)\hbar^2$ ), are in fact the constants  $\alpha_\phi$  and  $\alpha_\theta^2$ , and not the real values of the  $z$ -component

of angular momentum and the square of the magnitude of the angular momentum vector respectively. Now, we discuss about the cause of quantization of these quantities.

Before continuing, we consider a lemma from classical Hamilton-Jacobi theory. We know that when the classical potential is in the form of

$$V(r, \theta, \phi) = V_r(r) + \frac{V_\theta(\theta)}{r^2} + \frac{V_\phi(\phi)}{r^2 \sin^2 \theta} \quad (26)$$

then, the principal function  $S$ , becomes fully separable, in the form of

$$S(r, \theta, \phi) = W_r(r) + W_\theta(\theta) + W_\phi(\phi) - Et. \quad (27)$$

This is the case that we have here, because, considering (13), the effective potential  $V + Q$  is exactly in the form of (26). Therefore, the equation (27) is valid for our problem.

Similar to the one dimensional problem in previous section, here we must discuss about the condition of continuity. But, to begin with, it would be better to pay attention to several mathematical identities that clarify the relation between the continuity equation and the eigenvalue equations of quantum operators. First, we define the following expressions

$$f_r(r) = \frac{1}{R_r^2} \frac{\partial}{\partial r} (r^2 R_r^2 \frac{\partial W_r}{\partial r}) \quad (28)$$

$$f_\theta(\theta) = \frac{1}{R_\theta^2} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta R_\theta^2 \frac{\partial W_\theta}{\partial \theta}) \quad (29)$$

$$f_\phi(\phi) = \frac{1}{R_\phi^2} \frac{\partial}{\partial \phi} (R_\phi^2 \frac{\partial W_\phi}{\partial \phi}). \quad (30)$$

By these definitions the continuity condition becomes

$$0 = \frac{r^2}{R^2} \nabla \cdot (R^2 \nabla S) = f_r + f_\theta + \frac{f_\phi}{\sin^2 \theta}. \quad (31)$$

Writing  $\psi$  in the form  $\psi = R e^{iS/\hbar}$  we get, after some algebra

$$\frac{\hat{L}_z \psi}{\psi} = p_\phi + (-i\hbar) \frac{1}{R_\phi} \frac{\partial R_\phi}{\partial \phi} \quad (32)$$

$$\frac{\hat{L}_z^2 \psi}{\psi} = (p_\phi^2 + 2mQ_\phi) + (-i\hbar)f_\phi = \alpha_\phi^2 + (-i\hbar)f_\phi \quad (33)$$

$$\begin{aligned} \frac{\hat{L}^2 \psi}{\psi} &= \left( p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta} \right) + 2m \left( Q_\theta + \frac{Q_\phi}{\sin^2 \theta} \right) + \\ &(-i\hbar) \left( f_\theta + \frac{f_\phi}{\sin^2 \theta} \right) \\ &= \alpha_\theta^2 + (-i\hbar) \left( f_\theta + \frac{f_\phi}{\sin^2 \theta} \right) \end{aligned} \quad (34)$$

$$\frac{\hat{H} \psi}{\psi} = \left\{ \frac{(\nabla S)^2}{2m} + V + Q \right\} + \frac{(-i\hbar)}{2R^2} \nabla \cdot \left( R^2 \frac{\nabla S}{m} \right) = E. \quad (35)$$

We see that,  $E$  is eigenvalue of  $\hat{H}$  (because of (31), the imaginary part of (35) is zero), but the other constants of motion  $\alpha_\theta^2$  and  $\alpha_\phi^2$  are not necessarily eigenvalues of  $\hat{L}^2$  and  $\hat{L}_z^2$ , respectively. Also,  $p_\phi$  is not eigenvalue of  $L_z$ , necessarily.

If we want  $\psi$  to be the simultaneous eigenfunction of operators  $\hat{L}_z^2$ ,  $\hat{L}^2$  and  $\hat{H}$ , the different parts of the continuity equation must be separately zero, i.e.,

$$f_\phi = 0, f_\theta = 0, f_r = 0. \quad (36)$$

In other words, the validity of the Eqs. (36) is equivalent to  $\psi$  being the simultaneous eigenfunction of  $\hat{L}_z^2$ ,  $\hat{L}^2$  and  $\hat{H}$  (in a special case  $\psi$  may be the eigenfunction of  $\hat{L}_z$ ).

But from equation (31), we conclude that for some constants  $c_\phi$  and  $c_\theta$  we have

$$f_\phi(\phi) = c_\phi \quad (37)$$

$$f_\theta(\theta) + \frac{c_\phi}{\sin^2 \theta} = c_\theta \quad (38)$$

$$f_r(r) = -c_\theta \tag{39}$$

and from Eqs. (33) and (34) we get

$$\frac{\hat{L}_z^2 \psi}{\psi} = \alpha_\phi^2 - i\hbar c_\phi \tag{40}$$

$$\frac{\hat{L}^2 \psi}{\psi} = \alpha_\theta^2 - i\hbar c_\theta. \tag{41}$$

When the constants  $c_\theta$  and  $c_\phi$  are not necessarily zero,  $\psi$  is not necessarily an eigenfunction of  $\hat{L}_z^2$  and  $\hat{L}^2$ , whereas  $\alpha_\phi^2$  and  $\alpha_\theta^2$  are still constants of motion. Now, we consider the values of constants  $c_\phi$ ,  $c_\theta$  for a bound system.

When we have identically  $\dot{r} = 0$ , we get from Eq. (39) that  $c_\theta = 0$ . If the orbit of particle is such that  $\dot{r}$  is not equal to zero at all points, there necessarily exist at least one  $r_{min}$  and one  $r_{max}$  in the orbit of particle, for the system to be bound. If we integrate Eq. (39) from one  $r_{min}$  to the subsequent  $r_{max}$ , we obtain

$$\begin{aligned} & (r^2 R_r^2 \frac{\partial W_r}{\partial r})_{r_{max}} - (r^2 R_r^2 \frac{\partial W_r}{\partial r})_{r_{min}} = \\ & - c_\theta \int_{r_{min}}^{r_{max}} R_r^2 dr \end{aligned} \tag{42}$$

but at  $r_{min}$  and  $r_{max}$  we have  $p_r = \partial W_r / \partial r = 0$  and therefore the left hand side of the equality is zero. The integrand in the right hand side is positive and because  $\dot{r}$  is not identical to zero, we have  $r_{max} \neq r_{min}$ . Therefore, the integral becomes non-zero, and the equality holds when we have  $c_\theta = 0$ . The value of  $c_\theta$  is a constant for the whole path of the particle. Therefore, from (39) we obtain for the whole path of particle that

$$r^2 R_r^2 \frac{\partial W_r}{\partial r} = r^2 R_r^2 m \dot{r} = \lambda_r \tag{43}$$

in which  $\lambda_r$  is a constant. If  $\lambda_r \neq 0$  we have  $\dot{r} \neq 0$  for the whole path of the particle. This means that, there is no turning point in the trajectory

of the particle. The particle either approaches the center of potential or move away from it. This state can not be a bound one. Therefore, for a bound system we must have  $\lambda_r = 0$  and also identically  $\dot{r} = 0$ . Therefore,

"for conservative states of a bound system we *always* have  $c_\theta = 0$  and  $\dot{r} = 0$ ".

Now, consider the coordinate  $\theta$  and let  $c_\theta = 0$ . When we have identically  $\dot{\theta} = 0$ , we get from Eq. (38) that  $c_\phi = 0$ . If  $\dot{\theta}$  is not identically zero and system is bound, then there necessarily exists at least one  $\theta_{min}$  and one  $\theta_{max}$  in the path of particle. Putting  $c_\theta = 0$  in Eq. (38) and integrating from one  $\theta_{min}$  to the subsequent  $\theta_{max}$ , we obtain

$$\begin{aligned} & (\sin \theta R_\theta^2 \frac{\partial W_\theta}{\partial \theta})_{\theta_{max}} - (\sin \theta R_\theta^2 \frac{\partial W_\theta}{\partial \theta})_{\theta_{min}} = \\ & - c_\phi \int_{\theta_{min}}^{\theta_{max}} \frac{R_\theta^2}{\sin \theta} d\theta. \end{aligned} \quad (44)$$

Similar to the previous reasoning for  $c_\theta$ , this equality holds when  $c_\phi = 0$ . This means that we get from (38)

$$\sin \theta R_\theta^2 \frac{\partial W_\theta}{\partial \theta} = \sin \theta R_\theta^2 m r^2 \dot{\theta} = \lambda_\theta \quad (45)$$

in which  $\lambda_\theta$  is a constant. Now, if  $\lambda_\theta \neq 0$ , then  $\dot{\theta}$  can never vanishes. Thus,  $\theta$  either increases or decreases, and since  $\theta$  changes between 0 and  $\pi$ , it reaches its limits and exceeds them, but according to definition of  $\theta$ , this is impossible. Therefore, the continuation of motion necessitates to have  $\dot{\theta} = 0$  somewhere, which contradicts our assumption. Thus, we must have  $\lambda_\theta = 0$ , and since  $\sin \theta$  or  $R_\theta$  are not identical to zero, we must always have  $\dot{\theta} = 0$ . Therefore

"for conservative states of a bound system we *always* have  $c_\phi = 0$  and  $\dot{\theta} = 0$ ".

Putting  $c_\phi = 0$  in (37) we get

$$R_\phi^2 \frac{\partial W_\phi}{\partial \phi} = R_\phi^2 m r^2 \sin^2 \theta \dot{\phi} = \lambda_\phi \quad (46)$$

in which  $\lambda_\phi$  is a constant.

We summarize the results of vanishing  $c_\theta$ ,  $c_\phi$ ,  $\dot{r}$  and  $\dot{\theta}$  for stationary bound states. We obtain from (40) and (41) that the constants of motion  $\alpha_\theta^2$  and  $\alpha_\phi^2$  are eigenvalues of operators  $\hat{L}^2$  and  $\hat{L}_z^2$ , respectively. Formerly, we observed from Eq. (35) that the other constant of motion i.e. energy is eigenvalue of operator  $\hat{H}$ . Because of vanishing  $p_\theta$  and  $p_r$ , the Eqs. (21) and (22) reduce to angular and radial parts of Schrödinger equation.

There is no necessity to have  $\dot{\phi} = 0$  for the bound systems but it can occur. Indeed, there are two cases which are consistent with Eqs. (20) and (46). We can take  $\dot{\phi} = 0$  (i.e.  $p_\phi = 0$ ) or  $R_\phi = \text{const}$  (i.e.  $p_\phi = \alpha_\phi$ ). In the first case, we conclude from (20) that  $R_\phi$  is an eigenfunction of operator  $\hat{L}_z^2$ , and  $\alpha_\phi$  take values  $m\hbar$  for integer  $m$ . In the second case we conclude from (20) that  $p_\phi = \alpha_\phi$  and there is no a way to quantize  $\alpha_\phi$  but imposing

$$\oint \nabla S \cdot dx = \text{integer} \times 2\pi\hbar \quad (47)$$

which is equivalent to appealing to the wave function concept. According to Eq. (32) the wave function  $\psi$  becomes eigenfunction of  $\hat{L}_z$ . The condition (47) means that the  $S$  function is unique apart from additive constants. This condition allows us to introduce the single-valued wave function  $\psi = R \exp(iS/\hbar)$ . Without condition (47) we can not have a well-defined wave function.

Therefore, for quantized  $\alpha_\phi$  we can either drop the wave function or appeal to it. If you think in a quasi-Newtonian framework, you can accept the first case, and if you think about the complex wave function as a fundamental entity, you can accept the second case.

Consequently, one can explain the quantization of  $\alpha_\phi$  without appealing to the wave function. Of course, we must remember that in this case  $p_\phi$  and consequently  $\dot{\phi}$  are always zero. This with vanishing of  $\dot{r}$  and  $\dot{\theta}$ , leads to the conclusion that in the stationary states of central potentials, the particle is always at rest. Thus we can say that, contrary to what

is stated in the articles on ordinary Bohmian mechanics, the electron in hydrogen atom is at rest not only when the magnetic quantum number  $m$  is zero, but it is at rest even for  $m \neq 0$ . In this case, the function  $\psi$  is an eigenfunction of  $\hat{H}$  and  $\hat{L}^2$  and also of  $\hat{L}_z^2$ , rather than  $\hat{L}_z$  and becomes

$$\begin{aligned}\psi_e(r, \theta, \phi) &= R_{nl}(r)P_l^m(\cos \theta) \cos m\phi \\ \psi_o(r, \theta, \phi) &= R_{nl}(r)P_l^m(\cos \theta) \sin m\phi \quad (m \neq 0)\end{aligned}$$

rather than

$$\psi(r, \theta, \phi) = R_{nl}(r)P_l^m(\cos \theta)e^{im\phi}.$$

In the other words, assuming that  $\psi$  is an eigenfunction of  $\hat{L}_z^2$  rather than  $\hat{L}_z$ ,  $\psi$  becomes real and we reach the conclusion that the electron in all eigenfunctions of a central potential is at rest.

#### 4 Summary

As we observed in this paper, one can solve the quantum problems by Hamilton's canonical equations. We observed that considering Hamilton's canonical equation along with the continuity condition yield the quantization of energy and angular momentum in a natural way without appealing to the 'eigenvalue postulate'. This approach *is a new kind of quantization*, based directly and completely on the Bohmian mechanics. The presence of a non-trivial (Bohmian) quantum potential in Hamilton's equations permits the existence of stable conservative states  $\partial R/\partial t = 0$ , and the presence of continuity condition compels the energy and angular momentum for these states to be quantized. This fact shows that the Bohmian mechanics is on a correct route. This fact also shows the merit of writing Bohmian mechanics in the form of modified Hamilton-Jacobi and continuity equations.

According to this paper, the operator methods of ordinary quantum mechanics are often useful, but we should not consider them as the basis of theory of particle mechanics. We should consider the operators and operator algebra as merely useful mathematical tools for solving problems.

Here we did not consider the Schrödinger equation as the basis of our work, and we emphasized that the concept of wave function is not



necessarily a basic quantum concept, and that one can solve quantum problems without appealing to it. But, this does not mean that we are denying the practical value of the Schrödinger equation. We can combine the real equations of Bohmian mechanics in the form of complex Schrödinger equation and use it to solve large number of problems. Indeed, as we mentioned in previous paper, we must be aware that the form of quantum potential is a mathematical necessity for minimizing the total energy of ensemble (without referring to the wave function and Schrödinger equation), and the quantization of energy and angular momentum is a consequence of continuity condition for stationary states.

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