On the fallacies of Yvon-Takabayasi approaches to Dirac theory and their rectification

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ABSTRACT. Two disjunct universal factorizations of the Clifford algebra Cl_3 of biquaternions are derived. These respectively generate particle and antiparticle solutions of the Dirac equation. Due to their universality, they also provide parametrizations of every biquaternion $\psi \neq 0$ which is noninvertible, $\psi\widetilde{\psi} = 0$, where the approach of Yvon and Takabayasi fails. It is shown that for $\psi \neq 0$, $\psi\widetilde{\psi} = 0$ **all** *vectors of the tetrade of Takabayasi $f_\mu = \psi\gamma_\mu\widetilde{\psi}$, $\mu = 0,\ 1,\ 2,\ 3$, are lightlike $f_\mu \cdot f_\mu = 0$ but mutually orthogonal $f_\mu \cdot f_\nu = 0$, $\mu \neq \nu$, and the pseudoscalar $f_0 f_1 f_2 f_3$ vanishes. Thus the tetrade of Takabayasi then collapses and leads to wrong conclusions ! Making use of the two factorizations, the Dirac equation is solved for plane waves. In the limit of infinite momentum, or, equivalently, of vanishing restmass,* **the biquaternions of these solutions are non invertible and can not be obtained by Yvon-Takabayasi approaches !**

1 Introduction

A biquaternion is the most general element of the Clifford algebra Cl_3 of the euclidean space $\mathbb{R}^3$. Cl_3 is the even graded subalgebra of $Cl_{1,3}$, the Clifford algebra of the lorentzian time-space manifold $\mathbb{R}\otimes\mathbb{R}^3 = \mathbb{R}^{1,3}$. Basis vectors of degree 1 in $Cl_{1,3}$ are $\{\gamma_0,\ \gamma_1,\ \gamma_2,\ \gamma_3\}$, where $\gamma_0^2 = 1$, $\gamma_k^2 = -1$, $k = 1,\ 2,\ 3$, and $\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu$ for $\mu \neq \nu = 0,\ 1,\ 2,\ 3$. Basis vectors of degree 1 in the even graded subalgebra Cl_3 of $Cl_{1,3}$ are defined by $\{\vec{e}_k = \gamma_k\gamma_0,\ k = 1,\ 2,\ 3\}$, which satisfy the Clifford product rules $\vec{e}_k^{\,2} = 1$ and $\vec{e}_k\vec{e}_j = -\vec{e}_j\vec{e}_k$ for $j \neq k$. A vector of degree 1 in Cl_3, $\vec{a} \in \mathbb{R}^3 \subset Cl_3$ is a linear combination $\vec{a} = \sum_{k=1}^{3} a_k\vec{e}_k$, $a_k \in \mathbb{R} \subset Cl_3$, correspondingly $\vec{b} = \sum_{j=1}^{3} b_j \vec{e}_j$. With respect to $Cl_{1,3}$, $\vec{a}$ and $\vec{b}$ are vectors of degree 2. The product $i = \vec{e}_1\vec{e}_2\vec{e}_3 = \gamma_0\gamma_1\gamma_2\gamma_3$, of degree 3 in Cl_3 and of degree 4 in $Cl_{1,3}$, has the name pseudoscalar and $i^2 = -1$, $i\gamma_\mu = -\gamma_\mu i$, $i\vec{a} = \vec{a}\ i$. The equation $\vec{a}\vec{b} = \vec{a}\cdot\vec{b} + i(\vec{a}\times\vec{b})$ defines the conventional scalar- and crossproduct in $\mathbb{R}^3$ and $i_1 \overset{\text{def}}{=} \vec{e}_2\vec{e}_3 = i\vec{e}_1$, $i_2 \overset{\text{def}}{=} \vec{e}_3\vec{e}_1 = i\vec{e}_2$, $i_3 \overset{\text{def}}{=} \vec{e}_1\vec{e}_2 = i\vec{e}_3$. The elements of $\mathbb{R}$ and $i_1,\ i_2,\ i_3 = i_2 i_1$, $i_k^2 = -1$, $k = 1,\ 2,\ 3$ generate the even graded subalgebra Cl_2 of Cl_3, the algebra of **quaternions** $Q = q_0 + i\vec{q}$, $q_0 \in \mathbb{R}$, $\vec{q} \in \mathbb{R}^3$, $\gamma_0 Q = Q\gamma_0$ [1]. Thus any element $\psi \in Cl_3$ may be written in the universal form

$$\psi = A + iB\ ,\quad A = \alpha + i\vec{a}\ ,\quad B = \beta + i\vec{b}\ ,\quad \alpha,\beta \in \mathbb{R}\ ,\quad \vec{a},\vec{b} \in \mathbb{R}^3. \tag{1.1}$$

Because ψ is composed of the two quaternions A and B, the name biquaternion is appropriate. On $Cl_{1,3}$ there is an automorphism, which maps two elements $X,\ Y \in Cl_{1,3}$ in $\widetilde{X}, \widetilde{Y} \in Cl_{1,3}$ With respect to the Clifford operations it is defined by

$$(\widetilde{X+Y}) = \widetilde{X} + \widetilde{Y}\ ,\quad (\widetilde{XY}) = \widetilde{Y}\widetilde{X}. \tag{1.2}$$

The automorphism $\widetilde{\ }$ is fixed in $Cl_{1,3}$ and hence for the whole sequence of its mutually even-grades subalgebras

$$Cl_{1,3} \supset Cl_3 \supset Cl_2 \supset Cl_1 = \{\mathbb{R} + i_3\mathbb{R}\} = \mathbb{C} \supset C_0 = \mathbb{R} \tag{1.3}$$

by the definitions

$$\widetilde{\alpha} = \alpha \in \mathbb{R}\ ,\quad \widetilde{v} = v \in \mathbb{R}^{1.3} \subset Cl_{1,3} \tag{1.4}$$

This implies

$$\begin{aligned} \widetilde{i} &= i = \gamma_0\gamma_1\gamma_2\gamma_3 = \vec{e}_1\vec{e}_2\vec{e}_3 \\ \widetilde{\vec{e}_k} &= (\widetilde{\gamma_k\gamma_0}) = \gamma_0\gamma_k = -\gamma_k\gamma_0 = -\vec{e}_k. \end{aligned} \tag{1.5}$$

For any quaternion $Q = q_0 + i\vec{q}$, $q_0 \in \mathbb{R}$, $\vec{q} \in \mathbb{R}^3$ one finds

$$\widetilde{Q} = q_0 - i\vec{q} \tag{1.6}$$

and hence

$$Q\widetilde{Q} = q_0^2 + \vec{q}^{\ 2} = \widetilde{Q}Q \geqslant 0. \tag{1.7}$$

The positive quantity $Q\widetilde{Q}$ vanishes only if $Q = 0$. One defines therefore

$$|Q| = \sqrt{Q\widetilde{Q}} = \sqrt{\widetilde{Q}Q} \geqslant 0 \tag{1.8}$$

as the **magnitude** of Q. In contradiction to biquaternions $\psi \in Cl_3$, every quaternion $Q \neq 0$ is invertible

$$Q^{-1} = \frac{1}{Q} = \frac{\widetilde{Q}}{Q\widetilde{Q}} = \frac{\widetilde{Q}}{|Q|^2} \tag{1.9}$$

2 Universal factorizations of biquaternions

Let the quaternions A, B, C, $Q \in Cl_2 \subset Cl_3 \subset Cl_{1,3}$. Then there are **two** classes of biquaternions $\psi = A + iB \neq 0$, which admit universal factorizations into a simple biquaternion multiplied from the righthandside by a quaternion $Q \neq 0$. **Proof :**

$$\begin{aligned}
1.)\, & A \neq 0 \Rightarrow |A| > 0,\; A^{-1}|A|^2 = \widetilde{A} \Rightarrow \\
& \psi = A + iB = (1 + iBA^{-1})A = (1 + iC)A, \\
& C = c_0 - i\vec{c},\; c_0 \in \mathbb{R},\; \vec{c} \in \mathbb{R}^3 \;\Rightarrow \\
& \psi = [1 + i(c_0 - i\vec{c}\,)]A = [1 + ic_0 + \vec{c}\,]A = \\
& = \left(\frac{1 + ic_0}{\sqrt{1 + c_0^2}} + \frac{\vec{c}}{\sqrt{1 + c_0^2}}\right)A\sqrt{1 + c_0^2} = \\
& = (e^{i\eta} + \vec{n})Q \stackrel{\text{def}}{=} \psi_+,\; -\frac{\pi}{2} < \eta < \frac{\pi}{2},\; \vec{n} \in \mathbb{R}^3,\; Q \in Cl_2 - \{0\}.
\end{aligned} \tag{2.1}$$

$$\begin{aligned}
2.)\, & B \neq 0 \Rightarrow \psi = i(B - iA),\; 1.) \Rightarrow\; \psi = i\psi_+ = i(e^{i\eta} + \vec{n})Q \stackrel{\text{def}}{=} \psi_-, \\
& -\frac{\pi}{2} < \eta < \frac{\pi}{2},\; \vec{n} \in \mathbb{R}^3,\; Q \in Cl_2 - \{0\}.
\end{aligned} \tag{2.2}$$

q.e.d..

In the following section it is shown that in case of plane wave solutions of the Dirac equation **the ψ_+ class describes the particles and the ψ_- class describes the antiparticles**.

All non-invertible biquaternions $\psi \neq 0$, $\psi\widetilde{\psi} = 0$ are obtained from (2.1) and (2.2) by calculating

$$\psi_\pm\widetilde{\psi}_\pm = \pm(e^{i\eta} + \vec{n})Q\widetilde{Q}(e^{i\eta} - \vec{n}) = \pm|Q|^2(e^{2i\eta} - \vec{n}^2)$$

or :

$$\psi_\pm\widetilde{\psi}_\pm = \pm|Q|^2\big[\cos(2\eta) - \vec{n}^2 + i\sin(2\eta)\big],\ -\frac{\pi}{2} < \eta < \frac{\pi}{2}. \tag{2.3}$$

So, for $Q \neq 0$, $\psi_\pm \neq 0$ and $\psi_\pm\widetilde{\psi}_\pm = 0$ if and only if

$$\eta = 0 \text{ and } \vec{n}^{\ 2} = 1. \tag{2.4}$$

This ample six-parameter variety is lost in the Yvon [2]-Takabayasi [3] representation of Dirac theory. The same variety was used by G. Lochak [4]. In the next section it is demonstrated that plane solutions of the Dirac equation fulfill (2.4) exactly if the rest mass vanishes (luxons), or equivalently, if the magnitude of the momentum $|\vec{k}|$ tends to infinity.

Let us enumerate some further consequences of (2.4). Equations (2.1) and (2.2) yield the tetrade of Takabayasi

$$f_\mu \stackrel{\text{def}}{=} \psi_\pm\gamma_\mu\widetilde{\psi}_\pm = (e^{i\eta} + \vec{n})Q\gamma_\mu\widetilde{Q}(e^{i\eta} - \vec{n}) \in \mathbb{R}^{1,3},\ \mu = 0,\ 1,\ 2,\ 3 \tag{2.5}$$

$$f_\mu^2 = f_\mu \cdot f_\mu = \gamma_\mu^2|Q|^4\big[(1 - \vec{n}^{\ 2})^2 + 4\vec{n}^{\ 2}\sin^2\eta\big] \tag{2.6}$$

$$f_\mu \cdot f_\nu = 0 \text{ for } \mu \neq \nu = 0,\ 1,\ 2,\ 3 \tag{2.7}$$

$$f_\mu \wedge f_\nu = |Q|^2(e^{-2i\eta} - \vec{n}^{\ 2})(e^{i\eta} + \vec{n})Q(\gamma_\mu \wedge \gamma_\nu)\widetilde{Q}(e^{i\eta} - \vec{n}) \tag{2.8}$$

$$f_0f_1f_2f_3 = i|Q|^8\big[(1 - \vec{n}^{\ 2})^2 + 4\vec{n}^{\ 2}\sin^2\eta\big]^2 = f_0 \wedge f_1 \wedge f_2 \wedge f_3. \tag{2.9}$$

This tetrade collapses completely in case of (2.4) and thus leads to wrong conclusions when (2.4) is approached! In particular the Dirac current $j = f_0$ becomes lightlike in case of (2.4). The spinvector $s = f_3$ behaves in the same way if (2.4) holds :

$$j \cdot s = 0,\ s^2 = -j^2 = 0,\ j \wedge s = 0 \tag{2.10}$$

This behaviour, used in [4], is in strong contrast to the widespread belief that always $j^2 > 0$, $s^2 < 0$ and $j \wedge s \neq 0$. These are some of the most striking mistakes caused by the blind application of the approach of Yvon and Takabayasi!

3 Plane waves.

A formulation of Dirac theory which is independent of matrix representations and invariant with respect to the choice of time-space coordinates was discovered and developped by David Hestenes [5]. In his theory, the Dirac equation for a particle of rest mass

$$m = \mu m_0 \ , \quad o \leqslant \mu < \infty \ , \quad m_0 > 0 \tag{3.1}$$

and charge $q \in \mathbb{R}$ in an external vector potential

$$A = A(x) = [A_0(x) + \vec{A}(x)]\gamma_0 \in \mathbb{R}^{1,3} \tag{3.2}$$

at the point $x = (ct + \vec{r})\gamma_0 \in \mathbb{R}^{1,3}$ is given by

$$\hbar \partial_x \psi(x) i_3 - \frac{q}{c} A(x)\psi(x) = \mu m_0 c \psi(x)\gamma_0 \ , \ \psi(x) \in Cl_3. \tag{3.3}$$

When instead of the time $t \in \mathbb{R}$ and the position vector $\vec{r} \in \mathbb{R}^3$ the dimensionless variables z_0 and $\vec{z}$ are introduced by

$$ct = \lambda\!\!\!^{-} z_0 \ , \ \vec{r} = \lambda\!\!\!^{-} \vec{z} \ , \ \lambda\!\!\!^{-} = \frac{\hbar}{m_0 c} \tag{3.4}$$

the Dirac-Hestenes equation (3.3) is

$$\mu \psi \gamma_0 + a\psi = \partial \psi i_3 \ , \ \partial = \lambda\!\!\!^{-} \partial_x \ , \tag{3.5}$$

$$a = \frac{q}{m_0 c^2} A(\lambda\!\!\!^{-} z) = [a_0(z_0, \vec{z}\ ) + \vec{a}(z_0, \vec{z}\ )]\gamma_0 \tag{3.6}$$

After the gauge transform

$$\psi = \varphi e^{i_3 \chi} \ , \ \chi \in \mathbb{R} \ , \ \varphi \in Cl_3 \ , \tag{3.7}$$

and a split of time from space variables, (3.5) finally has the form

$$\mu \gamma_0 \varphi \gamma_0 + (g_0 + \vec{g})\varphi = (\partial_0 + \vec{\partial})\varphi i_3 \ , \ \varphi = \varphi(z_0, \vec{z}\ ) \in Cl_3 \ , \tag{3.8}$$

$$\partial_0 \varphi = \frac{\partial}{\partial z_0}\varphi(z_0, \vec{z}\ ) = \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha}\varphi(z_0 + \alpha, \vec{z}\ ) \ ,$$

$$\vec{\partial}\varphi = \lim_{\alpha \to 0} \sum_{k=1}^{3} \vec{e}_k \frac{\partial}{\partial \alpha}\varphi(z_0, \vec{z} + \alpha \vec{e}_k\ ) \ , \tag{3.9}$$

$$g_0 = \partial_0 \chi + a_0 \ , \ \vec{g} = \vec{\partial}\chi - \vec{a} \ , \ \chi = \chi(z_0, \vec{z}\ ) \in \mathbb{R}. \tag{3.10}$$

The biquaternion φ may according to (2.1) and (2.2) be factorized in the two universal ways

$$\varphi_+ = (e^{i\eta} + \vec{n}\ )Q \stackrel{\text{def}}{=} \Omega\ ,\quad \varphi_- = i\Omega\ , \tag{3.11}$$

and with the definition

$$\varepsilon = +1 \text{ if } \varphi = \varphi_+\ ,\quad \varepsilon = -1 \text{ if } \varphi = \varphi_-\ , \tag{3.12}$$

equation (3.8) then becomes

$$\varepsilon\mu\gamma_0\Omega\gamma_0 + (g_0 + \vec{g}\ )\Omega = (\partial_0 + \vec{\partial})\Omega i_3. \tag{3.13}$$

Plane wave solutions of (3.13) are obtained by putting

$$\begin{aligned} &a = 0\ ,\ \chi = k_0 z_0 + \vec{k}\cdot\vec{z}\ ,\quad k_0 = \text{ constant } \in \mathbb{R}\ , \\ &\vec{k} = \text{ constant } \in \mathbb{R}^3\ ,\ \Omega = \text{ constant } \in Cl_2\ . \end{aligned} \tag{3.14}$$

This leads in (3.10) to $g_0 = k_0$, $\vec{g} = \vec{k}$ and hence for $Q \neq 0$ in (3.11) with (3.13) to the biquaternion equation

$$\varepsilon\mu(e^{-i\eta} - \vec{n}\ ) + (k_0 + \vec{k}\ )(e^{i\eta} + \vec{n}\ ) = 0, \tag{3.15}$$

which determines $\vec{n}$, η and k_0 in terms of ε, μ and $\vec{k}$. In order to solve for $\vec{n}$ one may write (3.15) in the form

$$(\vec{k} + k_0 - \varepsilon\mu)\vec{n} = -[(\vec{k} + k_0)e^{i\eta} + \varepsilon\mu e^{-i\eta}]. \tag{3.16}$$

For $\eta = 0$ and $\vec{k} = \vec{0}$ this yields$(k_0 - \varepsilon\mu)\vec{n} = -(k_0 + \varepsilon\mu)$, whence

$$k_0 = -\varepsilon\mu\ ,\ \text{ if } \eta = 0 \text{ and } \vec{k} = \vec{0}. \tag{3.17}$$

Multiplying (3.16) from the left by $\vec{k} - k_0 + \varepsilon\mu$, one finds after some rearrangement and making use of (3.12), i.e., $\varepsilon^2 = 1$,

$$\begin{aligned} &[\vec{k}^{\ 2} - (k_0 - \varepsilon\mu)^2]\vec{n} \\ &= -\big[2\varepsilon\mu\vec{k}\cos\eta + (\vec{k}^{\ 2} - k_0^2)e^{i\eta} + \mu^2 e^{-i\eta} + 2i\varepsilon\mu k_0 \sin\eta\big]. \end{aligned} \tag{3.18}$$

The left hand side of equation (3.18) is a vector of degree 1 (in Cl_3). The right hand side of (3.18) in addition contains degree 0 and degree 3 summands, which have to vanish. With $-\frac{\pi}{2} < \eta < \frac{\pi}{2}$ and $\mu k_0 \neq 0$ this implies $\eta = 0$ and

$$k_0 = \pm\varepsilon w \ , \ w = \sqrt{\mu^2 + k^2} \geqslant \mu \ , \ k = |\vec{k}| \geqslant 0. \tag{3.19}$$

The sign of k_0 in (3.19) may be fixed by comparison with (3.17) for $k = 0$. In this way, one finally obtains the result

$$\vec{n} = \frac{\varepsilon\vec{k}}{\mu + w} \ , \ \eta = 0 \ , \ k_0 = -\varepsilon w \ , \ w = \sqrt{\mu^2 + k^2} \ , \ k = |\vec{k}| \ , \tag{3.20}$$

which with (3.14), (3.12), (3.11), (3.7) leads to the **particle** solution ψ_+ and **antiparticle** solution ψ_-

$$\begin{aligned} \psi_+ &= \Big(1 + \frac{\vec{k}}{\mu + w}\Big) Q e^{i_3(-wz_0 + \vec{k}\cdot\vec{z}\ )} \\ \psi_- &= i\Big(1 - \frac{\vec{k}}{\mu + w}\Big) Q e^{i_3(wz_0 + \vec{k}\cdot\vec{z}\ )}. \end{aligned} \tag{3.21}$$

Note the different signs of energies w and momenta $\vec{k}$ in $\psi_\pm$!

This section ends with a discussion of (3.21) in the infinite momentum limit $k = |\vec{k}| \to \infty$, or, equivalently rest mass $m = \mu m_0 \to 0$ ((3.1)). In these cases, according to (3.20), $|\vec{n}| = 1$, whence $\psi_\pm$ in (3.21) then are of the form (2.4). Although this limit behaviour is an exact result of the Dirac equation, it is excluded in all Yvon-Takabayasi approaches to Dirac theory because they presume $\psi_\pm\widetilde{\psi}_\pm \neq 0$!

References

[1] Leonard Euler published his quaternions in a letter to Goldbach at 4.5.1748. (57 years before W.R. Hamilton was born)

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[5] D. Hestenes : *Mysteries and Insights of Dirac Theory*, Ann. Fond. Louis de Broglie, **28** $n°3 - 4$, 367 (2003), Section III, eqs. (8)-(15) and the comment on page 375 about Takabayasi

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