The Dirac Equation through Geometric Algebra: some implications

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ABSTRACT. The Dirac equation for a free electron has a solution which is a localised field with spin and shuddering. Here we note that the solution enables us to estimate the spin angular velocity and the radius of the electron field.

Key words: geometric algebra, multivector, Clifford algebra, spinor, spacetime, Dirac equation, Klein-Gordon equation

1 Introduction

This note is a sequel to [10]. In that paper we studied Toyoki Koga's solution to the Dirac equation using geometric algebra. We showed that the Dirac equation for a free electron has a solution with terms exhibiting spinning and shuddering fields. This solution is a localised field. As Koga ([8]) did, we interpret it as deterministic and consider the electron to be nothing but its field. Here we mention some implications.

It is worth noting that according to Ohanian ([9]), Belinfante in 1939 established (with the standard interpretation of quantum mechanics) that the spin could be regarded as due to a circulating flow of energy, or a momentum density, in the electron wave field. He also states that Gordon proved in 1928 that the magnetic moment of the electron is due to the circulating flow of charge in the electron wave field.

In section 2 we describe the solution of the Dirac equation. The implications are discussed in Section 3 and summarised in Section 4.

2 The Dirac Equation

Our notation is as in [10]. The Dirac equation in geometric algebra, as given by Hestenes (see [3]) is (for a free electron)

$$\nabla \psi I \sigma_3 = m \psi \gamma_0. \tag{1}$$

Here ψ is an even multivector field (defined below) in the Clifford algebra of a 4 dimensional real (Minkowski) spacetime, spanned by orthogonal unit vectors $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ parallel to the coordinate axes. We take γ_0 to be timelike with $\gamma_0^2 = 1$ and γ_i spacelike, with square -1, for $i = 1, 2, 3$.

The Klein-Gordon equation is $\nabla^2 \varphi + m^2 \varphi = 0$ where φ is a multivector field. The Laplacian here is the dot product of the gradient ∇ defined earlier with itself.

From a solution of this equation we can get a solution of the Dirac equation as follows: if $\nabla^2 \varphi + m^2 \varphi = 0$ then $\psi = \nabla \varphi I \sigma_3 + m \varphi \gamma_0$ is a solution of the Dirac equation in STA. If φ is odd then ψ is even and vice versa ([2], Section 10.1).

Koga ([5], [6], [7], [8]) worked out a solution to the Dirac equation, starting with a solution to the Klein-Gordon equation. He started from an idea of de Broglie from the 1920s, but made different assumptions:

$$\varphi = a e^{iS} \tag{2}$$

where a and S are real scalar fields in spacetime. (For convenience, we choose units such that the Planck constant and the speed of light have the value 1.)

For a free electron, expressions can be written out for a and S:

$$S = -Et + \mathbf{p} \cdot \mathbf{r},$$
$$a = \exp(-\kappa |\mathbf{r}'|)/|\mathbf{r}'|$$

where $\mathbf{r}$ is the position of a general point (in 3-space),

$$\mathbf{r}' = (\mathbf{r} - \mathbf{u}t)/(1 - u^2)^{1/2}$$

where $\mathbf{u}$ is the velocity of the electron (in our inertial frame), $u = |\mathbf{u}|$, κ is a positive constant, $\mathbf{p} = \mathbf{u}E$ is the momentum and $E^2 = (m^2 - \kappa^2)/(1 - u^2)$ where E is the energy.

We now consider a frame in which the electron is at rest: $\mathbf{u} = 0$, $\mathbf{p} = 0$, $|\mathbf{r}'| = |\mathbf{r}|$. We take the centre of the electron as the origin.

We write $\phi = ae^{SI\sigma_3}$ for a solution to the Klein-Gordon equation. Then

$$\psi = (\nabla\varphi)I\sigma_3 + m\varphi\gamma_0 \quad \text{where} \quad \varphi = ae^{SI\sigma_3}\gamma_0.$$

We now have $S = -Et$ and

$$a = \exp(-\kappa r)/r \tag{3}$$

where $r = |\mathbf{r}|$, $\mathbf{r} = x^1\gamma_1 + x^2\gamma_2 + x^3\gamma_3$.

In order to get a formula for ψ, it suffices to observe that $\nabla a = a\mathbf{R}$ where $\mathbf{R} = \mathbf{r}\left(\frac{1}{r^2} + \frac{\kappa}{r}\right)$ and so

$$\psi = \mathbf{R}\varphi I\sigma_3 + (E + m)\varphi\gamma_0. \tag{4}$$

3 Implications

If we introduce $\hbar$ and c properly, then our solution to the Dirac equation can be written as

$$\psi = \hbar c\mathbf{R}\varphi I\sigma_3 + (Ec + mc^2)\varphi\gamma_0,$$

and the term expressing the spinning field is given by

$$e^{-\frac{1}{2}(\frac{S}{\hbar}+\frac{\pi}{2})I\sigma_3}(x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3)e^{\frac{1}{2}(\frac{S}{\hbar}+\frac{\pi}{2})I\sigma_3}.$$

Since $S = -Ect$, the angular velocity is

$$\omega = -\frac{Ec}{\hbar}$$

We have

$$E^2c^2 = m^2c^4 - \hbar^2c^2\kappa^2 > 0.$$

Here Ec stands for the energy of the electron. Thus we obtain

$$0 < \kappa < \frac{mc}{\hbar}.$$

Taking

$$m = 9.1094 \times 10^{-31} \text{ Kg },$$
$$c = 2.9979 \times 10^{8} \text{ m/S}$$
$$\text{and} \quad \hbar = 1.0546 \times 10^{-34} \text{ J.S.}$$

We find that $\kappa < 2.5896 \times 10^{12}$ per metre.

The theory assumes that κ is constant but the value of κ is not given by the theory. Now we can roughly calculate the spin angular velocity. With the above values of mass m and speed of light c, the value of m^2c^4 is of the order of 10^{-27}. Similarly the value of $\hbar^2c^2$ is of the order of 10^{-52}. Taking the value of κ in the above range, the spin angular velocity is roughly of the order of 10^{21} radians/second, by using the relation

$$\omega = \frac{Ec}{\hbar} = (\frac{m^2c^4 - \hbar^2c^2\kappa^2}{\hbar^2})^{1/2}.$$

Now we come to the well known concept of spin up and down states.

Let $c = 1$ and $\hbar = 1$. We note that

$$\phi = ae^{-iS}$$

is a solution of the (original) Klein-Gordon equation.

Then by employing the same procedure as in the last section, we get another solution to the Dirac-Hestenes equation:

$$\psi = \mathbf{R}\varphi I\sigma_3 + (-E + m)\phi\gamma_0$$

where $\phi = e^{-I\sigma_3 S}\gamma_0$.

This describes a spinning field with angular velocity $\omega = \frac{Ec}{\hbar}$. Thus we get two values of spin. Koga does not prove that spin is two-valued (at least for a free electron) and the question as to whether this theory implies it is open.

Finally, the present theory enables us to put a bound on the size of the electron field.

Assuming that speeds greater than c do not occur, we must have $\omega r < c$ for any point in the electron. Here r is the distance from the point to the axis of rotation. Thus

$$r < c/\omega.$$

4 Conclusions

Both Koga's theory and the conventional approach to quantum mechanics conclude that spin is due to a circulating flow of energy.

Koga's theory suggests bounds for the size of the electron and its spin angular velocity. It suggests that spin can have two opposite values but does not give a proof.

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