Non-Dispersing Free-Particle Solutions in a 4-Space Dirac Theory

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ABSTRACT. Free-particle solutions of a 4-space Dirac equation include soliton-like possibilities not found in conventional theory. Non-dispersing examples are given of plane waves and spherically symmetric wave packets.

1 Introduction

In parametrized relativistic quantum theory (PRQT) [1, 2, 3], the space-time coordinates $X^\alpha = (x\ ,\ y\ ,\ z\ ,\ c\,t)$ are all on an equal footing, i.e. both spatial position and time are regarded as observables. An invariant parameter τ , corresponding to the proper time of classical relativity, is used instead of t to describe the evolution of the wave function, which in the spin-$\frac{1}{2}$ theory proposed by the author [4, 5, 6] is a bispinor ψ satisfying a "4-space" Dirac equation that generalizes the conventional one.

Although this 4-space Dirac equation has solutions formally similar to all those of the conventional theory, it also allows solutions of a different nature in which the proper mass m_0 - an observable in the 4-space formulation - is not sharp. In addition, the 4-space solutions (notably those for a Coulomb potential) generally require a modified interpretation of their wave functions, because the 4-space picture includes contributions from both electrons and positrons, as follows. $\psi^\dagger(i\gamma^4)\psi \equiv F(\mathbf{X},\tau)$ is an invariant expected charge density in space-time, and $-\psi^\dagger\gamma^4\boldsymbol{\gamma}\psi \equiv \mathbf{J}(\mathbf{X},\tau)$ is a particle 4-current, so that $J^4 = \psi^\dagger\psi \geq 0$ implies motion in the positive time direction. In general, $F = F_1 - F_2$, where F_1 and F_2 are electron and positron densities, and $\mathbf{J}$ is the sum of the particle and antiparticle currents, which are assumed to have a common 4-velocity

$\mathbf{U}$ given by $c\mathbf{J} = (F_1 + F_2)\mathbf{U}$. A second invariant, $Q \equiv \psi^\dagger\gamma^0\psi$, where $\gamma^0 \equiv -i\gamma^1\gamma^2\gamma^3$, is related to F and $\mathbf{J}$ by $\mathbf{J} \cdot \mathbf{J} = -(F^2 + Q^2)$. As a consequence, one finds $F_1 = (\sqrt{F^2+Q^2}/2$ and $F_2 = (\sqrt{F^2+Q^2} - F)/2$: note that $F_2 = 0 \Leftrightarrow Q = 0$, and $F_1 = F_2 \Leftrightarrow F = 0$. These points are detailed in [4, 5], and illustrated in [6], where the 4-space equation is applied to Klein's paradox. It has also been used to describe neutral particles [7].

The proposed 4-space Dirac equation is in general

$$\boldsymbol{\gamma}\cdot[\boldsymbol{\partial} - \left(\frac{ie}{\hbar c}\right)\boldsymbol{\Omega}]\psi = \left(\frac{i}{c}\right)\frac{\partial\psi}{\partial\tau} \quad , \tag{1}$$

where the 4-component spinor ψ is a function of the spacetime coordinates $X^\lambda = (x^k, ct)$ and the invariant parameter τ . For the present purpose, the 4-vector potential $\boldsymbol{\Omega}$ is assumed zero: we wish to solve

$$\boldsymbol{\gamma}\cdot\boldsymbol{\partial}\,\psi = \left(\frac{i}{c}\right)\frac{\partial\psi}{\partial\tau} \quad . \tag{2}$$

The conventions used here are those of Ref. 5; in particular, the Lorentz metric tensor is $\eta_{\alpha\beta}$ =diag(1 , 1 , 1 , −1), and a chiral representation of the Dirac matrices γ^λ is used unless otherwise specified. In Ref. 4, m was used for proper mass, but m_0 is employed here.

2 General Results for Free Particles

Although more general solutions of (2) can easily be written down, for the present purpose it will be enough to begin with solutions of the form

$$\psi = \exp\{\mathbf{P}\cdot(i\mathbf{X}I + c\tau\boldsymbol{\gamma})/\hbar\}\,\zeta \quad . \tag{3}$$

where ζ is any constant bispinor. These have sharp 4-momentum $\mathbf{P}$, but since the proper mass m_0 is not sharp in this case, we cannot use $P = m_0 c$ here. (Only later will we need the usual mass-shell condition.) The expression (3) is bounded for all $\mathbf{X}$ and τ provided $\mathbf{P}$ is timelike (so that P is real). To make this boundedness more obvious, we can rewrite (3) as

$$\psi = e^{i\mathbf{P}\cdot\mathbf{X}/\hbar}\{I\cos(c\tau P/\hbar) + \frac{\mathbf{P}\cdot\boldsymbol{\gamma}}{P}\sin(c\tau P/\hbar)\}\,\zeta \quad . \tag{4}$$

We also note that if $\boldsymbol{\Omega}$ is constant in (1), then (3) can be generalized to the form

$$\psi = \exp\{(i\boldsymbol{\Pi}\cdot\mathbf{X}I + c\tau\mathbf{P}\cdot\boldsymbol{\gamma})/\hbar\}\,\zeta \quad , \tag{5}$$

where $\mathbf{\Pi} = \mathbf{P} + (e/c)\,\mathbf{\Omega}$ (kinetic momentum), though this result will not be pursued here.

A free wave packet in 4-space may be obtained from (3) by superposition:

$$\psi(\mathbf{X};\tau) = \int \exp(i\mathbf{P}\cdot\mathbf{X}/\hbar)\exp(c\tau\mathbf{P}\cdot\boldsymbol{\gamma}/\hbar)\varphi_0(\mathbf{P})d^4P \ , \tag{6}$$

where (apart from any physical constraints we may wish to impose, such as positive energy) φ_0 is an arbitrary bispinor. An alternative route to (6) is via the momentum representation, in which a bispinor wave function $\varphi(\mathbf{P};\tau)$ satisfies (cf. (2))

$$\mathbf{P}\cdot\boldsymbol{\gamma}\,\varphi = \frac{\hbar}{c}\frac{\partial\varphi}{\partial\tau} \ . \tag{7}$$

This has the general solution

$$\varphi(\mathbf{P};\tau) = \exp(c\tau\mathbf{P}\cdot\boldsymbol{\gamma}/\hbar)\varphi_0(\mathbf{P}) \ , \tag{8}$$

where φ_0 is an arbitrary bispinor, and a 4-space Fourier transform converts (8) to the coordinate representation, as in (6).

We note that (6), (7) and (8) are manifestly invariant: $\mathbf{P}\cdot\mathbf{X}$ is an invariant, while $\mathbf{P}\cdot\boldsymbol{\gamma}\varphi$ and $\exp(c\,\tau\mathbf{P}\cdot\boldsymbol{\gamma}/\hbar)\varphi_0$ are bispinors, with the same transformation law as φ and ψ. And because the integration in (6) is over 4-momentum space, it does not pick out a preferred reference frame.

Though (3) and (4) have plane symmetry, they do not represent the usual kind of plane wave. However, in the 4-space theory there is a proper mass operator, $\hat{m}_0 = (-i\hbar/c^2)\partial/\partial\tau$, and imposing the condition $\hat{m}_0\psi = m_0\psi$ on (3) gives

$$(\mathbf{P}\cdot\boldsymbol{\gamma} - im_0cI)\,\zeta = 0 \ . \tag{9}$$

This in turn simplifies (3) to the 4-space form of a conventional plane wave:

$$\psi = \exp\{i(\mathbf{P}\cdot\mathbf{X} + E_0\tau)/\hbar\}\,\zeta \ . \tag{10}$$

(The usual Dirac plane wave lacks the term $E_0\tau \equiv m_0c^2\tau$ in the exponential.) As in the more familiar 3-space Dirac theory, the constant bispinor ζ is now an eigenvector defined by (9): for the chiral representation we find

$$\zeta = \begin{bmatrix} i\alpha m_0 c \\ i\beta m_0 c \\ \alpha(P_3 - P_4) + \beta(P_1 - iP_2) \\ \alpha(P_1 + iP_2) - \beta(P_3 + P_4) \end{bmatrix} \ , \qquad (11)$$

where α and β are arbitrary complex constants. The existence of ζ requires the mass-shell condition

$$\mathbf{P} \cdot \mathbf{P} = -m_0^2 c^2 \ . \qquad (12)$$

Despite the formal similarity between these 4-space plane waves and their conventional analogues, there is a notable difference in their velocities of propagation. The phase 4-velocity of the wave (10) is $\mathbf{P}/m_0$, precisely that of the corresponding classical particle, whereas the usual Dirac plane wave has a phase 3-velocity of magnitude $E/|\mathbf{p}| > c$, where $\mathbf{p}$ is the 3-momentum. We also note that the charge density is strictly positive ($Q = 0, F_2 = 0$) in any plane wave given by (10) and (11) - this is easily verified by choosing a frame in which the spatial momentum components are zero. However, the same is not true of waves given by (3) or (4) [7].

We can generate wave packets with sharp proper mass by integrating (10) over a particular mass shell, i.e. by fixing m_0 in (12). (We use (12) to determine $P^4 \equiv E/c$, while in (11) α and β are functions of the 3-momentum $\mathbf{p}$.) The factor $\exp(i\ E_0\tau\ /\hbar)$ can be dropped from ψ at this point, and then in place of (6) we obtain the usual free Dirac wave packet in 3-space:

$$\psi(\mathbf{x};t) = \int \exp(i\mathbf{p}\cdot\mathbf{x}/\hbar)\exp(-iEt/\hbar)\,\varphi_0(\mathbf{p})\,d^3p \ , \qquad (13)$$

where $\mathbf{p}$ is the 3-momentum, $\mathbf{x}$ is the spatial position vector, and the integration is confined to the mass shell. This is made to appear Lorentz-invariant if we rewrite it in the form

$$\psi(\mathbf{x};t) = \int \exp(i\mathbf{P}\cdot\mathbf{X}/\hbar)\,\chi(\mathbf{p})\,(m_0c^2/E)\,d^3p \ , \qquad (14)$$

where $\chi = (E/m_0c^2)\varphi_0$. Here $\mathbf{P}\cdot\mathbf{X}$ and d^3p/E are Lorentz-invariant, but now we require χ, not φ_0 , to be a bispinor. It follows that in the usual 3-space formulation one cannot combine Lorentz invariance with symmetry between the coordinate and momentum representations - which should not be surprising, in view of the special role assigned to the time coordinate.

Unless E is also sharp, such wave packets are unphysical from the 4-space viewpoint, since they give a charge distribution that depends on t , but not on τ . Thus the distribution is spread out in time (as expected), but does not move forward in time as the evolution parameter τ changes. If E_0 and E are both sharp, we have a stationary solution that is valid in both the 3-space and 4-space forms of Dirac's theory. When interactions are included, this last type also covers the crucially important Coulomb-field solutions.

3 Non-Dispersing Plane Waves

Although the four-space plane waves given by (10) are eigenstates of both $\mathbf{P}$ and m_0 , they allow the generation of solutions with indeterminate proper mass. As a simple example, in (10)-(12) let $\mathbf{P} = \lambda\ \mathbf{K}$, where $\mathbf{K}$ is a fixed timelike 4-vector, i.e. $\mathbf{K}\cdot\ \mathbf{K} = -\ K^2 < 0$, so that $\mathbf{P}\cdot\ \mathbf{P} = -\lambda^2 K^2 = -\ m_0^2 c^2$ and $m_0 c = \lambda K$. Set $\hbar = 1$, let $\alpha = e^{-\lambda a}, \beta = 0$, in (11), where a is any positive real constant, and integrate (10) with respect to λ, over $0 \le \lambda < \infty$. The resulting solution describes the flow of a charge that is distributed in spacetime: to within a constant factor, we find that the wave function is

$$\psi = \frac{\zeta_0}{[a - i(\mathbf{K}\cdot\mathbf{X} + Kc\tau)]^2} \quad , \tag{15}$$

where

$$\zeta_0 = \begin{bmatrix} iK \\ 0 \\ K_3 - K_4 \\ K_1 + iK_2 \end{bmatrix} \quad . \tag{16}$$

This gives the charge density

$$F = F_1 = \frac{2K(K^3 + K^4)}{[a^2 + (\mathbf{K}\cdot\mathbf{X} + Kc\tau)^2]^2} \quad , \tag{17}$$

while $F_2 = 0$, since in this case $Q = 0$. We find that $\mathbf{J} = F_1\mathbf{K}/K$, in agreement with the 4-velocity $c\,\mathbf{K}/\ K$ implied by the term $\mathbf{K}\cdot\ \mathbf{X} +$

$Kc\tau$ in (17). (In the absence of antiparticles, $\mathbf{J}$ gives both the particle current and the charge current.) The solution represents a kind of plane wave, spread out both spatially (in the direction of its 3-velocity) and temporally.

However, the wave does not disperse as τ increases. This soliton-like behaviour is explained by the fact that all contributions to the wave function have the same 4-velocity, because of the relation $\mathbf{P} = \lambda\,\mathbf{K}$, where $\mathbf{K}$ is fixed. (Thus the non-dispersive property, which is usually associated with nonlinear wave equations, is a special case in the present context.) There is no requirement here for m_0 to be sharp, since a free particle has no need for a definite proper mass. We can find $\langle m_0\rangle$ using the operator $\hat{m}_0 = (-i/c^2)\partial/\partial\tau$: by choosing a frame in which $K_1 = K_2 = K_3 = 0$, and integrating only over time to obtain a finite expression, one can verify that the result is positive.

4 Non-Dispersing Spherical Wave Packets

It is sometimes useful to split the 4-component field equation into two coupled 2-component equations. Denote the first two components of ψ by ξ , and the last two by η . Then (2) becomes a pair of coupled equations: with the chiral representation of the Dirac matrices (see appendix to Ref. 5 for details), we obtain

$$\sigma_k\,\eta_{,k} + \frac{1}{c}\frac{\partial\eta}{\partial t} = \frac{i}{c}\frac{\partial\xi}{\partial\tau} \ , \tag{18}$$

$$\sigma_k\,\xi_{,k} - \frac{1}{c}\frac{\partial\xi}{\partial t} = \frac{i}{c}\frac{\partial\eta}{\partial\tau} \ . \tag{19}$$

In terms of the 2-component spinors ξ and η , the invariants F and Q that determine F_1 and F_2, the electron and positron densities (see §1), are, in the chiral representation,

$$F = i(\xi^\dagger\eta - \eta^\dagger\xi) \ , \tag{20}$$

$$Q = \xi^\dagger\eta + \eta^\dagger\xi \ . \tag{21}$$

Recalling that $F_2 = 0 \Leftrightarrow Q = 0$, and $F_1 = F_2 \Leftrightarrow F = 0$, we can look for solutions of the system (18), (19) with particular properties. For example, setting $\eta = -i\xi$ gives $Q = 0$, $F = 2\xi^\dagger\xi \geq 0$, and reduces (18), (19) to

$$\sigma_k\,\xi_{,k} = 0 \ , \tag{22}$$

$$\frac{\partial\xi}{\partial t} + \frac{\delta\xi}{\partial\tau} = 0 \ . \tag{23}$$

This implies that ξ is a function of $t-\tau$, so that the charge distribution moves forward in coordinate time t as the proper time τ increases. The precise dependence on $t-\tau$ is a matter of choice. For the moment we assume only that the variables may be separated so that $\xi = \upsilon(t-\tau)\mu(\mathbf{x})$, where μ is a 2-component spinor and υ is a scalar function.

On the other hand, if $\eta = i\xi$ we get $Q = 0, F = -2\xi^\dagger\xi \leq 0$, and (18), (19) are reduced to

$$\sigma_k\, \xi_{,k} = 0 \ , \tag{24}$$

$$\frac{\partial\xi}{\partial t} - \frac{\delta\xi}{\partial\tau} = 0 \ . \tag{25}$$

Now ξ is a function of $t+\tau$, and the charge distribution moves backward in time as the proper time τ increases. This illustrates the point that the model has been set up to give a natural description of particles rather than antiparticles [4, 5]. To obtain solutions in which a distribution of negative density moves forward in time (as τ increases) we may modify the field equation, changing the sign of the right-hand side in (1) and (2).

Returning to (22), (23), we can construct examples of non-dispersing solutions along the lines of the standard solutions for an inverse-square field. In the first place, we are looking for solutions of $\sigma_k\, \mu_{,k} = 0$, and we note that any such solution must also satisfy Laplace's equation. The simplest case, similar in mathematical form to the ground state of hydrogen, is (in spherical polar coordinates $r\ ,\theta, \varphi$)

$$\mu = \begin{bmatrix} f(r)\cos\theta \\ f(r)\sin\theta\, e^{i\varphi} \end{bmatrix} \ . \tag{26}$$

To use this we need to write $\sigma_k\, \partial_k$ in spherical form:

$$\begin{aligned} \sigma_k\, \partial_k = & \begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{bmatrix} \frac{\partial}{\partial r} \\ & + \begin{bmatrix} -\sin\theta & \cos\theta\, e^{-i\varphi} \\ \cos\theta\, e^{i\varphi} & \sin\theta \end{bmatrix} \frac{1}{r}\frac{\partial}{\partial\theta} \\ & + \begin{bmatrix} 0 & -e^{-i\varphi} \\ e^{i\varphi} & 0 \end{bmatrix} \frac{i}{r\sin\theta}\frac{\partial}{\partial\varphi} \ . \end{aligned} \tag{27}$$

We now find that (22) reduces to

$$f' + 2f/r = 0 \ , \tag{28}$$

and hence gives

$$f(r) = k/r^2 \ , \tag{29}$$

where we can assume that the constant k is real and positive. For υ, we only need to choose a plausible dependence on $t - \tau$; for example,

$$\upsilon = e^{iE(t-\tau)/\hbar}\, e^{-\frac{1}{2}\lambda(t-\tau)^2/\hbar} \ , \tag{30}$$

where E and λ are constants. The space-time charge density is now

$$F(r,t;\tau) = \frac{2k^2 e^{-\lambda(t-\tau)^2/\hbar}}{r^4} \ . \tag{31}$$

To ensure the convergence of $\int Fr^2 dr$, a cut-off at small r is needed, and we note that this does not contradict Lorentz invariance: the boundary is $r = r_0$, say, in the rest frame that we are using here, and can be transformed into a general frame. (By working in the rest frame we ensure that every contribution to the wave packet has the same 4-velocity, and this property, as in the example of Section 3, accounts for the non-spreading of the packet.) The constants r_0 and k may be chosen so that $4\pi \ \int F(r,t;\tau) r^2 dr dt = 1$.

The complex factor in (30) ensures that E is the expected value of both the total energy and the proper mass-energy, though the present solution is not an eigenstate of either. These results are found by direct application of methods and definitions outlined in Ref. 4; e.g.

$$\langle E \rangle = \int \psi^\dagger (i\gamma^4) \hat{E} \psi \, d^4X \ , \tag{32}$$

where as usual $\hat{E} = i\hbar\, \partial/\partial t$. Similarly for $\langle E_0 \rangle$, using $\hat{E}_0 = -i\hbar\, \partial/\partial\tau$.

5 Discussion

The increasing dispersion of free wave packets is a hallmark of conventional quantum mechanics, relativistic or otherwise. The inevitability of spreading follows immediately from the fixed mass of a particle: if there is a distribution of momentum, then there is also a distribution of velocity.

However, we have seen above that (from a theoretical viewpoint!) a 4-space approach opens up the possibility of non-dispersing wave packets in relativistic QED. The crucial new property that allows this is the observable (rather than parametric) nature of proper mass in the 4-space theory. In free-particle solutions, if the proper mass is not required to take a particular value, we can now have distributions of momentum and proper mass such that all components of the wave packet have the same velocity - and so it does not disperse. Along with the observable proper mass, we also have the invariant evolution parameter τ, which allows wave functions to be distributed in both space and time.

However, these theoretical wave packets raise obvious questions, in particular: should quantum mechanics encompass wave packets that do not disperse? Entanglement aside, particles seem to remain well localized, even after travelling long distances over extended time intervals. But can non-dispersing wave packets be prepared and tested in the laboratory? And how might they be formed?

This last question appears to lie beyond the scope of the scenario considered here. Although we have found *soliton-like* behaviour, it has not been produced by the interaction of dispersion and nonlinearity that is characteristic of solitons. The wave equation is strictly linear, and the 4-space approach has allowed all dispersion to be eliminated. However, we can suggest a possible mechanism for limiting dispersion, because one of the features [4] of the 4-space picture is the appearance of virtual particles and antiparticles whenever an interaction occurs. Although this effect (which increases with the intensity of the interaction) is usually very small, one could expect any polarization within a wave packet to act against dispersion. As for the source of the interaction, even the quantum vacuum might suffice. Thus solutions of the sort presented above may bear some resemblance to states existing within a more realistic interaction model.

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(Manuscrit reçu le 15 septembre 2009, révisé le 7 octobre 2010)