The Dirac Equation: an approach through Geometric Algebra

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ABSTRACT. We solve the Dirac equation for a free electron in a manner similar to Toyoki Koga, but using the geometric theory of Clifford algebras which was initiated by David Hestenes. Our solution exhibits a spinning field, among other things.

Key words: geometric algebra, multivector, Clifford algebra, spinor, spacetime, Dirac equation, Klein-Gordon equation, Zitterbewegung

1 Introduction

The Dirac equation in geometric algebra, as given by Hestenes (see [3]) is (for a free electron)

$$\nabla\psi I\sigma_3 = m\psi\gamma_0. \tag{1}$$

Here ψ is an even multivector field (defined below) in the Clifford algebra of a 4 dimensional real (Minkowski) spacetime, spanned by orthogonal unit vectors $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ parallel to the coordinate axes. We take γ_0 to be timelike with $\gamma_0^2 = 1$ and γ_i spacelike, with square -1, for $i = 1, 2, 3$.

The *spacetime algebra* (STA) is the real Clifford algebra generated by γ_μ ($\mu = 0, 1, 2, 3$), where for $\mu \neq \nu$,

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = \eta_{\mu\nu}.$$

where $\eta_{\mu\nu}$ is the Lorentz metric. The gradient $\nabla\psi$ is defined to be $\gamma_\mu \dfrac{\partial\psi}{\partial x^\mu}$ (the Einstein summation convention applies). A vector space basis for STA is given by

$$\{1\} \cup \{\gamma_\mu | \mu = 0, 1, 2, 3\} \cup \{\gamma_\mu\gamma_\nu | \mu < \nu\} \cup \{I\gamma_\mu | \mu = 0, 1, 2, 3\} \cup \{I\}$$

where $I = \gamma_0\gamma_1\gamma_2\gamma_3$ is called the *pseudoscalar.* The elements $\gamma_\mu\gamma_\nu$ are *bivectors* and $I\gamma_\mu$ are *trivectors.* The members of STA are called *multivectors.* In the basis above, the vectors and trivectors are *odd* and the rest are *even.* The even multivectors form a subalgebra of STA called the *even subalgebra.* It is generated by $\sigma_1, \sigma_2, \sigma_3$ where $\sigma_i = \gamma_i\gamma_0$ satisfy $\sigma_i^2 = 1$ and $\sigma_i\sigma_j + \sigma_j\sigma_i = 0$. Note that $I = \sigma_1\sigma_2\sigma_3$, $I\sigma_3 = \sigma_1\sigma_2 = \gamma_2\gamma_1$ and so on.

STA is called the *geometric algebra of (Minkowski) spacetime* while its even subalgebra is the *geometric algebra of (3-dimensional Euclidean) space.* We assume that the latter algebra describes the world as seen by an observer following a timelike path with unit speed, located at the origin of (3 dimensional) space, and σ_1, σ_2, σ_3 stand for unit vectors along the coordinate axes of 3 dimensional space.

We now describe the relation between the Dirac equation in its traditional matrix form and the the version given by Hestenes. Details are in [1] (and [3]).

Firstly, there is an isomorphism of real vector spaces between 2-dimensional complex spinor space and the 4-dimensional real vector space spanned by 1, $I\sigma_1$, $I\sigma_2$, $I\sigma_3$:

$$|\psi\rangle = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} \leftrightarrow \psi = a^0 + a^1 I\sigma_1 + a^2 I\sigma_2 + a^3 I\sigma_3.$$

This extends to an isomorphism between complex 4-spinor space and the even subalgebra of STA:

$$|\psi\rangle = \begin{pmatrix} |\varphi\rangle \\ |\eta\rangle \end{pmatrix} \leftrightarrow \psi = \varphi + \eta\sigma_3.$$

The Dirac equation in its standard form is

$$i\hat{\gamma}^\mu \frac{\partial}{\partial x^\mu}|\psi\rangle = m|\psi\rangle \tag{2}$$

where $\hat{\gamma}^\mu$ are the Dirac matrices, satisfying

$$(\hat{\gamma}^0)^2 = 1, \quad (\hat{\gamma}^i)^2 = -1 \quad (i = 1, 2, 3), \quad \hat{\gamma}^\mu\hat{\gamma}^\nu + \hat{\gamma}^\nu\hat{\gamma}^\mu = 0 \quad (\mu \neq \nu).$$

Note that $\hat{\gamma}_\mu = \eta_{\mu\nu}\hat{\gamma}^\mu$. The action of the various operators is translated as follows:

$$\hat{\gamma}_\mu|\psi\rangle \leftrightarrow \gamma_\mu\psi\gamma_0,$$
$$i|\psi\rangle \leftrightarrow \psi I\sigma_3.$$

The reader can easily verify the correspondence between (1) and (2).

2 The solution to the Dirac equation

The Klein-Gordon equation is $\nabla^2\varphi + m^2\varphi = 0$ where φ is a multivector field. The Laplacian here is the dot product of the gradient ∇ defined earlier with itself. It is well-known that from a solution of this equation we can get a solution of the Dirac equation.

In geometric algebra this works out as follows: if $\nabla^2\varphi + m^2\varphi = 0$ then $\psi = \nabla\varphi I\sigma_3 + m\varphi\gamma_0$ is a solution of the Dirac equation in STA. If φ is odd then ψ is even and vice versa ([2], Section 10.1).

Koga ([5], [6], [7], [8]) worked out a solution to the Dirac equation, starting with a solution to the Klein-Gordon equation, following an idea of de Broglie from the 1920s. He started with

$$\varphi = ae^{iS} \tag{3}$$

where a and S are real scalar fields in spacetime. (For convenience, we choose units such that the Planck constant and the speed of light have the value 1.)

For a free electron, expressions can be written out for a and S:

$$\begin{aligned} S &= -Et + \mathbf{p}\cdot\mathbf{r}, \\ a &= \exp(-\kappa|\mathbf{r}'|)/|\mathbf{r}'| \end{aligned}$$

where $\mathbf{r}$ is the position of the centre of the electron (in 3-space),

$$\mathbf{r}' = (\mathbf{r} - \mathbf{u}t)/(1-u^2)^{1/2}$$

where $\mathbf{u}$ is the velocity of the electron (in our inertial frame), $u = |\mathbf{u}|$, κ is a positive constant, $\mathbf{p} = \mathbf{u}E$ is the momentum and $E^2 = (m^2 - \kappa^2)/(1-u^2)$ where E is the energy.

We now consider a frame in which the electron is at rest: $\mathbf{u} = 0$, $\mathbf{p} = 0$, $|\mathbf{r}'| = |\mathbf{r}|$. We take the centre of the electron as the origin.

We write $\phi = ae^{SI\sigma_3}$ for a solution to the Klein-Gordon equation. We can do this because $(I\sigma_3)^2 = -1$ in STA. However, unlike de Broglie and Koga, by replacing $i = \sqrt{-1}$ with $I\sigma_3$ we are giving a special role to the x^3-axis, as we will see soon.

Since we want ψ to be an even multivector, we make our Klein-Gordon field odd by multiplying by γ_0. Thus

$$\psi = (\nabla\varphi)I\sigma_3 + m\varphi\gamma_0 \quad \text{where} \quad \varphi = ae^{SI\sigma_3}\gamma_0.$$

We now have $S = -Et$ and

$$a = \exp(-\kappa r)/r \tag{4}$$

where $r = |\mathbf{r}|$, $\mathbf{r} = x^1\gamma_1 + x^2\gamma_2 + x^3\gamma_3$.

In order to get a formula for ψ, it suffices to observe that $\nabla a = a\mathbf{R}$ where $\mathbf{R} = \mathbf{r}\left(\frac{1}{r^2} + \frac{\kappa}{r}\right)$ and so

$$\psi = \mathbf{R}\varphi\gamma_0 I\sigma_3 + (E+m)\varphi\gamma_0. \tag{5}$$

3 Interpretation of the solution

We now turn to the geometrical (kinematical) meaning of the expression we have obtained for the multivector field ψ.

For the physical interpretation of ψ, its relation to the Maxwell field etc., see Koga ([5]–[8]).

It should be emphasised that this is neither conventional quantum mechanics nor the de Broglie-Bohm theory ([4]).

Following Koga, we interpret φ and ψ above as indicating that the electron is a localised field in spacetime, rather than a sizeless particle. The real field $a = \exp(-\kappa r)/r$ has a singularity at $r = 0$. The physical meaning of this seems to be that that our expressions are approximate descriptions of reality; they are reasonably accurate at points that are not too close to the singular point.

The term $(E+m)\varphi\gamma_0$ is nothing but a Klein-Gordon field (which is now an even multivector as desired). The other term is $a\left(\frac{1}{r^2} + \frac{\kappa}{r}\right)\mathbf{r}\gamma_0 I\sigma_3$. Here $a\left(\frac{1}{r^2} + \frac{\kappa}{r}\right)$ is a spherically symmetric scalar field. So the first term on the right hand side of (5) is

$$a\left(\frac{1}{r^2} + \frac{\kappa}{r}\right)\mathbf{r}e^{SI\sigma_3}\gamma_0 I\sigma_3.$$

We note that γ_0 commutes with $e^{SI\sigma_3} = \cos S + (\sin S)I\sigma_3$ and

$$\begin{aligned}\mathbf{r}\gamma_0 &= (x^1\gamma_1 + x^2\gamma_2 + x^3\gamma_3)\gamma_0 \\ &= x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3.\end{aligned}$$

We will come to the last factor $I\sigma_3$ later. We are now concerned with

$$(x^1\sigma_1+x^2\sigma_2+x^3\sigma_3)e^{SI\sigma_3} = (x^1\sigma_1+x^2\sigma_2)e^{SI\sigma_3}+x^3\sigma_3+x^3\sigma_3(e^{SI\sigma_3}-1). \tag{6}$$

The first two of the three terms on the right hand side give

$$e^{-(S/2)I\sigma_3}(x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3)e^{(S/2)I\sigma_3} \tag{7}$$

which is the result of rotating the vector $x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3$ through the angle S in the $\sigma_1\sigma_2$ plane (or, in other words, about the σ_3 axis), where $S = -Et$ (see the discussion of rotations in [3], Sections 2.7 and 4.2). This stands for a field rotating with a constant angular velocity.

The factor $I\sigma_3 = \exp(\pi/2I\sigma_3)$ simply rotates the above expression (6) through a further angle of $\pi/2$.

We now consider the expression $x^3\sigma_3(e^{(S/2)I\sigma_3}-1)I\sigma_3$. This is a field whose value at each point (x^1, x^2, x^3) is independent of x^1 and x^2. It is proportional to x^3 and thus represents an oscillatory motion that gets larger with $|x^3|$. We conclude that this motion is similar or analogous to what Schrödinger called Zitterbewegung, which is still controversial in the sense that there is no agreement on whether it is related to electron spin or not. In our context, of course, it is multiplied by the scalar factor $a\left(\frac{1}{r^2}+\frac{\kappa}{r}\right)$ and hence is a localised field which vanishes at infinity.

4 Conclusion

Thus, we conclude that the solution ψ of the Dirac equation is the sum of three fields:

1. a field spinning about the x^3-axis (which, because of our choices, turns out to be the axis of symmetry of the electron),

2. an oscillatory motion similar to Zitterbewegung, and

3. a solution to the Klein-Gordon equation.

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