

# Geometrical Properties and Propagation for the Proca Field Theory

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RÉSUMÉ. Nous considérons le champ de Proca avec un terme dynamique donné par une dérivée externe par rapport à la connection la plus générale; les équations du champ de Proca les plus générales sont données: une discussion sur la propagation et sur les propriétés géométriques est présentée. Il est montré que cette généralisation est inconsistente et donc la théorie standard est déjà la théorie de Proca la plus générale qui soit compatible avec nos restrictions géométriques. *ABSTRACT. We consider the Proca field with dynamical term given by the exterior derivative with respect to the most general connection; the most general Proca field equations are given: a discussion about the propagation and the geometrical properties is presented. It is shown that this generalization is inconsistent and therefore the standard theory is already the most general Proca theory compatible with our geometrical restrictions.*

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## 1 Introduction

Despite what is widely believed, it was de Broglie who first propounded a theory for a photon with rest mass in order to maintain the coherence of the wave-particle duality (his account of a theory in which the propagation of light may occur with speeds that are subluminal started in 1934 with [1]); and it was upon these theoretical basis that later de Broglie's doctoral student Proca founded the model for the spin-1 fields with mass terms (the original paper may be found in [2]): since then the de Broglie-Proca theory was simply called the Proca model now known

to describe the dynamics of vector bosons with massive degrees of freedom. Eventually, the de Broglie-Proca theory gave rise to what we now accept as the correct description not of a massive photon but of the massive vector bosons, the fields that arise as a consequence of the spontaneous breakdown of the gauge symmetry within the standard model of elementary particles; thus the Proca model became a very important model in particle physics after the  $Z^0$  and  $W^\pm$  particles were discovered. On the other hand beyond the phenomenological reasons for its introduction and the experimental evidence for its confirmation, there are nevertheless further theoretical reasons for which the Proca field is that important in physics and mathematics.

These reasons come from the fact that spin- $s$  fields have  $2s + 1$  degrees of freedom while massless condition implies the loss of 1 degree of freedom, and therefore spin-1 massive fields have 3 degrees of freedom and none less; this means that since vector fields have originally 4 degrees of freedom then the reduction to 3 degrees of freedom is obtained whenever 1 additional constraint is postulated to ensure consistency, according to the Fermi prescription. Now such a constraint is required as a subsidiary condition imposed from the exterior of the model in the case of the Fermi field whereas it is developed from within the model itself in the case of the Proca field, so that it is in terms of a Fermi model autonomously defined that we can think at the Proca model, and this is the sense in which the Proca model is considered to be more elegant.

The Proca field is a vector field whose dynamical term is the divergence of the curl of the vector field and it has a mass term, and this is what gives to its field equations the structure they need to automatically provides the subsidiary condition. However, the fact that this field has dynamical term written in terms of the antisymmetric part of the derivative, which can be defined without torsion, does not prevent us to generalize it up to the the exterior derivative calculated with respect to the most general connection, in which torsion is present in a natural way.

In this paper we will consider such a generalization, deriving its consequences and discussing its implications regarding the propagation and its most important geometrical properties.

## 2 Fundamental Definitions

In a given geometry, the metric structure is given in terms of two symmetric metric tensors  $g_{\alpha\beta}$  and  $g^{\alpha\beta}$  that are one the inverse of the other,

and differential operations  $D_\mu$  are defined through the connections  $\Gamma_{\alpha\beta}^\rho$ ; the metric tensor are to be such that they can be locally reduced to the Minkowskian form, and the covariant derivatives applied upon the metric tensors are required to vanish according to what is called metricity condition  $D_\mu g = 0$ , as it has been discussed in reference [3]. Then, requiring this condition of metricity for any connection leads to the complete antisymmetry of Cartan torsion tensor  $Q_{\alpha\mu\rho}$ , as explained in reference [4]. Further, we will define Riemann tensor with  $G_{\alpha\beta\mu\nu}$  antisymmetric in both the first and the second couple of indices, with one independent contraction given by the Ricci tensor  $G_{\alpha\lambda}^\lambda = G_{\alpha\beta}$ , whose contraction is given by the Ricci scalar  $G_{\alpha\beta}g^{\alpha\beta} = G$  and this will set our convention up.

Riemann curvature tensor, Ricci curvature tensor and scalar, together with Cartan torsion tensor verify the identities

$$D_\rho Q^{\rho\mu\nu} + \left( G^{\nu\mu} - \frac{1}{2}g^{\nu\mu}G \right) - \left( G^{\mu\nu} - \frac{1}{2}g^{\mu\nu}G \right) \equiv 0 \quad (1)$$

and

$$D_\mu \left( G^{\mu\rho} - \frac{1}{2}g^{\mu\rho}G \right) - \left( G_{\mu\beta} - \frac{1}{2}g_{\mu\beta}G \right) Q^{\beta\mu\rho} + \frac{1}{2}G^{\mu\kappa\beta\rho}Q_{\beta\mu\kappa} \equiv 0(2)$$

which are geometric identities, called Jacobi-Bianchi identities.

We remark that from the metric tensor it is possible to define the Levi-Civita tensor  $\varepsilon_{\beta\mu\rho\sigma}$  for which  $D_\mu \varepsilon_{\beta\mu\rho\sigma} = 0$  identically; in turn, since torsion is completely antisymmetric then we can write

$$Q^{\beta\mu\rho} = \varepsilon^{\beta\mu\rho\sigma} W_\sigma \quad (3)$$

in terms of what is known to be the axial torsion vector, useful in the following.

Within this background, to define matter fields that can be classified according to the value of their spin we have to consider that a given matter field of spin  $s$  possesses  $2s + 1$  degrees of freedom, which have to correspond to the  $2s + 1$  independent solutions of a system of equations that specify the highest-order time derivative for all components of the field, called system of matter field equations; however, since it may happen that field equations are not enough to determine the correct rank of the solution, restrictions need to be imposed in terms of equations

in which all components of the field have highest-order time derivatives that never occur, called system of constraints: these constraints can be imposed in two ways, either being assigned as subsidiary conditions that come along with the field equations or being implied by the field equations.

Before proceeding we remind the reader that in order to check causal propagation of the wave fronts, the general method consists in considering the full system of field equations with constraints neglecting all terms that are not the highest-order temporal derivatives of the field, then formally replacing the derivatives with the normal vector  $n$  obtaining a matricial equation in terms of  $n$  of which we have to demand singularity to get an equation in terms of  $n$  that is called characteristic equation, whose solutions are the normal to the characteristic surfaces representing the wave fronts: if the temporal component of the normal vector is real then the wave front propagates; if there is no time-like normal vector then there is no acausal propagation of the wave front itself.

Now, although the procedure for which a field is defined in terms of field equations implying their own constraints appears to be more elegant, nevertheless whenever interactions are present it can give rise to the fact that the presence of the interacting fields could increase the order derivative of the constraining equation up to the same order derivative of the field equations themselves, creating the possibility that highest-order time derivatives of some component occur which convert the constraint into a field equation spoiling the number of degrees of freedom; and anyway, even if in the constraining equation the highest-order time derivative never appeared, or if it actually appeared but could be removed by means of field equations, then the counting of degrees of freedom is preserved, but nevertheless the highest-order temporal derivative terms entering the characteristic equations may influence the propagation of the wave fronts by either boosting them of the light-cone, or by making them fail to propagate at all, as it has been explained and studied with several examples in reference [5].

Once this analysis is performed, causal propagation of wave fronts is checked and the exact number of degrees of freedom of the matter field solution is established, the last requirement for this system of matter field equations is that they have to ensure the complete antisymmetry of the spin, so that taking the spin  $S^{\nu\sigma\rho}$  with the energy  $T^{\sigma\rho}$  they have

to be such that the relationships

$$D_\rho S^{\rho\mu\nu} + \frac{1}{2} (T^{\mu\nu} - T^{\nu\mu}) = 0 \quad (4)$$

and

$$D_\mu T^{\mu\rho} - T_{\mu\beta} Q^{\beta\mu\rho} - S_{\beta\mu\kappa} G^{\mu\kappa\beta\rho} = 0 \quad (5)$$

are verified, implying the whole set of field equations

$$G^{\sigma\rho} - \frac{1}{2} g^{\sigma\rho} G = -\frac{1}{2} T^{\sigma\rho} \quad (6)$$

and

$$Q^{\nu\sigma\rho} = S^{\nu\sigma\rho} \quad (7)$$

to be such that the conservation laws (1) and (2) are satisfied automatically.

This determines the setting of the fundamental field equations in minimal coupling, that is taking the least-order derivative in the field equations.

### 3 Propagation and Geometrical Properties

Having settled this background, and because the background has restrictions then matter fields will be restricted correspondingly, as also explained in [6].

To consider which matter vector fields could possibly be defined within this background, we see that in the case of a vector  $V_\mu$  it is possible to define beside the standard covariant derivative given in terms of the connection another most special differential operation written in the form  $Z_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$  in terms of no additional field and called exterior derivative or curl; this curl can be generalized up to the differential operator given by  $Z_{\rho\mu} = D_\rho V_\mu - D_\mu V_\rho$  that is formally the curl but written with respect to the most general connection with torsion: according to this most general dynamical term, the most general Proca Lagrangian will have kinetic terms given not only by the standard  $Z^{\mu\alpha} Z_{\mu\alpha}$  term but also by the additional  $Z_{\eta\rho} Z_{\mu\alpha} \varepsilon^{\mu\alpha\eta\rho}$  term whose contributions are relevant because in presence of torsion it cannot be written as a total divergence in the Lagrangian and therefore neglected. The most general

action will then be defined in terms of two parameters given by a parameter  $\lambda$  accounting for the additional terms of this generalization and the mass  $m$  of the matter field.

By varying this action with respect to the field involved we get the Proca matter field equations

$$D_\mu Z^{\mu\alpha} + \frac{\lambda}{2} D_\mu Z_{\eta\rho} \varepsilon^{\mu\eta\rho\alpha} + m^2 V^\alpha = 0 \quad (8)$$

which specify the second-order time derivative for only the spatial components, but which also develop the constraint

$$\begin{aligned} m^2 D_\mu V^\mu - \frac{\lambda}{4} Q_{\rho\mu\nu} D^\rho Z_{\alpha\beta} \varepsilon^{\alpha\beta\mu\nu} - \frac{1}{2} Q^{\rho\alpha\beta} D_\rho Z_{\alpha\beta} - \\ - \frac{\lambda}{2} D_\mu Q^\rho{}_{\beta\nu} Z_{\rho\alpha} \varepsilon^{\alpha\beta\mu\nu} - \frac{1}{2} D_\rho Q^{\rho\alpha\beta} Z_{\alpha\beta} = 0 \end{aligned} \quad (9)$$

in which the terms with the second-order time derivative of spatial components can be removed by means of field equations showing that this is a real constraint, which can then be plugged back into the field equations allowing them to specify the second-order time derivative of all components showing that these are true field equations; the conserved quantities are given by the energy

$$\begin{aligned} T^{\alpha\mu} = -\frac{1}{2} g^{\alpha\mu} m^2 V^2 + \left( \frac{1}{4} g^{\alpha\mu} Z_{\rho\eta} Z^{\rho\eta} - Z^{\mu\theta} Z^\alpha{}_\theta \right) + \\ + D_\rho V^\mu \left( Z^{\rho\alpha} + \frac{\lambda}{2} Z_{\sigma\theta} \varepsilon^{\sigma\theta\rho\alpha} \right) \end{aligned} \quad (10)$$

and the spin

$$S^{\rho\alpha\beta} = \frac{1}{2} \left[ V^\alpha \left( Z^{\rho\beta} + \frac{\lambda}{2} Z_{\sigma\theta} \varepsilon^{\sigma\theta\rho\beta} \right) - V^\beta \left( Z^{\rho\alpha} + \frac{\lambda}{2} Z_{\sigma\theta} \varepsilon^{\sigma\theta\rho\alpha} \right) \right] \quad (11)$$

so that, whereas the condition

$$V^\alpha \left( Z^{\rho\beta} + \frac{\lambda}{2} Z_{\sigma\theta} \varepsilon^{\sigma\theta\rho\beta} \right) + V^\rho \left( Z^{\alpha\beta} + \frac{\lambda}{2} Z_{\sigma\theta} \varepsilon^{\sigma\theta\alpha\beta} \right) = 0 \quad (12)$$

ensures the complete antisymmetry of the spin: this form of the spin with the energy is such that one the field equations are considered then the conservation laws (10) and (11) are satisfied and thus the Jacobi-Bianchi identities verified.

We notice that the field equations for the spin express the spin in terms of the covariant derivatives of the field written in terms of the

spin itself, and therefore the field equations for the spin actually express the spin as an implicit relationship that can eventually be made explicit giving as result the inverted relationship

$$S^{\rho\alpha\beta} = \left( \frac{1 + \lambda^2}{3\lambda} \right) (\partial_\sigma V_\nu - \partial_\nu V_\sigma) V^\sigma \varepsilon^{\nu\rho\alpha\beta} \quad (13)$$

in which the complete antisymmetry of the spin is manifest, and hence torsion is writeable in terms of the torsionless derivatives of the field; then it can be plugged back into the field equations to account for the back-reaction of torsion onto the field, or equivalently the autointeraction of the field with itself.

Further we can consider the condition of completely antisymmetric spin in its only independent contraction, of which we can take the divergence, thus obtaining two expressions from which it is possible to derive the conditions

$$V^\alpha V_\alpha = 0, \quad Z_{\rho\beta} Z^{\rho\beta} + \frac{\lambda}{2} Z_{\rho\beta} Z_{\sigma\theta} \varepsilon^{\sigma\theta\rho\beta} = 0 \quad (14)$$

and also

$$V_\rho S^{\rho\alpha\beta} = 0, \quad Z_{\rho\beta} S^{\rho\beta\alpha} = 0 \quad (15)$$

or equivalently

$$V_\rho W_\beta - V_\beta W_\rho = 0, \quad Z_{\rho\beta} W_\alpha + Z_{\beta\alpha} W_\rho + Z_{\alpha\rho} W_\beta = 0 \quad (16)$$

which are orthogonality conditions of the field and its exterior derivatives with the torsion tensor; these orthogonality conditions will considerably simplify the expression of the field equations we are going to work out.

Now, to study the propagation, it is possible to see that we have the number of degrees of freedom still correct; then, the calculation of the characteristic equation gives

$$3m^2 n^2 - (1 + \lambda^2) Z^{\rho\alpha} Z_{\rho\beta} n_\alpha n^\beta = 0 \quad (17)$$

from which we have that causal propagation is always possible, but by introducing  $Z^{0i} = E^i$  and  $Z^{ij} = -\varepsilon^{ijk} B_k$  then whenever

$$(n^0)^2 |\vec{E}|^2 - 2n^0 (\vec{n} \cdot \vec{E} \times \vec{B}) - (|\vec{n} \cdot \vec{E}|^2 - |\vec{n} \times \vec{B}|^2) < 0 \quad (18)$$

we have that acausal propagation may occur and whenever

$$\begin{aligned} & |\vec{n} \cdot \vec{E}|^2 |\vec{E}|^2 + 2(\vec{n} \cdot \vec{E})(\vec{n} \cdot \vec{B})(\vec{E} \cdot \vec{B}) - \\ & - |\vec{n} \cdot \vec{E}|^2 |\vec{B}|^2 - |\vec{n}|^2 |\vec{E} \cdot \vec{B}|^2 < 0 \end{aligned} \quad (19)$$

there is no propagation whatsoever for the present Proca massive vector field.

On the other hand, to study the geometrical properties, we notice that torsion imposes through the condition of complete antisymmetry

$$V^\alpha \left( Z^{\rho\beta} + \frac{\lambda}{2} Z_{\sigma\theta} \varepsilon^{\sigma\theta\rho\beta} \right) + V^\rho \left( Z^{\alpha\beta} + \frac{\lambda}{2} Z_{\sigma\theta} \varepsilon^{\sigma\theta\alpha\beta} \right) = 0 \quad (20)$$

an additional constraint that does reduce to 2 the highest number of degrees of freedom, leaving this generalization of the Proca massive vector field with the same degrees of freedom of a massless vector field and therefore overdetermined.

Therefore, although not in terms of the propagator but rather in terms of the torsional restrictions, this present generalization of the Proca massive vector field is not defined correctly, and the most general Proca matter field is to be considered that in its original form.

## 4 Conclusion

In this paper, we have considered the Proca vector field in which the dynamical term is written in terms of the exterior derivative or equivalently the curl, calculated with respect to the most general metric connection with completely antisymmetric Cartan torsion tensor; also, another dynamical term has been included in the field equations: this more general field equations have been discussed from the perspective of propagation and the geometrical properties. The model is defined in terms of the parameter  $\lambda$  beside its mass  $m$  as usual.

It has been shown that, regarding the propagation, such a field may always have propagating failure or be acausal, while about the geometric properties, it never has the correct number of degrees of freedom.

As this discussion has underlined, any attempt to pursue a generalization of the Proca massive vector field is inconsistent, and the most general Proca matter field is the one originally introduced.



**Références**

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