Lorentzian Symmetry in an Expanding Hyperbolic Geometry: The Connection with Minkowski Space.

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ABSTRACT. In a hyperbolic universe the 4-vectors and subspaces of velocity and position are isomorphic examples of Lorentzian symmetry and pseudospherical geometry. Light is shed on this connection by examining a hyperbolic 3-space as the subspace of an O(1,3) 4-space subject to the constraint of a pseudosphere of imaginary radius $R = i\rho$. This symmetry persists if the hyperradius of position space expands with cosmological time, as in the Hubble distance $\rho_{\mathrm{H}} = ct_{\mathrm{H}}$. Such an expanding noneuclidean space with a negative curvature changing in time leads automatically to the appearance of a differential 4-vector with an imaginary timelike component $ic\Delta t$. This provides a new understanding of the origin of the time-varying coordinate in the Minkowski space-time four-vector. The position-velocity symmetry supports an enlarged Lorentz group based on the direct product of separate Lorentz subgroups in position and velocity spaces. The geometric isometry of negative curvature in both position and velocity spaces now provides additional evidence for a previously demonstrated reciprocal relationship between the Lorentz transformation phenomenology and the Hubble velocity-distance law.

1. Special relativity in a position space of homogeneous negative curvature.

It is largely through their symmetry behaviors that we can recognize the isomorphism that links the Lorentz group properties of relativistic velocity space with those of a hyperbolic position space. To exploit these effectively, it is necessary to express the geometries and kinematics on both sides of the phase space in a common, symmetrical notation, so they can be compared and used together smoothly. Historically, in special relativity the Lorentz group properties of both velocity and position-time spaces have been expressed in the conventions of 4-vector and tensor expressions, and the 4-dimensional space-time of the Minkowski world is necessarily taken as flat. On the other hand, the understanding of hyperbolic space in mathematics came about through the development of the non-Euclidean geometry of the hyperbolic plane and three-space founded in the work of Gauss, Bolyai and Lobatchevskii. This geometry was known to be that of a generalized sphere of imaginary radius. The common thread, recognized by Minkowski in the electrodynamics of moving media as developed by Lorentz, Einstein and Poincaré, is that such operations as the Lorentz transformation or Einstein's addition of velocities have all the properties of generalized rotations in such a space. The group of these rotations is O(1,3), the abstract group isomorphic with the homogeneous Lorentz group.

The 4-vector notation for a flat 4-space universally used in special relativity makes no obvious contact with the conceptual notions of a hyperbolically curved 3-space used in non-Euclidean geometry, or even in cosmological models taking their origin in general relativity.

The relationships between the two superficially different embodiments of the mathematical group O(1,3) can be expressed in an illuminating way by using a vector notation, adapted to geodesic vectors in hyperbolic 3-space, and appropriate for both position and velocity space expressions. This supplements and connects closely with the 4-space notation that is standard for dealing with Minkowski space. The connection will be demonstrated and discussed in Sections 1, 3 and 4 of this paper. A number of useful expressions of the algebra of hyperbolic vectors will be presented in Section 2.

Using these concepts, a different light is shed on old relationships. In particular, a new understanding emerges of a connection between the geometry of a hyperbolic position space expanding in time, and the relationships between differential intervals in space and in time as they are modified by such processes as the Lorentz transformation under a velocity boost. This offers a new explanation for the appearance of the imaginary timelike element $ic\Delta t$ in the four-space introduced by Poincaré [1] and systematically exploited by Minkowski [2]. It also confirms a reciprocal relationship previously demonstrated between the Lorentz transformation and the Hubble expansion and reinforces the resulting extension of special relativity to incorporate the direct product of two Lorentz subgroups $\mathrm{L}^2 = \mathrm{L}_{\mathrm{vel}} \otimes \mathrm{L}_{\mathrm{pos}}$ and a modified Poincaré group [5].

a. Local Minkowski 4-space and hyperbolic geometry:

The formalism that is now universally used to express the relationships of Lorentz symmetry in special relativistic applications exploits the pseudosymmetry between space and time in a 4-space with the Minkowski metric tensor

$$g_{\iota\kappa} = g_{\mathrm{diag}}\left(-1,+1,+1,+1\right) \tag{1}$$

while assuming that the coordinate x_0 is related to time as the other three are to space:

$$x_0 = -x^0 = c\tau\,, \tag{2}$$

where c is taken to be invariant and τ is a time measured with respect to a local, recent origin. Minkowski [2] and Sommerfeld [3] recognized the connection between linear transformations in this 4-space geometry and the rotations of a hypersphere of imaginary radius $i\rho$ with the metric (1):

$$x^\iota x_\iota = g_{\iota\kappa} x^\iota x^\kappa = -x_0^2 + x_1^2 + x_2^2 + x_3^2 = \left(i\rho\right)^2. \tag{3}$$

That this is equivalent to an operation in a noneuclidean (hyperbolic) geometry was explicitly recognized by others immediately after the appearance of Minkowski's first paper [4].

The development of special relativity in practice followed the course of treating Minkowski space as a flat 4-space with the metric tensor of Eq. (1), and applying it as a local and approximate source of symmetries on a microscopic scale in the background of the gravitational universe of general relativity. In that case, Eq. (3) is taken in its differential form,

$$ds^2 = c^2 d\tau^2 - dr^2 = c^2 d\tau^2 - \left(dx_1^2 + dx_2^2 + dx_3^2\right), \tag{4}$$

and no particular significance is given to the hyperbolic curvature length ρ implied by Eq. (3). I shall here follow the opposite course, of exploring the consequences of the model of a pure hyperbolic geometry when extended to a macroscopic and even cosmological scale.

b. Lorentz symmetry and the coordinates of hyperbolic geometry:

Eq. (3) is a constraint equation, expressing the fact that the four variables x^ι or x_ι are confined to three degrees of freedom orthogonal to the pseudoradius ρ. The four variables x_ι with the constraint then provide a parametric description of the entire hyperbolic 3-space, just as the three coordinates (x_1, x_2, x_3) with the constraint condition $\Sigma_i x_i^2 = R^2$ provide a full parametrization of the 2-space of the surface of a sphere of radius R.

In the case of a hyperbolic geometry, the coordinates of the three degrees of freedom compatible with the constraint of Eq. (3) can be made explicit by introducing an orthogonal curvilinear coordinate system $(\rho, \xi, \theta, \phi)$ adapted to the hyperbolic geometry of the constraint:

$$\begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \rho \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \rho \begin{pmatrix} \cosh\xi \\ \sinh\xi \sin\theta \sin\phi \\ \sinh\xi \sin\theta \cos\phi \\ \sinh\xi \cos\theta \end{pmatrix}. \tag{5}$$

The pseudoradius ρ is clearly invariant under Lorentz velocity boosts, under rotations, and under geodesic translations (pseudorotations) in the 3-space of the hyperbolic variables of arc and axis (ξ, θ, ϕ). Under all these operations the Minkowski 4-vector $\{\xi_\iota\}$ is covariant, but its variation is subject to the invariant constraint of Eq. (3),

$$\xi^\iota \xi_\iota = -1. \tag{6}$$

The hyperbolic constraint is thus compatible with all the symmetries of the homogeneous Lorentz group. The hyperbolic 3-space orthogonal to ρ can be parametrized either by the arc and axis coordinates (ξ,θ,ϕ) or by the components (ξ_1,ξ_2,ξ_3) of a hyperbolic three-vector which will here be denoted by $\widehat{\boldsymbol{\xi}}$:

$$\widehat{\boldsymbol{\xi}} = \mathbf{x}/\rho = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} / \rho = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \sinh\xi \begin{pmatrix} \sin\theta\sin\phi \\ \sin\theta\cos\phi \\ \cos\theta \end{pmatrix}. \tag{7}$$

A special notation $\widehat{\boldsymbol{\xi}}$ is adopted to distinguish hyperbolic 3-vectors from Euclidean ones because their squared norm is not $\boldsymbol{\xi}\cdot\boldsymbol{\xi} \neq \xi^2$, but $\widehat{\boldsymbol{\xi}}\cdot\widehat{\boldsymbol{\xi}} = \sinh^2\xi$. Like Euclidean vectors, hyperbolic 3-vectors such as $\widehat{\boldsymbol{\xi}}$ are not covariant under velocity boosts, they must be transformed as part of the 4-vector $\{\xi_\iota\} = \begin{pmatrix} \xi_0 \\ \widehat{\boldsymbol{\xi}} \end{pmatrix}$. They are vectors in the rest system or within any constant velocity frame.

The description of a hyperbolic position space by the 3-vector $\widehat{\boldsymbol{\xi}} = (\xi_1,\xi_2,\xi_3)$ or (ξ,θ,ϕ), Eq. (7), together with a fixed curvature length ρ, is fully equivalent to the standard treatment by the 4-vectors $\{\xi_\iota\}$ of Eq. (5) with the constraint of Eq. (6). The 4-vector notation $\{\xi_\iota\}$ provides a parametric description of the 3-space subject to the constraint. The 3-vector notation is particularly useful in the transition between non-relativistic and relativistic geometry and kinematics, because the vector $\mathbf{x} = \rho\widehat{\boldsymbol{\xi}}$ extrapolates directly to the non-relativistic Newtonian and Galilean limit. In a hyperbolic geometry the fourth Minkowski coordinate $x_0 = \rho\cosh\xi$ or $\xi_0 = \cosh\xi$ is redundant, and can always be expressed in closed form as a function of the independent variables.

A simple extension of the vector algebra of Euclidean space and Galilean kinematics to hyperbolic geometry and to the corresponding Lorentzian kinematics, which will be developed in Section 2 below, provides tools especially adapted to extend relativistic kinematics into the expanding hyperbolic geometry of the open Friedman, Robertson, Walker (FRW) cosmological model.

c. Time dependence in an expanding cosmological geometry:

The Minkowski 4-space structure of Eq. (5) when described by the four orthogonal coordinates (ρ,ξ,θ,ϕ) or (ρ,ξ_1,ξ_2,ξ_3) is well suited to describing the FRW cosmological geometry expanding in time; this can be assumed to be spatially isotropic and homogeneous from the point of view of any observer at any cosmological time *t*,

$$\rho = \rho(t) \cong ct \, , \tag{8}$$

with *t* measured from the big bang. This cosmic time must be carefully distinguished from the local, observable, frame-dependent time variable τ of Eq. (2). The connection between the variables *t* and τ is part of a well-defined change in coordinate systems between the global hyperbolic coordinates and the local Minkowski ones. This will be given below, in subsection 1.d.

In contrast to τ, the cosmic time *t* itself, like ρ, is a Lorentz invariant. It is orthogonal to the hyperbolic 3-space described by the coordinate system (ξ,θ,ϕ) or (ξ_1,ξ_2,ξ_3). It is exclusively in that 3-dimensional subspace of space-time that all the operations of Lorentz transformation by velocity boosts, of rotation of axes, and of geodesic translations in the position space take effect.

In practical applications describing local systems the radial length scale ρ at the present epoch is measured by the cosmological Hubble length $\rho_{\mathrm{H}} = ct_{\mathrm{H}}$, or equivalently by the Hubble age of the universe t_{H}. In the laboratory or rest system, the range of variation of the position variable $|\Delta\bar{x}|$ sets an upper bound on the range of the hyperbolic arc ξ :

$$\left|\Delta\bar{\xi}\right|_{\text{rest}} \le |\Delta\bar{x}| / \rho_{\text{H}} \ . \tag{9}$$

Being the ratio of a local length to the Hubble distance of cosmology, $\bar{\xi}$ or $\Delta\bar{\xi}$ is exceedingly small and $\cosh\bar{\xi} \to 1$. This is true, however, only in the rest system. When the phenomenology involves relativistic velocity effects at large fractions of *c*, significantly large changes in ξ may result from a velocity boost between the observer and the system observed,

$$\hat{\xi}_{(\text{rest})} \xrightarrow{(\text{boost})} \hat{\xi}'_{\text{obs}} \ . \tag{10}$$

The magnitude of an observed time interval can be expressed as:

$$\Delta\tau_{ab} = c^{-1}\left(x_0\left[a\right] - x_0\left[b\right]\right) = c^{-1}\left[\rho\left(t_a\right)\cosh\xi_a - \rho\left(t_b\right)\cosh\xi_b\right]. \tag{11}$$

In the rest system, this will be essentially independent of the factors $\cosh\xi_a$ and $\cosh\xi_b$ because, by Eq. (9), their arguments $\left(\xi_a, \xi_b\right)$ in a local region are negligibly small. Time intervals in the rest system—proper time intervals—are therefore governed by the cosmic time variable,

$$\left(\Delta\tau_{ab}\right)_{\text{rest}} = \Delta t_{ab} \ . \tag{12}$$

If this time interval is being observed in a system in motion with respect to the observer, ξ_a and ξ_b will both be altered by the velocity boost,

$$\left(\xi_a, \xi_b\right) \xrightarrow{(\text{boost})} \left(\xi'_a, \xi'_b\right), \tag{13}$$

and the $\cosh\xi'$ factors will become greater than 1, thus increasing the observed time interval by the usual Lorentz time dilatation; this can be evaluated at the average value ξ'_{ave} :

$$\Delta\tau_{ab}\left(\xi'_{\text{ave}}\right) = \Delta t_{ab}\cosh\xi'_{\text{ave}} \ge \left(\Delta\tau_{ab}\right)_{\text{rest}} \ . \tag{14}$$

d. The connection between cosmological hyperbolic coordinates and those of local Minkowski space:

The natural symmetry-adapted coordinates of time and position in a cosmologically expanding hyperbolic geometry are the variables (t,ξ,θ,ϕ) or $(t,\boldsymbol{\xi})$ of Eqs. (1.5) and (1.8), i.e., of the 4-vector

$$\begin{pmatrix} x_0 \\ \mathbf{x} \end{pmatrix} = ct\begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = ct\begin{pmatrix} \cosh\xi \\ \sinh\xi\sin\theta\sin\phi \\ \sinh\xi\sin\theta\cos\phi \\ \sinh\xi\cos\theta \end{pmatrix}. \tag{15}$$

The Minkowski 4-space description uses local coordinates $(\tau,\mathbf{r}) = (\tau,r,\theta_{\mathbf{r}},\phi_{\mathbf{r}})$, adapted to describing a small local region of space-time, with a differential 4-vector of the form

$$\begin{pmatrix} c\tau \\ \mathbf{r} \end{pmatrix} = \begin{pmatrix} c\tau \\ r\sin\theta_{\mathbf{r}}\sin\phi_{\mathbf{r}} \\ r\sin\theta_{\mathbf{r}}\cos\phi_{\mathbf{r}} \\ r\cos\theta_{\mathbf{r}} \end{pmatrix}. \tag{16}$$

We seek the connection between these two sets of coordinates. The dependence on the directions (θ,ϕ) is nonessential, and the connection can be examined in the case of a single spatial dimension. The global 4-vector Eq. (15) then reduces to

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = ct\begin{pmatrix} \cosh\xi \\ \sinh\xi \end{pmatrix}, \tag{17}$$

while the Minkowski 4-vector of Eq. (16) becomes simply

$$\begin{pmatrix} c\tau \\ r \end{pmatrix}. \tag{18}$$

These two expressions allow us to establish a connection between the local Minkowski coordinates and the global hyperbolic ones in two alternative forms, a general one valid over a wide region, and a local one valid especially in the differential limit.

(1) The connection in its general form:

While the Minkowski space-time vector, Eq. (18), is usually employed locally, the connection implied by a comparison with Eq. (17) can be established in such a way as to be valid generally.

In the presence of the cosmological expansion, the use of a cosmic time coordinate with a remote and approximately datable origin is obviously called for. The measure of its remoteness from us is the Hubble time t_{H}, about 1.37×10^{10} y. The Minkowski origin of time is always assumed to be fixed locally at some convenient recent time—a condition that is required for precision measurements and estimates. For position space, the space of the variables $\mathbf{x}, \mathbf{r}$, and $\boldsymbol{\xi}$, a nearby origin $\mathbf{x}=\mathbf{r}=0$ can always be chosen. To connect the coordinate sets we are at liberty to choose two matching conditions, which I take to be that (a) at $\mathbf{r}=0$ the zero of τ coincides with $t=t_{\mathrm{H}}$, and (b) at the space and local time origin $(\mathbf{r}=0, \tau=0)$, clocks in both time systems must run at the same rate:

$$\left(\tau = t - t_{\mathrm{H}} = \Delta t\right)_{\mathbf{r}=0}, \text{ and } \left(\frac{dx_0}{dt} = \frac{dr_0}{d\tau}\right)_{\mathbf{r}=0,\tau=0}. \tag{19}$$

These conditions are satisfied if

$$\begin{aligned} \tau &= \left(t - t_{\mathrm{H}}\right)\cosh\xi, \\ r &= \rho_{\mathrm{H}}\sinh\xi = ct_{\mathrm{H}}\sinh\xi. \end{aligned} \tag{20}$$

This allows us to complete the matching conditions for locations remote from the local origin.

(2) The connection in the local limit:

When the Minkowski 4-vector is used locally, in the spirit of Eq.(4), Eq. (18) will represent a difference between two events, and in that case can be written

$$\Delta\begin{pmatrix} c\tau \\ r \end{pmatrix} \rightarrow \begin{pmatrix} c\Delta\tau \\ \Delta r \end{pmatrix}. \tag{21}$$

This must be compared not with the global hyperbolic vector of Eq. (17), but with its differential form as a vector between two events. If we consider these to be observed at the average time and location $\left(\bar{t} = t_{\mathrm{H}}, \bar{\xi}\right)$, i.e., between two events at

$$\begin{pmatrix} x_0' \\ x_1' \end{pmatrix} = c\left(t_{\mathrm{H}} + \Delta t\right)\begin{pmatrix} \cosh\left(\bar{\xi} + \Delta\xi\right) \\ \sinh\left(\bar{\xi} + \Delta\xi\right) \end{pmatrix} \text{ and } \begin{pmatrix} x_0'' \\ x_1'' \end{pmatrix} = ct_{\mathrm{H}}\begin{pmatrix} \cosh\bar{\xi} \\ \sinh\bar{\xi} \end{pmatrix}, \tag{22}$$

we obtain, to first order,

$$\Delta\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = c\Delta t\begin{pmatrix} \cosh\bar{\xi} \\ \sinh\bar{\xi} \end{pmatrix} + ct_{\mathrm{H}}\Delta\xi\begin{pmatrix} \sinh\bar{\xi} \\ \cosh\bar{\xi} \end{pmatrix}. \tag{23}$$

Using Eq. (9) we can set $\bar{\xi} \rightarrow 0$ and evaluate Eq. (23) as

$$\Delta\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = c\begin{pmatrix} \Delta t \\ t_{\mathrm{H}}\Delta\xi \end{pmatrix}. \tag{24}$$

This is exactly in the form of the Minkowski expression of Eq. (21), and requires us to make the identification

$$\Delta\tau = \Delta t, \qquad \Delta r = ct_{\mathrm{H}}\Delta\xi \,. \tag{25}$$

These agree with Eq. (20) in the limit as $\xi \to 0$.

We see thus that the Minkowski space-time vector of the form of Eq. (21)—or, more generally, Eq. (16)—can be identified with the differential expression, Eq. (24), derived from the parametric 4-space vector, Eq. (15), of a hyperbolic position space whose curvature length

$$\rho = ct = \rho_{\mathrm{H}} + c\Delta t \tag{26}$$

is increasing linearly in cosmic time.

This identification of the specific form of the Minkowski space-time 4-vector with the differential expression of a parametric 4-coordinate description of a hyperbolic 3-space in which a negative curvature is changing in time is an unexpected conclusion. It implies the view that the Lorentz transformation itself carries two important pieces of information: (a) that position 3-space is itself on average hyperbolically curved, and (b) that the magnitude of that curvature is changing in time.

2. The algebra of 3-vectors in hyperbolic triangles:

In a hyperbolic geometry the 4-vector structure of the coordinates of position and time, Eq. (5), is formally fully adapted to the application of the Lorentz transformation under velocity boosts as a linear operation in the Minkowski 4-space. However, a full employment of the 4-space apparatus associated with Eq. (5) runs into the possibility of calculational difficulties because of the extraordinary mismatch of magnitudes between x_0, dominated by the cosmological Hubble length ρ_{H} itself, and the components x_i of the 3-vector $\mathbf{x}$ or $\Delta\mathbf{x}$ appropriate to a local application. It is therefore advantageous to make use of a treatment based entirely in the 3-vector space of the hyperbolic vectors $\hat{\boldsymbol{\xi}} = (\xi_1, \xi_2, \xi_3)$. It is easy to show that all the important linear operations of rotation and pseudorotation in the Minkowski 4-space can be described by a nonlinear extension of the vector algebra of Euclidean 3-space to the analogous composition of hyperbolic vectors like $\hat{\boldsymbol{\xi}} = (\xi_1, \xi_2, \xi_3)$ in a hyperbolic space. The operations of the resulting hyperbolic vector algebra can all be expressed in the language of hyperbolic trigonometry as it is applied to the composition of sides and angles in hyperbolic triangles.

a. Vector addition in hyperbolic triangles:

We denote as a hyperbolic vector $\hat{\boldsymbol{\alpha}}$ a geodesic arc in a hyperbolic space, normalized to the pseudoradius of curvature of that space ρ_{α} as in Eq. (7), and with the arc-and-angle coordinates $(\alpha, \theta_{\alpha}, \phi_{\alpha})$ and the 3-vector components

$$\hat{\boldsymbol{\alpha}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \sinh\alpha \begin{pmatrix} \sin\theta_{\alpha}\sin\phi_{\alpha} \\ \sin\theta_{\alpha}\cos\phi_{\alpha} \\ \cos\theta_{\alpha} \end{pmatrix}. \tag{27}$$

The addition of vectors as the sum of sides in a triangle remains applicable in hyperbolic geometry, where it is carried out by the rules of hyperbolic trigonometry. Like the addition of sides in spherical triangles, such addition is no longer commutative. Hyperbolic vector addition will be identified here by the special summation notation $\hat{+}$. In the triangle

$$\hat{\boldsymbol{\alpha}} \,\hat{+}\, \hat{\boldsymbol{\beta}} \,\hat{+}\, \hat{\boldsymbol{\gamma}} = 0, \tag{28}$$

where the interior angles opposite those sides are *A*, *B*, and *C*, the addition of sides follows the hyperbolic law of cosines:

$$\cosh\gamma = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta\cos C. \tag{29}$$

In velocity space this vector addition in a hyperbolic triangle reproduces the results of Einstein vector addition of velocities. This can be supplemented by the hyperbolic law of sines to obtain the necessary angles:

$$\frac{\sin A}{\sinh \alpha} = \frac{\sin B}{\sinh \beta} = \frac{\sin C}{\sinh \gamma} = \kappa_{\alpha\beta\gamma}\,, \tag{30}$$

where

$$\kappa_{\alpha\beta\gamma} = \frac{\left[-1 + \cosh^2 \alpha + \cosh^2 \beta + \cosh^2 \gamma + 2\cosh\alpha \cosh\beta \cosh\gamma\right]^{1/2}}{\sinh\alpha \sinh\beta \sinh\gamma}$$

and (31)

$$\kappa_{\alpha\beta\gamma}^{-1} = \frac{\left[1 - \cos^2 A - \cos^2 B - \cos^2 C - 2\cos A \cos B \cos C\right]^{1/2}}{\sin A \sin B \sin C}.$$

b. The rotations of parallel transport:

In hyperbolic as in spherical trigonometry, a sequence of non-collinear translations has a secondary effect in addition to the vector addition in the triangle: the transport of a directed object over a finite nongeodesic path induces an apparent rotation. In the product of Lorentz matrices

$$K\left(\widehat{\boldsymbol{\beta}}\right) K\left(\widehat{\boldsymbol{\gamma}}\right) = L\left(\widehat{\boldsymbol{\beta}}, \mathbf{0}\right) L\left(\widehat{\boldsymbol{\gamma}}, \mathbf{0}\right) = L\left(\widehat{\boldsymbol{\alpha}}, \breve{\boldsymbol{\omega}}_{\alpha\beta\gamma}\right) = K\left(\widehat{\boldsymbol{\alpha}}\right) R\left(\breve{\boldsymbol{\omega}}_{\alpha\beta\gamma}\right) \tag{32}$$

the second order rotation $\breve{\boldsymbol{\omega}}_{\alpha\beta\gamma}$ adjusts for the angular defect of the hyperbolic triangle, measured by its directed area. This area can be expressed as a hyperbolic vector product, for which we can use the symbol " $\hat{\times}$ ":

$$\breve{\boldsymbol{\omega}}_{\alpha\beta\gamma} = \left[\widehat{\boldsymbol{\alpha}} \,\hat{\times}\, \widehat{\boldsymbol{\beta}}\right]/2 = \left[\widehat{\boldsymbol{\beta}} \,\hat{\times}\, \widehat{\boldsymbol{\gamma}}\right]/2 = \left[\widehat{\boldsymbol{\gamma}} \,\hat{\times}\, \widehat{\boldsymbol{\alpha}}\right]/2. \tag{33}$$

This rotation arises in any curved space as a consequence of the parallel transport of a local vector describing the orientation of an infinitesimal test body carried over a nongeodesic path. Its exact magnitude $\omega_{\alpha\beta\gamma}$ is given by Euler's formula,

$$\cos\left(\omega_{\alpha\beta\gamma}/2\right)=\frac{1+\cosh\alpha+\cosh\beta+\cosh\gamma}{4\cosh\left(\alpha/2\right)\cosh\left(\beta/2\right)\cosh\left(\gamma/2\right)}, \tag{34}$$

and equivalently by an expression that illustrates the connection with the area of the triangle and establishes the sign of $\omega_{\alpha\beta\gamma}$:

$$\sin\left(\omega_{\alpha\beta\gamma}/2\right)=\frac{\sinh\alpha\sinh\beta\ \sin C}{4\cosh\left(\alpha/2\right)\cosh\left(\beta/2\right)\cosh\left(\gamma/2\right)}. \tag{35}$$

In velocity space a similar rotation is responsible for the well-known Thomas precession correction in atomic hyperfine spectra. This is a physical consequence of the velocity-space rotation:

$$\delta\breve{\boldsymbol{\omega}}_{\text{vel}}=\oint\widehat{\boldsymbol{\zeta}}\,\hat{\times}\,d\widehat{\boldsymbol{\zeta}}/2\cong\oint\mathbf{v}\times d\mathbf{v}/\left(2c^{2}\right). \tag{36}$$

The rotation factor $R\left(\breve{\boldsymbol{\omega}}_{\alpha\beta\gamma}\right)$ in Eq (32) appears in the important product and commutator relationship

$$K\left(\widehat{\boldsymbol{\alpha}}\right)K\left(\widehat{\boldsymbol{\beta}}\right)=\left[K\left(\widehat{\boldsymbol{\alpha}}\right),K\left(\widehat{\boldsymbol{\beta}}\right)\right]/2=K\left(\widehat{\boldsymbol{\alpha}}\,\hat{+}\,\widehat{\boldsymbol{\beta}}\right)R\left(\widehat{\boldsymbol{\alpha}}\,\hat{\times}\,\widehat{\boldsymbol{\beta}}/2\right). \tag{37}$$

In practice, the induced rotation will be very small, because it contains the curvature-dependent factor $\left(\alpha\beta\cong\Delta x_{\alpha}\Delta x_{\beta}/\rho^{2}\right)$. Its role is nonetheless very important for an understanding of the physics of the system. In describing the effect of displacements in a hyperbolic position or velocity space we shall therefore make use of Lorentz matrices of the general form

$$L\left(\boldsymbol{\eta},\boldsymbol{\omega}_{\eta}\right)=K\left(\boldsymbol{\eta}\right)R\left(\boldsymbol{\omega}_{\eta}\right), \tag{38}$$

where the subscript on the rotation vector $\boldsymbol{\omega}_\eta$ identifies its association with $\boldsymbol{\eta}$ as part of the six-parameter entity $\left(\boldsymbol{\eta},\boldsymbol{\omega}_\eta\right)$.

c. A corollary: vector algebra in spherical triangles:

Precisely the same considerations apply in spherical triangles as in hyperbolic ones. We shall use the notation $\breve{\boldsymbol{\omega}}$ for a geodesic vector in a spherical geometry. The analytic properties

$$\sin i\alpha = i\sinh\alpha, \qquad \cos i\alpha = -\cosh\alpha\ , \tag{39}$$

make it easy to convert expressions between spherical and hyperbolic trigonometry. Some changes in notation must follow. We can now assume a spherical triangle

$$\breve{\boldsymbol{\chi}} \breve{+} \breve{\boldsymbol{\psi}} \breve{+} \breve{\boldsymbol{\omega}} = 0 \tag{40}$$

where the interior angles opposite those sides are *A*, *B*, and *C*, and the addition of sides follows the spherical law of cosines:

$$\cos\omega = \cos\chi\cos\psi - \sin\chi\sin\psi\cos C\ . \tag{41}$$

The spherical law of sines is:

$$\frac{\sin A}{\sin\chi} = \frac{\sin B}{\sin\psi} = \frac{\sin C}{\sin\omega} = \kappa_{\chi\psi\omega}\ , \tag{42}$$

where

$$\begin{aligned} \kappa_{\chi\psi\omega} &= \frac{\left[1-\cos^2\chi-\cos^2\psi-\cos^2\omega+2\cos\chi\cos\psi\cos\omega\right]^{1/2}}{\sin\chi\sin\psi\sin\omega} \quad \text{and} \\ \kappa_{\chi\psi\omega}^{-1} &= \frac{\left[1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C\right]^{1/2}}{\sin A\sin B\sin C}\ . \end{aligned} \tag{43}$$

For the spherical vector product we can use the symbol "$\breve{\times}$", and write

$$\breve{\boldsymbol{\lambda}}_{\chi\psi\omega} = [\breve{\boldsymbol{\chi}} \breve{\times} \breve{\boldsymbol{\psi}}]/2 = [\breve{\boldsymbol{\psi}} \breve{\times} \breve{\boldsymbol{\omega}}]/2 = [\breve{\boldsymbol{\omega}} \breve{\times} \breve{\boldsymbol{\chi}}]/2. \quad (44)$$

It is the area of the spherical triangle, and its magnitude $\lambda_{\chi\psi\omega}$ is given by Euler's formula,

$$\cos\left(\lambda_{\chi\psi\omega}/2\right) = \frac{1+\cos\chi+\cos\psi+\cos\omega}{4\cos(\chi/2)\cos(\psi/2)\cos(\omega/2)}. \quad (45)$$

3. Four-vectors and the geometries of velocity and position in curved 3-spaces:

The geometries of velocity and position spaces can now be presented in a uniform notation. Following Minkowski, this can be based on the model of the geometry of negative curvature as the geometry of a sphere of imaginary radius.

a. The hyperbolic four-vector of velocity:

In a 1907 lecture [2c] Minkowski recognized that the velocity vector **v** in the electrodynamics of special relativity can be extended to be part of a covariant 4-vector, generating a noneuclidean manifold. He soon improved his derivation, showing that the velocity four-vector takes the form

$$\mathbf{w} = (w_1, w_2, w_3, w_4) = \left(1 - v^2/c^2\right)^{-1/2}\left(v_x, v_y, v_z, ic\right), \quad (46)$$

and that it satisfies the constraint

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = -c^2 \quad (47)$$

of a vector to the 3-surface of a 4-sphere of imaginary radius ic in velocity space [2a]. The fact that the geodesics on a sphere of imaginary radius were known to generate Lobachevsky and Bolyai's noneuclidean geometry established the connection with the noneuclidean manifold that Minkowski made explicit in [2c]. Unable to identify a comparable geometry in position space, he displayed these equations prominently in [2a] but omitted any textual discussion of this velocity symmetry either there or in [2b]. His sudden death in January, 1909, deprived him of any opportunity to take up the matter later.

The similarity between the structure of relativistic velocity space and that of a hyperbolic position space such as that of Eq. (5) is brought out by rewriting Eq. (46) in terms of the rapidity variable ε of relativistic kinematics, defined by:

$$\tanh\varepsilon = v / c \tag{48}$$

so that

$$\beta = \left(1 - v^2 / c^2\right)^{-1/2} = \cosh\varepsilon, \quad \sinh\varepsilon = \beta v / c,$$

and

$$\mathbf{w} = \mathbf{w}\left(c; \varepsilon, \theta_{\text{vel}}, \phi_{\text{vel}}\right) = c\left(\hat{\mathbf{v}}\left[\theta_{\text{vel}}, \phi_{\text{vel}}\right]\sinh\varepsilon, i\cosh\varepsilon\right). \tag{49}$$

The components of the four-vector can then be written as

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = c\begin{pmatrix} \sinh\varepsilon\sin\theta_{\text{vel}}\sin\phi_{\text{vel}} \\ \sinh\varepsilon\sin\theta_{\text{vel}}\cos\phi_{\text{vel}} \\ \sinh\varepsilon\cos\theta_{\text{vel}} \\ i\cosh\varepsilon \end{pmatrix}. \tag{50}$$

These are redundant, satisfying the constraint (47). They represent the magnitude and direction of a geodesic three-vector of rapidity

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\text{vel}} = \left(\varepsilon_{\text{vel}}, \theta_{\text{vel}}, \phi_{\text{vel}}\right) \tag{51}$$

in a homogeneously curved velocity space of negative Gaussian curvature,

$$K_{\text{vel}} = -c^{-2}\,. \tag{52}$$

Comparison of the Minkowski velocity 4-vector expression in Eq. (50) with the hyperbolic position 4-vector in Eq. (5) shows that they are isomorphic in physical content but differ seriously in notation. This is shown by the contrast between the absence of the factor "i" in the term "$x_0 = \cosh\xi$" in Eq. (5) and its presence in "$w_4 = i\cosh\varepsilon$" in Eq. (50). To perfect the parallel, I shall reexpress the description of hyperbolic position space in Minkowski's real-and-imaginary notation instead of in the four-space tensor notation that led to Eq. (5).

b. The hyperbolic four-vector of position:

I shall now write a position four-vector for a hyperbolic geometry characterized by the imaginary spherical radius $R = i\rho$ in the form

$$\mathbf{s} = \left(s_1, s_2, s_3, s_4\right) = \left(s_x, s_y, s_z, iq\right) \tag{53}$$

where the constraint is

$$s_1^2 + s_2^2 + s_3^2 + s_4^2 = -\rho^2\,. \tag{54}$$

The analogue in position space of the rapidity ε in velocity space is the separation η, defined by

$$\sinh\eta = r/\rho\,. \tag{55}$$

It is numerically equal to the parameter ξ in Eq. (5). The fourth component of the 4-vector is

$$-is_4 = q = \rho\sqrt{1 + r^2 / \rho^2} = \rho \cosh \eta \,. \tag{56}$$

The connection between the variables of Eqs. (53) to (56) and those of Eqs. (3) to (6) is

$$x_0 = -x^0 = -is_4 = q = \rho \cosh \eta, \quad x_i = x^i = s_i \,(i = 1,2,3), \quad \xi = \eta \,. \tag{57}$$

The four components of the position 4-vector are now

$$\mathbf{s} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \rho \begin{pmatrix} \sinh \eta \sin \theta_{\text{pos}} \sin \phi_{\text{pos}} \\ \sinh \eta \sin \theta_{\text{pos}} \cos \phi_{\text{pos}} \\ \sinh \eta \cos \theta_{\text{pos}} \\ i \cosh \eta \end{pmatrix} . \tag{58}$$

They are, of course, redundant, and represent the magnitude and direction of a geodesic three-vector of separation

$$\boldsymbol{\eta} = \boldsymbol{\eta}_{\text{pos}} = \left(\eta_{\text{pos}}, \theta_{\text{pos}}, \phi_{\text{pos}}\right) \tag{59}$$

in a homogeneous position three-space whose Gaussian curvature is negative:

$$K_{\text{pos}} = -\rho^{-2} \,. \tag{60}$$

In this form the parallel with the velocity four-vector of Eq. (50) is complete.

c. Four-vectors and the description of homogeneously curved three-spaces:

The four-component structures of Eqs. (50) and (58) have the form of four-vectors but are effectively limited by the respective constraints of Eqs. (47) and (54) to three degrees of freedom in each case. The situation is analogous to the employment of a Cartesian coordinate system to parametrize the surface of a sphere. To represent in a Cartesian framework a space of homogeneous curvature, either positive or negative, requires embedding the curved space in a Cartesian space of one higher dimension. This dimension will appear with real or imaginary signature, depending on whether the sign of the curvature is positive or negative. Equivalently, the change in sign in the diagonal Minkowski metric tensor is necessarily correlated with the appearance of negative curvature in an embedded subspace of spherical symmetry.

In the established tradition of both special and general relativity, *c* is treated as invariant and all the time-dependence of the red-shift is attributed to the cosmological expansion in length, Eq. (26). As a result, the four-vector form for the velocity has always appeared to have an anomalous role in special relativity, with only three real degrees of freedom. That anomaly now takes on a different aspect, and a three-dimensional symmetry between negatively curved spaces of both position and velocity can be recognized instead.

d. The effect of spatial expansion:

When ρ is not constant, but depends on the cosmic time and the Hubble expansion, we know that our best approximation to the hyperbolic length scale expands linearly in the cosmic time variable, $\rho = ct = c\left(t_{\mathrm{H}} + \delta t\right)$. Using this the position 4-vector becomes

$$\begin{aligned} \mathbf{s} &= ct_{\mathrm{H}}\left(1+\delta t / t_{\mathrm{H}}\right)\left(\hat{\mathbf{r}}\sinh\eta, i\cosh\eta\right) \\ &= \left(1+\delta t / t_{\mathrm{H}}\right)\left(x, y, z, i\left[c^2 t_{\mathrm{H}}^2 + r^2\right]^{1/2}\right). \end{aligned} \tag{61}$$

This represents an expanding hypersphere of imaginary radius $R(t) = ict$.

Since all physical observations are made locally at small values of the ratios $\mathbf{r}_j / ct_{\mathrm{H}}$ and $\delta t_j / t_{\mathrm{H}}$, we can use a double expansion in those variables,

$$\begin{aligned}\mathbf{s}_j &= ct\left(\hat{\mathbf{r}}_j \sinh\eta_j, i\cosh\eta_j\right) = \left(1+\delta t_j / t_\mathrm{H}\right)\left(\mathbf{r}_j, ict_\mathrm{H}\cosh\eta_j\right) \\ &= ict_\mathrm{H} + \left(\mathbf{r}_j + ic\delta t_j\right) + \left(\mathbf{r}_j\delta t_j / t_\mathrm{H} + ir_j^2 / 2ct_\mathrm{H}\right) + \ldots,\end{aligned} \tag{62}$$

and evaluate the difference 4-vector between two events $\mathbf{s}_2$ and $\mathbf{s}_1$ to first order as

$$\Delta\mathbf{s} = \mathbf{s}_2 - \mathbf{s}_1 = \left(\Delta\mathbf{r}, ic\Delta t\right) = \left(\Delta x, \Delta y, \Delta z, ic\Delta t\right) \tag{63}$$

where, obviously,

$$\Delta\mathbf{r} = \left(\Delta x, \Delta y, \Delta z\right) = \mathbf{r}_2 - \mathbf{r}_1$$

and

$$\Delta t = t_2 - t_1 = \delta t_2 - \delta t_1 .$$

The local times $\delta t_2, \delta t_1$ are measured from a convenient common local zero where the cosmic time is taken as the Hubble time t_H.

Eq. (63) confirms and provides more detail on the conclusion already attained in Eq. (24). The imaginary factor *i* in fourth component $ic\Delta t$ of the Minkoski space-time four-vector is now seen to be a signal of the negative curvature of position space, while its magnitude and time dependence $c\Delta t$ describe the effect of the Hubble expansion with the velocity *c*.

4. Lorentz symmetries in a system with twofold negative curvature:

The strong parallelism that exists between the geometric structure of relativistic velocity space and that of the hyperbolic position space of the Hubble expanding universe invites further exploitation in the group theory of special relativity. It has long been known that the homogeneous Lorentz group O(3,1) is also the isometry group of noneuclidean geometry in hyperbolic space, but the connection has not been fully exploited in the context of special relativity. I have recently shown [5] that this structural symmetry can be incorporated in a new extended form of special relativity founded on the direct product of two Lorentz subgroups, one generated by the familiar Lorentz velocity boost operator and the other by an analogous operator producing geodesic translations in hyperbolic position space. The double Lorentz group produced by this direct product is also a symplectic group, and leads to a new exploitation of Hamiltonian symmetry in relativistic dynamics. A brief summary of some of the consequences can be given here.

a. The direct product double Lorentz group: Lorentz transformation and Hubble expansion:

In a hyperbolic position geometry, translations in hyperbolic position space generate their own Lorentzian subgroup, with a boost-like operator whose parameter is not the velocity ratio $\Delta v / c$ but a ratio of lengths $\Delta r / \rho_{\mathrm{H}} = \Delta r / ct_{\mathrm{H}}$. For this reason we have not just the velocity Lorentz group $\mathrm{O}(3,1)_{\mathrm{vel}}$ generated by velocity boosts $\Delta v / c$, but also a second Lorentz group $\mathrm{O}(3,1)_{\mathrm{pos}}$ generated by translational shifts in the curved position space measured by $\Delta r / \rho_{\mathrm{H}}$. These two Lorentz groups can now be taken as subgroups, and a larger group can be formed as their direct product,

$$\mathrm{L}^2 = \mathrm{O}(3,1)_{\mathrm{pos}} \otimes \mathrm{O}(3,1)_{\mathrm{vel}} . \qquad (64)$$

This group, the double Lorentz group, has remarkable symmetry properties including that of symplectic symmetry. It is the basis for an enlarged special relativity. One of its important features is a reciprocity between the Lorentz transformation, where the operator describing a boost in velocity space produces observed displacements also in position space, and the Hubble effect, where a translational shift in position space produces a change in the observed velocity. This reciprocal structure as well as other properties and consequences of the enlarged Lorentz group and its associated Poincaré group have been developed and studied in considerable detail in [5].

To form the hyperbolic Poincaré group the group L^2 is extended further by a one-dimensional translational subgroup in the cosmic time t_{cosm},

$$P_{hyp} = L^2 \otimes T(1)_{time} = O(3,1)_{pos} \otimes O(3,1)_{vel} \otimes T(1)_{time}\,. \tag{65}$$

This extended Poincaré group is shown in [5] to extrapolate smoothly to the usual Poincaré group in the appropriate limit of flat position space.

b. The matrices of the double Lorentz group:

A thorough treatment of the double Lorentz group L^2 is given in [5]. The properties of this direct product group are unfamiliar, so the essential steps in its creation will be briefly sketched here.

The fundamental matrices representing operators of the direct product group $O(3,1)_{pos} \otimes O(3,1)_{vel}$ can be constructed from those of the constituent subgroups by forming larger block-diagonal matrices in the pattern

$$\begin{aligned} \mathsf{L}_{pos}\left(\eta_{pos},\omega_{pos}\right) &= \begin{pmatrix} L_{pos}\left(\eta_{pos},\omega_{pos}\right) & 0 \\ 0 & 1 \end{pmatrix}, \\ \mathsf{L}_{vel}\left(\varepsilon_{vel},\omega_{vel}\right) &= \begin{pmatrix} 1 & 0 \\ 0 & L_{vel}\left(\varepsilon_{vel},\omega_{vel}\right) \end{pmatrix}. \end{aligned} \tag{66}$$

In these matrices the first four rows and columns label the coordinates (three real and one imaginary) needed to describe the hyperbolic position subspace, and the second set of four does the same for the velocity subspace.

As in Eq. (38), the matrices of the constituent Lorentz subgroups in (66) can each be written as a product of the matrix of a pure hyperbolic translation $K(\alpha)$ in position or velocity, and an associated pure 3-space rotation matrix $R(\omega_\alpha)$:

$$L_{pos}\left(\eta_{pos},\omega_{pos}\right) = K\left(\eta_{pos}\right)R\left(\omega_{pos}\right),\ L_{vel}\left(\varepsilon_{vel},\omega_{vel}\right) = K\left(\varepsilon_{vel}\right)R\left(\omega_{vel}\right)\,. \tag{67}$$

The variables $(\eta_{\text{pos}},\omega_{\text{pos}})$ and $(\varepsilon_{\text{vel}},\omega_{\text{vel}})$ are those of the respective geometries of position and velocity.

The usual procedure for forming the general operators of a direct product group is to use the product of the matrices of Eq. (66):

$$\Lambda_{\text{geom}}(\eta_{\text{pos}},\omega_{\text{pos}},\varepsilon_{\text{vel}},\omega_{\text{vel}}) = \mathsf{L}_{\text{pos}}(\eta_{\text{pos}},\omega_{\text{pos}})\mathsf{L}_{\text{vel}}(\varepsilon_{\text{vel}},\omega_{\text{vel}}). \tag{68}$$

I shall call this a geometric representation of the product group, because its parameters are the coordinates of the respective geometries, positional and velocity. By the construction of Eq. (66) the operators $\mathsf{L}_{\text{pos}}(\eta_{\text{pos}},\omega_{\text{pos}})$ and $\mathsf{L}_{\text{vel}}(\varepsilon_{\text{vel}},\omega_{\text{vel}})$ are seen to commute, and neither of them will have the property required of a Lorentz boost operator, of operating both on position space, to accomplish a Lorentz transformation, and on velocity space, to accomplish the addition of velocities.

To form an appropriate velocity boost operator in the double Lorentz group, we form a boost operator as the block diagonal matrix

$$\mathsf{K}_{\text{boost}}(\varepsilon_{\text{boost}}) = \begin{pmatrix} K(-\varepsilon_{\text{boost}}) & 0 \\ 0 & K(\varepsilon_{\text{boost}}) \end{pmatrix}. \tag{69}$$

This has the appropriate properties—the first of its subblocks carries out a linear transformation on the position subspace, with the appropriate sign for the Lorentz transformation, and the second performs an Einstein vector addition in the velocity subspace.

The operator complementary to the boost of Eq. (69) can now be taken as

$$\mathsf{K}_{\text{shift}}(\eta_{\text{shift}}) = \begin{pmatrix} K(\eta_{\text{shift}}) & 0 \\ 0 & K(\eta_{\text{shift}}) \end{pmatrix}. \tag{70}$$

The first of its subblocks performs the operation of vector addition in the hyperbolic position space, and the second subblock adds a positional shift to a vector in the velocity subspace—this is the operational description of the Hubble effect of distance on an observed velocity.

Each of the operators $\mathsf{K}_{\text{boost}}$ and $\mathsf{K}_{\text{shift}}$ now acts interactively on both the position and the velocity subspaces.They are the generating operators of the interactive representation of the double Lorentz group. The coordinates parametrizing them are related to the separation and rapidity vector coordinates of the geometric representation by hyperbolic vector relations of the type described in Section 3 above,

$$\boldsymbol{\eta}_{\text{shift}} = \left(\boldsymbol{\varepsilon}_{\text{vel}} \mathbin{\hat{+}} \boldsymbol{\eta}_{\text{pos}}\right)/2, \quad \boldsymbol{\varepsilon}_{\text{boost}} = \left(\boldsymbol{\varepsilon}_{\text{vel}} \mathbin{\hat{-}} \boldsymbol{\eta}_{\text{pos}}\right)/2. \tag{71}$$

It remains to take into account the rotational portions of the general Lorentz operators. We now introduce the following pair of matrix operators:

$$\mathsf{J}\left(\boldsymbol{\omega}_{\text{int}}\right) = \begin{pmatrix} R\left(\boldsymbol{\omega}_{\text{int}}\right) & 0 \\ 0 & R\left(\boldsymbol{\omega}_{\text{int}}\right) \end{pmatrix}, \quad \mathsf{Q}\left(\boldsymbol{\upsilon}\right) = \begin{pmatrix} R\left(-\boldsymbol{\upsilon}\right) & 0 \\ 0 & R\left(\boldsymbol{\upsilon}\right) \end{pmatrix}. \tag{72}$$

Of these, J represents the usual angular momentum operator in its incarnation in the double Lorentz group, and it generates the usual R(3) rotational subgroup. Its parameter vector is a spherical arc-and-angle vector $\boldsymbol{\omega}_{\text{int}}$ of the interaction representation, formally combining the constituent subvectors in the expression

$$\boldsymbol{\omega}_{\text{int}} = \left(\boldsymbol{\omega}_{\text{pos}} \mathbin{\breve{+}} \boldsymbol{\omega}_{\text{vel}}\right)/2. \tag{73}$$

The operator Q, on the other hand, is a previously unrecognized type of generalized angular momentum (or spin) operator, the contra-angular momentum, without an analogue in the kinematics of the usual Lorentz and Poincaré groups. The possibilities for its physical role and detection are discussed in [5]. The argument of Q, the spherical vector $\boldsymbol{\upsilon}$, depends not only on the arc-and-angle variable

$$\boldsymbol{\omega}_{\text{conj}} = \left(\boldsymbol{\omega}_{\text{vel}} \breve{-} \boldsymbol{\omega}_{\text{pos}}\right) / 2 \tag{74}$$

conjugate to $\boldsymbol{\omega}_{\text{int}}$ but also on the vector products of the variable pairs

$$\begin{aligned} \boldsymbol{\beta} &= \left(\boldsymbol{\eta}_{\text{pos}} \hat{\times} \boldsymbol{\varepsilon}_{\text{vel}}\right) / 2 = \left(\boldsymbol{\eta}_{\text{shift}} \hat{\times} \boldsymbol{\varepsilon}_{\text{boost}}\right), \\ \boldsymbol{\gamma} &= \left(\boldsymbol{\omega}_{\text{pos}} \breve{\times} \boldsymbol{\omega}_{\text{vel}}\right) / 2 = \left(\boldsymbol{\omega}_{\text{int}} \breve{\times} \boldsymbol{\omega}_{\text{conj}}\right). \end{aligned} \tag{75}$$

The variable $\boldsymbol{\upsilon}$ can then be expressed as an expansion beginning with the terms

$$\boldsymbol{\upsilon} = \boldsymbol{\omega}_{\text{conj}} \breve{-} \boldsymbol{\beta} \breve{-} \frac{1}{2}\boldsymbol{\gamma} \breve{+} \left(\boldsymbol{\omega}_{\text{int}} \breve{\times} \boldsymbol{\beta}\right) \ldots \tag{76}$$

The matrix expressing a general operator of the double Lorentz group in its interaction representation can now be written as

$$\Lambda_{\text{int}}\left(\boldsymbol{\eta}_{\text{shift}}, \boldsymbol{\varepsilon}_{\text{boost}}, \boldsymbol{\omega}_{\text{int}}, \boldsymbol{\upsilon}\right) = \mathrm{K}_{\text{shift}}\left(\boldsymbol{\eta}_{\text{shift}}\right) \mathrm{K}_{\text{boost}}\left(\boldsymbol{\varepsilon}_{\text{boost}}\right) \mathrm{J}\left(\boldsymbol{\omega}_{\text{int}}\right) \mathrm{Q}\left(\boldsymbol{\upsilon}\right). \tag{77}$$

5. Discussion:

This work was undertaken with the purpose of improving the understanding of the underlying group-theoretical similarity in symmetry structure between the Lorentz group L as regularly used in special relativity in local contexts and as the underlying background to the gravitational structure developed in general relativity, and the embodiment of the abstract group O(1,3) isomorphic to L as the isometry group of the geometry of hyperbolic 3-space and also the embodiment of that geometry in the expanding hyperbolic geometry of the open FRW cosmology. The tools of 3-vector algebra from nonrelativistic dynamics turn out to be particularly useful for these purposes, and have been systematically extended in this paper and in reference [5] into the hyperbolic domain.

An unexpected consequence indicating the value of this alternative 3-vector viewpoint has immediately been brought to light. The familiar

Minkowski construction of a local, differential, 4-vector of position and time turns out to be identifiable as the differential increment specifically belonging to a hyperbolic spatial geometry expanding with time—thus providing a new viewpoint to understand what in 1908, with Minkowski's announcement that an imaginary time was equivalent to a fourth real spatial dimension, was an extraordinary source of wonderment [4].

As stated earlier, this new insight into the connection between an expanding hyperbolic geometry and observable consequences as to local, measurable intervals in time or space, can then be turned around. The existence of the Lorentz transformation and its observable consequences as they can be expressed in Minkowski space now provide a strong argument for two important pieces of information: (a) that position 3-space is itself on average hyperbolically curved, and (b) that the magnitude of that curvature is changing in time.

These deductions demonstrate a tight connection between the Lorentz transformation and the Hubble expansion. This evidence is independent of, and consistent with, the pattern of strict symplectic reciprocity which I demonstrated in reference [5] to exist between the Hubble effect and the Lorentz transformation in the double Lorentz group L^2.

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(Manuscrit reçu le 16 mai 2009)