Mechanics and Thermodynamics of the "Bernoulli" oscillators (uni-dimensional closed motions) Part I : Historical review and recent assessments

G. Mastrocinque

Dipartimento di Scienze Fisiche dell'Università di Napoli
"Federico II" - Facoltà di Ingegneria - P.le Tecchio - 80125 Napoli

ABSTRACT. In this paper, we join two different theoretical approaches to the problem of finding a classical-like interpretation of quantum effects : the "fluctuating energy" model and the "hidden variables" model. We show that they merge together into a more powerful, comprehensive one. A basic assumption in the last is that a "vacuum interaction" is responsible for energy fluctuations, and splits the classical motion into two different components. These are as it was first identified, on general grounds, by Kapitza in a famous theorem : a "super-oscillation" of the particle position and velocity around a center ; and the motion of the center itself, respectively. These motions define what we call the hidden degree(s) of freedom HDF, figuring in the energy theorem through a peculiar potential that we show correlated with a mass effect. The implicated functions for each energy level are called "the mass eigenfunctions". Classical oscillators submitted to these vacuum perturbations exhibit a quantum-like behavior. We name them the Bernoulli oscillators, because their properties came out in a frame where the mass-flow theorem takes a dominant role. A brief historical review and recent assessments are given in the present Part I of the work. Two concrete examples will be solved numerically in the following Part II ; the results will also allow us to give insight into the classical limit peculiar to the model, so that this last will be found there expounded afterwards.

RÉSUMÉ. Dans cet article, nous allons joindre deux approches théoriques différents au problème de trouver une interpretation de type classique à des effets quantiques : le modèle des "fluctuations d'énergie" et le modèle des "variables cachées". Nous montrons qu'ils donnent lieu à un modèle plus puissant. Une hypothèse de base dans ce dernier est que

une "intéraction de vide" cause les fluctuations d'énergie, le mouvement classiques se partageant en 2 composantes. Celles ci sont comme l'avait déja trouvé, sur une base générale, Kapitza dans un théorème connu : une "super-oscillation" de la position et de la vitesse de particule au tour d'un centre, et le mouvement du centre lui-meme, respectivement. Ces mouvements definissent ce que nous appellons les degrés de liberté cachés HDF, apparaissant dans le théorème d'énergie par des potentiels particuliers dont nous montrons la correlation avec un effet de masse. Les fonctions impliquées sont appellées "fonctions de masse". Les oscillateurs soumis à une telle perturbation due au "vide" montrent un comportement quantique. Nous les appellons les oscillateurs de Bernoulli, parce que leur proprietés s'engendrent dans un cadre où le théorème du flux de masse prends un role dominant. Dans la présente Partie I de l'article, on donne une brève revue hystorique ainsi que des développements récents. Deux exemples concrets seront résolus numériquement dans la suivante Partie II ; les résultats nous permettrons d'envisager la limite classique, speciale à ce modèle, que l'on trouvera donc examinée en fin de travail.

PACS. 45.50.-j - Dynamics and kinematics of a particle and a system of particles
PACS. 03.65.Ta - Foundations of Quantum Mechanics

1 Introduction

Still in recent years, a number of papers have been dedicated to discover and discuss relations enlightening the elusive consistency between classical and quantum mechanics. Many of these relations are well known, and historically consolidated, as (orthodox) approximations of quantum mechanics to the classical (JWKB, quasi-classical approximations) [1-3]. Many others have recently been produced within modern theoretical frames, as f.i. random matrix theory, trace formula, periodic orbits, chaos and recurrence spectroscopy [4÷12]. A number of them are rather heuristic, but interesting findings, with great potential for further developments (from the famous Fermi-Pasta-Ulam subject [13] to the symmetrization rules delivering detailed balance to the classical (inelastic) cross-sections [14-15]). A variety of now classical [16÷20] and original [21-23] approaches is available, and even a quite specific experimental insight has been given into the classical behavior which can be found embedded into deep quantum phenomena [24]. Amongst other, attention has been drawn to very basic features of classical physics which

are strictly linked to quantum appearances : e.g. Carati and Galgani [25] have pointed out that in molecular physics we can find some classical-like action parameters, which by their recurrence - and (even numerical) similarity to the Planck's constant, are worth a role as classical precursors to this last. The same authors have also given models able to reproduce, at least conceptually, some peculiar quantum effects as tunnelling and weak reflection. This has been done using classical non-unique solutions of the Dirac equation of motion [26].

On another hand, the historical framework of the so called old quantum theory has been, since almost one century, abandoned. Yet it looks having been so only because a much more powerful theory - the matter wave mechanics - has been developed, and brought to dominance. So the potential of that incomplete model has been finally neglected.

During the century, another strong "ideology" has also independently grown, up to shadow closely the now orthodox quantum mechanics. It is based on the assumption of a (so-called) hidden degree of freedom (HDF) playing the supplemental actor in classical mechanics, and being responsible for quantum behaviors. This investigation domain has been initiated by de Broglie [27÷28]. Energetically carried out by Bohm [29÷32], and nowadays by a number of authors [33], it has given rise to a great expectation that a classical motion concept and a particle velocity are hidden in the wave equation, and may be taken out by various techniques [5, 34÷38].

A HDF framework has also been developed [39÷42] by the author of this paper, moving from a number of different observations, and specifically enclosing elements of the old quantum theory (e.g. the Bohr-Sommerfeld rules, reinterpreted). Yet the physical model essentially starts from another one, already available in the history, whose origin looks dated back to Einstein [43], Kapitza [44-45] and Vigier [30] . This is the so-called fluctuation model, based on the assumption that an external perturbation (coming from the quantum "vacuum") causes the classical energy of a system to fluctuate ; and just aimed at discovering the action able to bring the system to quantum behavior indeed.

As it will be shown here with more details, the recent developments of this model have brought indeed the author in the previously quoted papers to consider the possibility that mass fluctuations are also involved in particle dynamics and very effective to the end of explaining quantum-like results by classical reasoning. Actually, from the Higgs boson to the electromagnetic mass, up to cosmological effects in a Machian scheme,

the mass concept has lately got wide reconsideration in the literature [46-50]. That it can be involved as a hidden degree of freedom into both classical and quantum dynamics has been recently pointed out, with interesting results in Hydrogen orbital modelling and related magnetic properties, also by X. Oudet at the de Broglie Foundation [51-54].

Within this quoted framework (which is by now limited to the analysis of uni-dimensional motions), a definite interpretation of the classical/quantum physics relations is developed. A specific, classical-like motion equation is also proposed as a possible issue to what - many people think - is only paradoxically an antagonism between the two mechanical theories.

In the present paper, we expound the more complete model that we have obtained by re-elaborating, and further developing, the most important amongst previous results. To introduce it here, we first review the main features of the fluctuation model, reporting its essential approach. We provide the connection with the HDF model, what brings us to introduce mass effects and new potentials, both in the classical energy theorem and in the matter wave equation. We introduce upgraded model equations, and solve them for the typical cases of the rectangular well and harmonic potential (Part II of this work).

So we hope having provided the reader with a synthetic view of the relationships emerging from the assumptions of both - the hidden degree of freedom and the fluctuating energy models - being used together ; in the attempt to wrap (at least some) basic quantum appearances by the means of a classical-like formalism.

We start here with, very briefly, recalling Einstein's interpretation of Planck's law (recently well resumed in [25]). The de Broglie, Kapitza, Vigier and Bohm views are also recalled, followed by a description of more recent developments and results.

2 Einstein, Kapitza and Vigier : the historical origin of the fluctuation model

Although very schematically, we might resume Einstein's view [43,25] by the following equations :

$$\mathrm{E}_{BE} = \frac{\Delta \mathrm{E}}{\exp(\Delta \mathrm{E}/\mathrm{T}) - 1} \tag{1}$$

$$\sigma_E^2 = \mathrm{E}_{BE}(\mathrm{E}_{BE} + \Delta \mathrm{E}) = \mathrm{T}^2 \frac{d\mathrm{E}_{BE}}{d\mathrm{T}} \tag{2}$$

Here E_{BE} is the Bose−Einstein energy distribution expression, $\Delta E/h = \nu_c$ is the classical oscillator frequency, and σ_E^2 is defined as a variance. Einstein interpreted equation (2) indeed as a peculiar relationship between the mean value E_{BE} of the distribution and the variance. Then he thought that a deep interpretation of quantum mechanics should involve a fluctuating energy model.

Later in the history, Kapitza gave a famous theorem [44−45] where the mean effect of a fast fluctuating force, perturbing a classical oscillator, was calculated. The fast force imposes a"super-oscillation", of amplitude $\eta(t)$, to the particle motion around a center. This is at a position x in a uni-dimensional space. The energy theorem given by Kapitza has the form

$$\frac{1}{2}mv^2 + \Phi(x) + \Phi_{pert}(x) = \frac{1}{2}mv^2 + \Phi(x) + < \frac{1}{2}m\,\dot{\eta}(t)^2 > = E_n \qquad (3)$$

Here $\Phi(x)$ is the classical potential energy characterizing the oscillator, mass m, (absolute) velocity v. The additional potential $\Phi_{pert}(x)$ acting on the particle is shown by Kapitza to be a time-averaged value, at the position x, of the kinetic energy pertaining to the super-oscillation ([1]).

Approximately in the same period, Vigier [30] expressed the sound opinion that quantum behaviors might be explained by the effects of some chaotic perturbation coming from the "vacuum".

3 De Broglie and Bohm : the search for "hidden parameters" subtending quantum mechanics

On more general grounds Vigier and Bohm [30], basing upon former ideas by de Broglie [27,28], started with investigating the possibility that vacuum fluctuations may reverse into classical motions in the form of "hidden" parameters. Actually Bohm [29÷32], since 1952, has systematically developed a wide reference framework for the very definite idea that an hidden degree of freedom should be accounted for, within classical mechanics, to explain quantum behaviors ; and possibly be brought to evidence. We report here the key relations funding this view, in the form useful to our paper (based on the Madelung equations, uni-dimensional stationary case) :

[1] To our purposes, we since now identify the energy constant as the n-state eigenvalue E_n.

$$\Phi_B(\mathrm{x}) = -\frac{\hbar^2}{2\mathrm{m}}\frac{A(\mathrm{x})''}{A(\mathrm{x})} = \mathrm{E}_n - \Phi(\mathrm{x}) - \frac{\nabla S^2(\mathrm{x})}{2\mathrm{m}} \tag{4}$$

$$A(\mathrm{x})^2\nabla S(\mathrm{x}) = \rho(\mathrm{x})\nabla S(\mathrm{x}) = J(\mathrm{x}) \tag{5}$$

$$\mathrm{v}(\mathrm{x}) \in \frac{\nabla S(\mathrm{x})}{\mathrm{m}} \tag{6}$$

As is well known, these typical equations subtend a quantum mechanical wavefunction (space-dependent part) of the form

$$\Psi_n(\mathrm{x}) = A(\mathrm{x})\exp(iS(\mathrm{x})/\hbar) \tag{7}$$

$$\oint \nabla S(\mathrm{x})d\mathrm{x} = 2\pi(\mathrm{n}-1)\hbar \tag{8}$$

with $S(\mathrm{x}) \equiv$ phase function and $J(x)/m \equiv$ probability current density ([2]). They are submitted to the Bohr postulate (we have a comment on this matter, in the conclusive section of the paper in Part II). The potential $\Phi_B(\mathrm{x})$ is the Bohm potential, or quantum potential. The function $\rho(\mathrm{x})$ is the nth-state quantum density, m the particle mass, E_n the energy eigenvalue. Apices stand for x-derivatives. As is clear, all the quoted quantities pertain to a quantum state n : but for the sake of simplicity, we omit the index when no confusion may arise. We have resumed Bohm's view by the means of equation (6), stating that a particle velocity field v(x) is implicated into the quantum framework. Very often in the literature, one finds actually v(x) $defined$ as $\nabla S(\mathrm{x})/\mathrm{m}$; but in this paper, different roles will be given to these two quantities.

Following Bohm's perception of the physical problem, it can be said that discovering the "implicated order" within these equations (i.e. taking out the particle velocity field expression) is the challenge for people believing into a causal interpretation of quantum mechanics.

It is not to say here that by "causal interpretation" a restored - although differently featured - determinism, and Newtonian-like motion concept, are intended as the possible "explication of order" subtending the matter wave behavior.

[2]In standard quantum mechanics of stationary states, the divergence of the current density is zero. In our theory, we introduce a "mass eigenfunction" concept implying an additional Hamiltonian term in the wave equation. It causes a dependence of the current on x. These circumstances will be found better discussed in a next section.

A number of techniques [23,26,34÷38] have been developed to calculate fields (6) and to show consistency of the Bohm's view. A number of now historical results as in [16÷20,55÷57] produced even by different approaches or viewpoints (stochastic mechanics, quantum macrodynamics, measurement theory, dissipation theory, classical electrodynamics) are also useful, and available in the literature.

To the same end, in papers [39÷42,58÷60] the (mechanical and thermodynamical) properties of the so-called Bernoulli oscillators have been investigated, and correlated to both classical and quantum behaviors. In the next section, we will review the essential results of that work : this also provides a reassessment to the advantage of the upgraded model next introduced.

4 The Bernoulli oscillators model

4.1 Basic features

The Bernoulli oscillators are classical entities, submitted to a (generalized) Kapitza (or Vigier)-like external force producing energy fluctuations. It excites locally a sort of parametric "super-oscillation" of both the particle position and velocity around a central position, so driving the motion of the center itself. We specifically intend, in our model, that the perturbation comes from the quantum vacuum and is responsible for quantum behaviors. All the motions components are treated classically in the fluctuation model. The super-oscillation is assumed classical-like and harmonic in a first approximation ; in a statistical sense, it is able to fit the Heisenberg incertitude relations [40]. We leave out here the question whether it is "realistic" to approach the Heisenberg principle by a classical-like description ([3]). We rather stress that the super-oscillation forms some kind of physical interface between the vacuum and the oscillator, behaving as a supplemental potential. In this way, the perturbational components of the oscillators motion form what we have called the "hidden degree(s) of freedom" HDF. On a stationary point of view, turning to eq.(3), the perturbational potential energy $\Phi_{pert}(\mathrm{x})$ there appearing can be now identified with the potential $\Phi_{\mathrm{HDF}}(\mathrm{x},\xi_{hid}(\mathrm{x}))$ as defined in [39÷42]([4]). This was calculated by a procedure where - essentially - the Bohm potential in eq. (4) is identified with an indicated work (which is

[3] A number of very interesting discussions of this subject have left historical traces ; one may f.i. refer to [3].

[4] We name here ξ_{hid} the same variable called ξ in [39÷42].

the reason for emblematically entitling the model to D. Bernoulli). So our reference expression of the energy theorem from the quoted papers is

$$\frac{1}{2}\mathrm{mv}^2(\mathrm{x}) + \Phi(\mathrm{x}) + \Phi_{\mathrm{HDF}}(\mathrm{x},\xi_{hid}(\mathrm{x})) = \mathrm{E}_n \tag{9}$$

$$\Phi_{\mathrm{HDF}}(\mathrm{x},\xi_{hid}(\mathrm{x})) = -\frac{1}{2}\delta\mathrm{m}(\mathrm{x})\ \mathrm{v}^2(\mathrm{x}) - \frac{\hbar^2}{2\mathrm{m}}\frac{\mathrm{v}(\mathrm{x})^{2''}}{\mathrm{v}(\mathrm{x})^2} + \mathrm{C}(\mathrm{v}^2) \tag{10}$$

Here $\mathrm{v}(\mathrm{x})$ is the velocity field pertaining to a single particle and $\delta\mathrm{m}(\mathrm{x})$ is the induced mass due to the vacuum action. The HDF nature can be reported indeed to a mass effect. $\mathrm{C}(\mathrm{v}^2)$ is a function to be enlightened in the next section.

Note that equations (9) and (10) bring us to define the effective mass $\mathrm{m}_{eff}(\mathrm{x})$, which we call "the mass eigenfunction" :

$$\mathrm{m}_{eff}(\mathrm{x}) = \mathrm{m} - \delta\mathrm{m}(\mathrm{x}) \tag{11}$$

In the section entitled to it, a primary model to express this function and its properties will be drawn out.

In order to approach the basic quantum-mechanical equation starting with equation (9), two branches of the model now appear : we have a many particles (MP) or statistical-ensemble frame, and a single-particle (SP) frame. The statistical ensemble will be found defined in the next section. Some outlines of the SP view are also given in the sequel, because they are essential to our model ; in this paper, however, only the stationary energy theorem expression and the ensemble properties will be actually expounded with details and brought to numerical solution. Giving explicit solutions to the time-dependent SP equations does not imply conceptual difficulties but is a task demanded to future papers.

4.2 Mechanical and Statistical framework

4.2.1 Energy theorem and single-particle densities

The term named $\mathrm{C}(\mathrm{v}^2)$ in eq. (10) has an expression we have now available by new investigations. In principle, it reveals rather involved - but we are also able to produce here a very simple approximation, stressing the physical effect.

As seen in [42, eq.(53)] already, the term including the second derivative of the squared velocity in eq. (10) implies solving equation (9)

by a family of functions $v(x,x_{0n}(E))$ ([5]) for each quantum number. They correspond to an ensemble of stopping points $x_{0n}(E)$ where different energy values E are involved, depending on the value assigned to $v(x)^{2''}/v(x)^2$ when $v \to 0$. Then as a main effect, that term correlates to the energy broadening, and actually defines the fluctuation interval. For a simple description of this fact, taking into account an appropriate energy constant, we are brought to write

$$\Phi(x_{0n}(E)) = E - \frac{E_{nf} - E_{ni}}{2} \tag{12}$$

$$-\frac{\hbar^2}{2m}\frac{v(x,x_{0n}(E))^{2''}}{v(x,x_{0n}(E))^2} + C(v^2) \simeq E_n - E + \frac{E_{nf} - E_{ni}}{2} + \phi(x) \tag{13}$$

$$\frac{1}{2}\, m_{eff}(x)\; v(x,x_{0n}(E))^2 + \Phi(x) + \phi(x) + \frac{E_{nf} - E_{ni}}{2} = E \tag{14}$$

Comparing with eq. (10), we see that the function $\phi(x)$ is a residual component of the potential $\Phi_{HDF}(x,\xi_{hid}(x))$, and will be determined in a next section. By now we define a total effective potential energy

$$\Theta(x) = \Phi(x) + \phi(x) + \frac{E_{nf} - E_{ni}}{2} \tag{15}$$

This is to the purpose of conciseness.

To define the statistical ensemble, we can think to the fluctuation energies E (each of them playing the motion constant) as distributed to a particles packet. These energy values range across the fluctuation interval ($E_{ni} \leqslant E \leqslant E_{nf}$), where E_{nf} and E_{ni} are the maximum and the minimum energy values attainable by the fluctuation, respectively. Then a collection of energy states, with probability density (see eq.(18)) normalized to unity, defines the statistical properties of an "average" representative particle in the ensemble view. If we name $v_{_D}(x)$ the velocity of this particle (it is coincident with the center of mass or group velocity of the packet), and $\nu_n(x)$ the numerical flow function characterizing the ensemble in the n-th state, we have the constitutive equation

$$\nu_n(x) = \frac{v_{_D}(x)\rho(x)}{2} \tag{16}$$

[5] Since now, we may use the expression $v(x,x_0(E))$ in place of $v(x)$ to stress that every velocity field at energy E ends in a specific turning point $x_0(E)$.

Starting with the same equation (14), a single-particle approach can be also taken ; but then the quantity E must be considered a fluctuating, time-dependent energy affecting the particle. The correlation between the statistical ensemble and the SP case is obtained through the ergodic assumption (in a next section).

Each of the energy states (14) is characterized by a period T(E), a frequency ν(E), and a density $\rho_{_{SP}}$(x, x_{0n}(E)) (classically defined quantities).

Taking the v(x) expression out of eq.(14), we find easily

$$\rho_{_{SP}}(\mathrm{x},\mathrm{x}_{0n}(\mathrm{E})) = \frac{2}{\mathrm{T(E)v(x},\mathrm{x}_{0n}(\mathrm{E}))} = \nu(\mathrm{E})\sqrt{\frac{2\mathrm{m}_{eff}(\mathrm{x})}{\mathrm{E}-\Theta(\mathrm{x})}} \qquad (17)$$

4.2.2 Statistical properties

In our model, the nth quantum state is thought as a statistical ensemble of E-states, each of them represented with probability P(E). The quantum-state density is a statistical superposition of the classical-like densities (17) included in the microcanonical ensemble. Each of the velocity fields pertaining to the ensemble is submitted to the classical-like energy theorem of the form (14).

For consistence with the corresponding SP time-dependent framework, an ergodic assumption must be used to determine the probability density. The density P(E) must be taken proportional to the lifetime of every state E. Then we assume that it is just proportional to the period T(E) ([6]) :

$$\mathrm{P(E)} = \frac{1}{c_n \mathrm{h}}\mathrm{T(E)} \qquad (18)$$

$$\int_{\mathrm{E}_{ni}}^{\mathrm{E}_{nf}} \mathrm{P(E)}d\mathrm{E} = \frac{1}{c_n \mathrm{h}} \int_{\mathrm{E}_{ni}}^{\mathrm{E}_{nf}} \mathrm{T(E)}d\mathrm{E} = 1 \qquad (19)$$

The normalization condition here implies that all the statistical averages we may take over the statistical ensemble are intended "per particle". As far as the period T(E) is concerned, note that it is always a classically

[6]The density given here is an upgraded form compared to the one given in [41], with an additional coefficient c_n. This normalization coefficient is to be determined when solving the equations numerically case by case.

calculated quantity, but relates to the effective potential appearing in equation (14) and is not coincident in principle with the period $T_c(E)$ corresponding to the classical potential $\Phi(x)$ except when we actually go to the classical limit.

The theory implying an effective mass concept, we have now to understand the relationship of this last with the classical mass m ; and the reason why - if a physical reality - the mass effect may have escaped to experimental observation until now. The very reasonable interpretation seems to us that the induced mass δm in equation (11) probably consists in a fluctuation around a zero value. We ask therefore to our model that the statistical mean value of $m_{eff}(x)$ is kept equal to the classical mass m.

This requirement brings us to the following condition :

$$< m_{eff}(x) >= \frac{1}{2} \oint \rho(x)\ m_{eff}(x) dx = m \tag{20}$$

We also ask that the mean statistical value of the fluctuation energy (both in the SP time-dependent view and in the ensemble scheme) are kept equal to the corresponding quantum energy eigenvalue, for each of the quantum states of interest. So we write as a second condition

$$< E >_n = \frac{1}{c_n h} \int_{E_{ni}}^{E_{nf}} T(E) E dE = E_n = \frac{E_{nf} + E_{ni}}{2} \tag{21}$$

On general grounds, we would ask to our theory that all the physical quantities take the correct (i.e., as given by quantum mechanics) average values during the fluctuation. But here important remarks must be done. The fluctuation model rests on the very definite interpretation that the space co-ordinate x is the center position of the super-oscillation induced by the vacuum. In orthodox quantum mechanics, the space co-ordinate is instead a (probabilistic) variable representing the effective particle positions in space. Moreover, as enlightened in the sequel, we join our classical-like theory to a modified quantum-mechanical scheme, where variable current-density effects are accounted for. Specifically, we use a non-standard Hamiltonian (see next eq.(39)) in our calculations to find out the wavefunction $\Psi_n(x)$ and the density $\rho(x)$. These quantities will turn out (although not too much) different from the ones calculated by the use of the standard Hamiltonian. Therefore one should not expect that the model is able to reproduce exactly all the orthodox quantum

mechanics results. This is a very important point when searching sound differences to be brought to evidence, and possibly submitted to experimental checks. So many developments and tests for this expectation are still to come in the future.

We can integrate the density (17) with the energy distribution (18) and we obtain the statistical average

$$< \rho_{_{SP}}(\mathrm{x},\mathrm{x}_{0n}(\mathrm{E})) > \; = \frac{1}{c_n \mathrm{h}} \,\mathrm{Re} \int_{\mathrm{E}_{ni}}^{\mathrm{E}_{nf}} \mathrm{T}(\mathrm{E}) \rho_{_{SP}}(\mathrm{x},\mathrm{x}_{0n}) d\mathrm{E} =$$

$$= \frac{2\sqrt{2\mathrm{m}_{eff}(\mathrm{x})}}{c_n \mathrm{h}} \left(\sqrt{\mathrm{E}_{nf} - \Theta(\mathrm{x})} - \mathrm{Re}\,\sqrt{\mathrm{E}_{ni} - \Theta(\mathrm{x})} \right) \tag{22}$$

To be clear, note explicitly that we submit the previous expression to the Bohr condition

$$< \rho_{_{SP}}(\mathrm{x},\mathrm{x}_{0n}(\mathrm{E})) > \; = \rho(\mathrm{x}) = A(\mathrm{x})^2 \tag{23}$$

Now the following remark must be done. Equation (22) cuts the space in 2 Regions.

This is because in the particles ensemble only the ones with greater energy will go further in space before stopping and turning back ; the particle flow then starts with decreasing (see eqs. (29),(30)) at the space position x_n where the particle with minimal energy E_{ni} stops ([7]) :

$$\Theta(\mathrm{x}_n) = \mathrm{E}_{ni} \tag{24}$$

We call Region I or "internal region" the part of space where

$$\mathrm{x} : \, \Theta(\mathrm{x}) < \mathrm{E}_{ni} \; \Rightarrow (0 \leqslant \mathrm{x} < \mathrm{x}_n \,) \quad \Longleftrightarrow \quad \mathrm{x} \in \; \{\text{Region I}\} \tag{25}$$

We name instead Region II or "external region" the part where the contrary occurs :

$$\mathrm{x} : \, \Theta(\mathrm{x}) \geq \mathrm{E}_{ni} \; \Rightarrow (\mathrm{x}_n \leqslant \mathrm{x} \leqslant \mathrm{x}_0^* \,) \quad \Longleftrightarrow \quad \mathrm{x} \in \; \{\text{Region II}\} \tag{26}$$

Note that in refs. [39÷42], "Region I" and "Region II" were defined differently. But in the present paper, we find that is more appropriate to

[7] For the sake of simplicity, we assume symmetry of all potentials and other functions around $\mathrm{x} = 0$. So only the "right-half-space" with $\mathrm{x} > 0$, and the "right hand" specifications x_0, x_n etc. are essentially referred to in this paper.

turn to equations (25) and (26) (see a comment to the next equations (29) and (30)). The coordinate x_0^* is actually the extreme boundary attainable by the density. For a rectangular well of length L, centered in 0, $\mathrm{x}_0^* = \mathrm{L}/2$. In the harmonic oscillator case, $\mathrm{x}_0^* \to \infty$ ([8]). The abscissa x_n divides the two regions : all the stopping points $\mathrm{x}_{0n}(\mathrm{E})$ are obviously located in Region II. In this context, we also define a quantity

$$\mathrm{E}_{\min}(\mathrm{x}) = (\mathrm{E}_{ni} - \Theta(\mathrm{x}))\,\mathrm{UnitStep}[\mathrm{x}_n - \mathrm{x}] + \Theta(\mathrm{x}) \tag{27}$$

This is for use as an integration limit when calculating local averages over the energy interval. So it is easily seen that many of the functions we deal with in this paper assume different expressions in the two regions. For instance, as far as the flow function is regarded, we have :

$$< \nu\,(\mathrm{E}) > = \nu_n\,(\mathrm{x}) = \int_{\mathrm{E}_{\min}(\mathrm{x})}^{\mathrm{E}_{nf}} \frac{\mathrm{P}(\mathrm{E})}{\mathrm{T}(\mathrm{E})} d\mathrm{E} \tag{28}$$

$$\begin{aligned} \nu_n\,(\mathrm{x}) \to \nu_{n\mathrm{I}}\,(\mathrm{x}) = & \int_{\mathrm{E}_{ni}}^{\mathrm{E}_{nf}} \frac{\mathrm{P}(\mathrm{E})}{\mathrm{T}(\mathrm{E})} d\mathrm{E} \\ & = \frac{\mathrm{E}_{nf} - \mathrm{E}_{ni}}{c_n \mathrm{h}} = \nu_{n0} \quad \{\text{Region I}\} \end{aligned} \tag{29}$$

$$\begin{aligned} \nu_n\,(\mathrm{x}) \to \nu_{n\mathrm{II}}\,(\mathrm{x}) = & \int_{\Theta(\mathrm{x})}^{\mathrm{E}_{nf}} \frac{\mathrm{P}(\mathrm{E})}{\mathrm{T}(\mathrm{E})} d\mathrm{E} \\ & = \frac{\mathrm{E}_{nf} - \Theta(\mathrm{x})}{c_n \mathrm{h}} \qquad \{\text{Region II}\} \end{aligned} \tag{30}$$

Note here that, when the flow $\nu_n(\mathrm{x})$ is a constant as in Region I, the particles packet is a closed one. The contrary occurs when the flow gradient is different from zero, as in Region II. Referring to a previous remark, we see that it is appropriate to define the two regions of space in terms of the flow behavior : in the whole Region I the flow takes a constant value and the packet is ("dynamically") closed ; in Region II the flow is a variable function and the particles packet is open.

[8] Note here already that in the classical limit, (see eqs.(22)÷(25) in Part II), the wavefunction will come uncoupled from the particles density. It can be shown that the last will then be exhausted at the finite point $\mathrm{x}_0^* = \sqrt{\frac{\mathrm{n}\hbar}{\pi \mathrm{m} \nu_c}}$ in the harmonic oscillator case.

Concerning the expression of the flow function in Region II, we have found a well refined one in [42, eq.(80)] already. Yet in the present paper, for the sake of direct physical insight, we want to give another one, which is also simpler on a mathematical point of view. To this end, first consider the following. In a "second solution" (de Broglie-like) scheme, we assume eqs. (9) and (10) to hold not only for a single particle case, but also for the statistically equivalent particle with average square velocity $<\mathrm{v(x,x_{0n}(E))}^2>$. This means that, provided the function $\mathrm{C(v^2)}$ is uninfluential (what happens in Region II [42, eq.(65)]) we can identify the second-order derivative terms in (10) and (4) by means of the substitution

$$\mathrm{v(x,x_{0n}(E))}^2 \rightarrow < \mathrm{v(x,x_{0n}(E))}^2 > \equiv \mathrm{const}\sqrt{\rho(\mathrm{x})} \quad \{\text{Region II}\} \qquad (31)$$

Now since the difference between $\frac{1}{2}\mathrm{m} <\mathrm{v(x,x_{0n}(E))}^2>$ and $\frac{1}{2}\mathrm{m}\ \mathrm{v}_{D}^2(\mathrm{x})$ is well represented by the incertitude interval $c_n \mathrm{h}\nu_n(\mathrm{x})$, we can write easily ([9])

$$\frac{1}{2}\mathrm{m}\mathrm{v}_{D}^2(\mathrm{x}) \ + \tau_n \ c_n \mathrm{h}\nu_n(\mathrm{x}) = \mu_n \sqrt{\rho(\mathrm{x})} \qquad \{\text{Region II}\} \qquad (32)$$

where τ_n and μ_n are appropriate constants so that

$$\nu_n(\mathrm{x}) = \frac{-\tau_n c_n \mathrm{h}\ \rho(\mathrm{x})^2 + \sqrt{8\,\mu_n \rho(\mathrm{x})^{5/2} + \tau_n^2 c_n^2 \mathrm{h}^2 \rho(\mathrm{x})^4}}{4\mathrm{m}} \{\text{Region II}\} \qquad (33)$$

This expression also satisfies, as it must [42], the asymptotic behavior $\nu_n(\mathrm{x}) \rightarrow \rho(\mathrm{x})^{5/4}$ when $\mathrm{x} \rightarrow \mathrm{x}_0^*$ ([10]). The coefficients ν_{n0}, τ_n , μ_n must be determined for each specific case when solving the equations by numerical techniques.

Note now that using equations (16), (29), (30), equation (22) can also be put in the form (holding in both Regions)

$$\frac{1}{2}\mathrm{m}_{eff}(\mathrm{x})\mathrm{v}_{D}^2(\mathrm{x}) + \frac{c_n^2\,\mathrm{h}^2 \rho(\mathrm{x})^2}{32\mathrm{m}_{eff}(\mathrm{x})} + \Phi(\mathrm{x}) + \phi(\mathrm{x}) + \frac{c_n \mathrm{h}\nu_n(\mathrm{x})}{2} = \mathrm{E}_n \qquad (34)$$

[9] Note that the turning point corresponding to the group velocity is always the farest one at the space border, i.e. $\mathrm{x}_{0n}(\mathrm{E}_{nf})$.

[10] The exponent changes from 5/4 into a value 2 in the classical limit, see eq.(20) in Part II of this work, and eq. (13) in [41].

Using a procedure similar to the one shown in equations (12)÷(31) of ref.[41] ([11]), this equation can be usefully compared to what we get by performing a statistical average of eq.(14) :

$$\mathrm{K}_{cm}(\mathrm{x}) + <\mathrm{K}'> \ + \mathrm{K}_{\mathrm{vmre}} + \Theta(\mathrm{x}) = \mathrm{E}_n \tag{35}$$

This is actually the general form taken by the classical energy theorem when statistically averaged over a particles ensemble. Here $\mathrm{K}_{cm}(\mathrm{x})$ is the center of mass energy of the particles packet, $<\mathrm{K}'>$ the residual kinetic energy in the c.o.m. frame and $\mathrm{K}_{\mathrm{vmre}}$ the reactive potential due to inflow-outflow of particles from the open packet in Region II. Now the $<\mathrm{K}'>$ and $\mathrm{K}_{\mathrm{vmre}}$ expressions actually depend on the specific particles statistics inside the packet traveling with velocity $\mathrm{v}_{_D}(\mathrm{x})$, and moreover are affected by the fluctuating mass assumption we have done. So their precise constitutive expressions are actually unknown to us, still wanting extensive analysis in this direction ; but taking all the particles over space as belonging to the packet [42], we have in any case :

$$\mathrm{K}_{cm}(\mathrm{x}) = \frac{1}{2}\mathrm{m}_{eff}(\mathrm{x})\mathrm{v}_{_D}^2 \tag{36}$$

$$\begin{aligned} <\mathrm{K}'> \ + \mathrm{K}_{\mathrm{vmre}} - \mathrm{E}_n + \frac{\int_{\mathrm{E}_{\min(\mathrm{x})}}^{\mathrm{E}_{nf}} \mathrm{EP(E)dE}}{\int_{\mathrm{E}_{\min(\mathrm{x})}}^{\mathrm{E}_{nf}} \mathrm{P(E)dE}} \\ = \frac{c_n^2\, \mathrm{h}^2 \rho(\mathrm{x})^2}{32\mathrm{m}_{eff}(\mathrm{x})} \qquad \{\text{Region I+II}\} \end{aligned} \tag{37}$$

and specifically

$$<\mathrm{K}'> \ - \frac{c_n^2\, \mathrm{h}^2 \rho(\mathrm{x})^2}{32\mathrm{m}_{eff}(\mathrm{x})} \ = - \mathrm{K}_{\mathrm{vmre}} = 0 \qquad \{\text{Region I}\} \tag{38}$$

These equations are enough to pursue up a consequent physical interpretation in the sequel.

A number of other different properties or findings add to the scheme reported in the present section. Next (and crucial) to say, is that the (classical-like) energy theorem (9) and the Madelung equations (4)÷(6)

[11] The essential differences with the quoted analysis in [41] are that we have to put $\mathrm{m}_{eff}(\mathrm{x})$ in place of m and $\Theta(\mathrm{x})$ in place of $\Phi(\mathrm{x})$.

can be made consistent with each other, provided we assume appropriate functions $\mathrm{m}_{eff}(\mathrm{x})$, $\phi(\mathrm{x})$, $J(\mathrm{x})$ in all these equations. With specific reference to eq. (5), this is tantamount to consider "variable-current quantum states". They can be produced by inserting an extra Hamiltonian term as a divergence source in the quantization procedure. We illustrate the matter with details in the very next section.

4.2.3 Variable current-density quantum states and source potential

Probability current densities of unperturbed stationary states have zero divergence in the orthodox quantum mechanical formalism. But the point here is that accounting for the mass effect identified in eq. (10) makes a difference. Then with reference to the quantization procedure we propose to consider an Hamiltonian :

$$\hat{\mathrm{H}} = \frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} + \Phi(\mathrm{x}) + i\,\Phi_{im}(\mathrm{x}) \tag{39}$$

Here $\hat{\mathrm{p}}$ is the known quantum momentum operator $i\,\mathrm{h}/(2\pi)\nabla_x$. When compared to the standard Hamiltonian, expression (39) includes an additional imaginary term, actually the source of the divergence. The new potential $\Phi_{im}(\mathrm{x})$ is to express physically the deviation from the constant mass model. By our investigations, we have found an expression for it and we want to show in this paper, by the numerical calculations given in Part II, that this is quite the one able to insure good physical consistence to the model.

The expression is

$$\Phi_{im}(\mathrm{x}) = \frac{\mathrm{h}\,\mathrm{v}_{_D}(\mathrm{x})}{4\mathrm{m}\,\pi}\frac{d}{d\mathrm{x}}\left(\mathrm{m}_{eff}(\mathrm{x}) - \frac{c_n \mathrm{h}\nu_n(\mathrm{x})}{2\mathrm{v}_{_D}(\mathrm{x})^2}\right) \tag{40}$$

Now we use expression (40) in the quantum Hamilton equation

$$\hat{\mathrm{H}}\Psi_n(\mathrm{x}) = \mathrm{E}_n\Psi_n(\mathrm{x}) \tag{41}$$

and obviously equations (4) and (5) are recovered. Moreover we obtain :

$$\frac{\hbar}{2\mathrm{m}}\ \big(\rho(\mathrm{x})'\nabla S(\mathrm{x}) + \rho(\mathrm{x})S''(\mathrm{x})\big) = \rho(\mathrm{x})\Phi_{im}(\mathrm{x}) \tag{42}$$

or explicitly, in both Regions I and II :

$$J(\mathrm{x}) = \rho(\mathrm{x})\nabla S(\mathrm{x}) = 2\nu_n(\mathrm{x})\ \left(\mathrm{m}_{eff}(\mathrm{x}) - \frac{c_n \mathrm{h}\nu_n(\mathrm{x})}{2\mathrm{v}_{_D}(\mathrm{x})^2}\right) \tag{43}$$

To insure validity of this equation, also note the following. In a next section, we will give the expression (54) for $\mathrm{m}_{eff}(\mathrm{x})$, to hold in Region II. This expression is exactly the one bringing current density and imaginary potential to be both zero. Then the known quantum-mechanical stationary (i.e. with constant phase S) solution of (41) is recovered by our model in Region II. As a last remark here, is easily seen by eq. (43) that in the purely classical limit, when $\mathrm{m}_{eff}(\mathrm{x})\rightarrow\mathrm{m}$ and h is assumed negligeable, the current density becomes a constant as it must.

4.2.4 Wave momentum and energy

Equation (43) can be written in the form

$$\nabla S(\mathrm{x}) = \mathrm{m}_{eff}(\mathrm{x})\mathrm{v}_{D}(\mathrm{x}) - \frac{c_n \mathrm{h}}{4}\rho(\mathrm{x}) = p_1(\mathrm{x}) - p_2(\mathrm{x}) \tag{44}$$

We interpret this as the constitutive equation for $\nabla S(\mathrm{x})$ in terms of two characteristic momenta. Then the quantum wave with wave-vector $\nabla S(\mathrm{x})/\hbar$ interacts with two other waves (wave-vectors $\mathrm{m}_{eff}(\mathrm{x})\mathrm{v}_{D}(\mathrm{x})/\hbar$ and $c_n\pi\rho(\mathrm{x})/2$) much alike, for instance, three waves are coupled in a phonon interaction. Then we can calculate the propagating energy E_p of the quantum wave as follows :

$$\mathrm{E}_p = \frac{p_1(\mathrm{x})^2}{2\mathrm{m}_{eff}(\mathrm{x})} - \frac{p_2(\mathrm{x})^2}{2\mathrm{m}_{eff}(\mathrm{x})} = \frac{1}{2}\mathrm{m}_{eff}(\mathrm{x})\mathrm{v}_{D}^2 - \frac{c_n^2\,\mathrm{h}^2\rho(\mathrm{x})^2}{32\mathrm{m}_{eff}(\mathrm{x})} \tag{45}$$

The same equation can also be written

$$\mathrm{E}_p = \nabla S(\mathrm{x})\mathrm{v}_{D}(\mathrm{x}) - \frac{\left[4\mathrm{v}_{D}(\mathrm{x})\mathrm{m}_{eff}(\mathrm{x}) - c_n\,\mathrm{h}\rho(\mathrm{x})\right]^2}{32\mathrm{m}_{eff}(\mathrm{x})} \tag{46}$$

This expression is interesting because it defines the matter wave group velocity in the form

$$\frac{\partial \mathrm{E}_p}{\partial \nabla S} = \mathrm{v}_{D}(\mathrm{x}) \tag{47}$$

Concerning equation (45), the remarkable fact here is that in Region I (see equation (38)) the propagating energy E_p appears to us as the difference between the center of mass energy $\mathrm{K}_{cm}(\mathrm{x})$ and the residual energy in the group velocity frame $<\mathrm{K}'>$. Since each of these quantities correlates to a classical degree of freedom, we conclude that the

matter wave propagates the amount of energy exceeding the balance typical of classical equipartition. Equation (37) shows us that in Region II, however, different potentials merge all together into the expression $p_2(\mathrm{x})^2/(2\mathrm{m}_{eff}(\mathrm{x}))$. Yet by the very form of this last term, still we can think to it as the mean energy taken by a single, "equivalent" degree of freedom in the open packet ; thus reinforcing the previous interpretation. These circumstances also seem to indicate a physical explanation for the current density divergence : when equipartition is not achieved, an extra current flows out of the volume and the wave is a traveling one with $\nabla S \neq 0$. As far as physical parameters change along the way, the current density decreases down to the zero value attained in Region II, where both the propagated energy and the wave-vector $\nabla S/\hbar$ are zero and the wave turns into a stationary tail. Incidentally, the traveling wave coming from Region I merges with continuity into the stationary one, right at the boundary x_n defined in equation (24).

As a general rule, all the quantities we produce in this paper are everywhere continuous with their first and second derivatives, with the exception of $\nu_n(\mathrm{x})$ (and of $\mathrm{m}_{eff}(\mathrm{x})$, depending on it) suffering a small discontinuity in their second derivative when crossing the border at the abscissa x_n. This is only because we chose in this work to rest on expression (33) instead of the more elaborated [42, eq.(80)]. Then even this discontinuity could be easily eliminated either by upgraded model or simply a mathematical refinement of eq. (32) around x_n ; at present, conclusively, we keep it as it stands due to simplicity and direct physical insight provided to the reader by this last equation.

4.2.5 Explicit expression for the mass eigenfunction

In reference [42] we outlined a model of the interaction between particles and vacuum in terms of velocity fields. This came out as a non-local model, and we found an expression for the characteristic distance $<|\xi^*_{hid}|>$ of the interaction :

$$< |\xi^*_{hid}| > |_{\mathrm{x}} \approx \frac{\gamma \sqrt{|1 - \mathrm{c}_n|}\mathrm{h}}{\mathrm{v}_{_D}(\mathrm{x})\sqrt{\mathrm{m} \mid \delta\mathrm{m}(\mathrm{x}) \mid}} \qquad \{\text{Region I}\} \qquad (48)$$

Here γ is a positive coefficient and the reason why we expose the term $|1 - \mathrm{c}_n|$ is explained next (see a comment to eq. (57)). Physically, equation (48) is limited to a domain where $\mathrm{v}_{_D}(\mathrm{x})$ is not too small (Region

I), and we can exploit it to find an expression of $\mathrm{m}_{eff}(\mathrm{x})$ therein. Next we'll give an expression holding in Region II (this region includes the stopping points where all the velocity fields are small).

The expression (48) looks very similar to a de-Broglie wavelength but includes the mass defect $\delta\mathrm{m}(\mathrm{x})$. In order to find an equation for this last quantity, it can be noted that the characteristic distance should be of the same order of the distance across which (relative) density variations are observed, i.e.

$$\frac{\gamma\sqrt{|1-\mathrm{c}_n|}\mathrm{h}}{\mathrm{v}_{_D}(\mathrm{x})\sqrt{\mathrm{m}\mid\delta\mathrm{m}(\mathrm{x})\mid}} \approx Abs[\frac{\rho(\mathrm{x})}{\rho'(\mathrm{x})}] \tag{49}$$

To refine this equation, we insert a complemental function $f(\mathrm{x})$ into it and write

$$\mid\delta\mathrm{m}(\mathrm{x})\mid = \frac{\gamma^2\ |1-\mathrm{c}_n|\,\mathrm{h}^2\ \ |1+f(\mathrm{x})|}{\mathrm{m}\ \mathrm{v}_{_D}(\mathrm{x})^2}\frac{\rho'(\mathrm{x})^2}{\rho(\mathrm{x})^2} \tag{50}$$

We need now an expression for $f(\mathrm{x})$. We have used the following (a rough one, but very simple and effective for consistence of our model) :

$$f(\mathrm{x}) = sign[\rho'(\mathrm{x})]\frac{const}{\rho'(\mathrm{x})^2}\frac{\mathrm{m}_{eff}(\mathrm{x}_n)g(\mathrm{x})-\mathrm{m}}{\mathrm{m}_{eff}(\mathrm{x}_n)-\mathrm{m}} \tag{51}$$

Indeed, if $g(\mathrm{x})$ is a regular function, this expression is able to avoid divergence of $<|\xi^*_{hid}|>|_{\mathrm{x}}$when $\rho'(\mathrm{x})\rightarrow 0$. The function $g(\mathrm{x})$ will be determined, for each case, by the numerical procedure described in the Part II of this work : we anticipate here that the constant value $g(\mathrm{x})=1$ will bring us very near to the right solutions, so that numerical calculations will start with this value to provide subsequent refinements. Now we can write simply

$$\begin{aligned}&\mathrm{m}_{eff}(\mathrm{x}) = \mathrm{m}_{eff\mathrm{I}}(\mathrm{x}) = \\ &= \mathrm{m}_{eff}(\mathrm{x}_n)g(\mathrm{x}) + sign[\rho'(\mathrm{x})]\frac{\gamma^2\ |1-\mathrm{c}_n|\,\mathrm{h}^2}{4\mathrm{m}}\frac{\rho'(\mathrm{x})^2}{\nu_n(\mathrm{x})^{\ 2}}\end{aligned} \tag{52}$$

{Region I}

Concerning Region II, we first use eq. (22) and find

$$\rho(\mathrm{x}) = \frac{1}{\mathrm{c}_n\mathrm{h}}\sqrt{8\mathrm{m}_{eff}(\mathrm{x})}\sqrt{\mathrm{E}_{nf}-\Theta(\mathrm{x})} \qquad \{\text{Region II}\} \tag{53}$$

By eq.(30) we get moreover

$$\mathrm{m}_{eff}(\mathrm{x}) = \mathrm{m}_{eff\,\mathrm{II}}(\mathrm{x}) = \quad \frac{\mathrm{c}_n \mathrm{h}\ \nu_n(\mathrm{x})}{2\,\mathrm{v}_{_D}(\mathrm{x})^2} \qquad \{\text{Region II}\} \quad (54)$$

4.2.6 Explicit expressions for the phase gradient and density current

Equations (43), (52) and (54) can be re-elaborated giving

$$\nabla S(\mathrm{x}) = \\ = \frac{\mathrm{c}_n \mathrm{h}}{4} \left\{ \frac{\rho(\mathrm{x}_n)^2}{\rho(\mathrm{x})} g(\mathrm{x}) - \rho(\mathrm{x}) + sign[\rho'(\mathrm{x})]\ \sigma_n\, |1 - \mathrm{c}_n|\, \frac{\rho'(\mathrm{x})^2}{\rho(\mathrm{x})} \right\} \qquad (55)$$
$$\times\ \mathrm{UnitStep}[\mathrm{x}_n - \mathrm{x}]$$

where by continuity in x_n we ask that $g(\mathrm{x}_n) = 1$ and the constant σ_n is a cross-section ($const \equiv [length^{-4}]$) of the form

$$\sigma_n = \frac{2\mathrm{h}\gamma^2}{\mathrm{c}_n \mathrm{m}\ \nu_{n0}} = \frac{8\,(\mathrm{m}_{eff}(\mathrm{x}_n) - \mathrm{m})\,\nu_{n0}}{\mathrm{c}_n\, const\ \mathrm{h}\,|1 - \mathrm{c}_n|} \qquad (56)$$

The term $|1 - c_n|$ appeared in equation (52) already : we displayed it as such in the mass expression for the purpose of the classical limit. Indeed, the continuity of expression (55) in x=x_n also calls for the condition

$$|1 - \mathrm{c}_n|\, \rho'(\mathrm{x}_n) = 0 \qquad (57)$$

so that - as it will be better explained in a next section - when we will turn to the classical limit we will have to satisfy this last condition just setting $\mathrm{c}_n = 1$, $\mathrm{m}_{eff}(\mathrm{x})$=m ; $\rho'(\mathrm{x}_n)$= 0 is instead the condition to impose when we search for quantum solutions. In our model, these circumstances represent mathematically the change of step to be made when crossing the boundary between the classical and quantum world.

4.2.7 Calculation of the potential ϕ(x)

Using eq. (43), (4), (34), we find

$$\phi(\mathrm{x}) = \frac{\nabla S^2(\mathrm{x})}{2\mathrm{m}} - \frac{\nabla S^2(\mathrm{x})}{2\mathrm{m}_{eff}(\mathrm{x})} - c_n \mathrm{h}\nu_n(\mathrm{x}) - \frac{\hbar^2}{2\mathrm{m}} \frac{A(\mathrm{x})''}{A(\mathrm{x})} \qquad (58)$$

so that eq.(14) writes

$$\frac{1}{2}\,\mathrm{m}_{eff}(\mathrm{x})\;\mathrm{v}(\mathrm{x})^2+\Phi(\mathrm{x})\;+\frac{c_n\mathrm{h}\nu_{n0}}{2}-c_n\mathrm{h}\nu_n(\mathrm{x})\;-\frac{\nabla S^2(\mathrm{x})}{2\mathrm{m}_{eff}(\mathrm{x})}\;+$$

$$+\frac{\nabla S^2(\mathrm{x})}{2\mathrm{m}}-\;\frac{\hbar^2}{2\mathrm{m}}\;\frac{A(\mathrm{x})''}{A(\mathrm{x})}\;=\;\mathrm{E} \tag{59}$$

or

$$\frac{1}{2}\,\mathrm{m}_{eff}(\mathrm{x})\;\mathrm{v}(\mathrm{x})^2+\mathrm{E}_{nf}-c_n\mathrm{h}\nu_n(\mathrm{x})\;-\frac{\nabla S^2(\mathrm{x})}{2\mathrm{m}_{eff}(\mathrm{x})}\;=\;\mathrm{E} \tag{60}$$

These last are just the final energy theorem expressions we promote in our paper to describe the physical behavior of what we have called the Bernoulli oscillators : i.e. classical oscillators submitted to the vacuum action. This action is described by different terms going back to the known Bohm potential + the few other mass and flow effects specifically featured in equation (59) itself. To compare with the standard expression generally used in Bohmian mechanics, we can state therefore that our "quantum potential" is not simply the Bohm potential -$\hbar^2$/2m $A(\mathrm{x})''/A(\mathrm{x})$ but

$$\Phi_{\mathrm{quant}}(\mathrm{x})=-\frac{1}{2}\,\delta\mathrm{m}(\mathrm{x})\;\mathrm{v}(\mathrm{x})^2+\frac{c_n\mathrm{h}\nu_{n0}}{2}-c_n\mathrm{h}\nu_n(\mathrm{x})+$$

$$+\frac{\nabla S^2(\mathrm{x})}{2\mathrm{m}}-\frac{\nabla S^2(\mathrm{x})}{2\mathrm{m}_{eff}(\mathrm{x})}-\;\frac{\hbar^2}{2\mathrm{m}}\;\frac{A(\mathrm{x})''}{A(\mathrm{x})} \tag{61}$$

4.2.8 Single-particle time-dependent behavior

In the time-dependent branch of the model, equation (14) writes

$$\frac{1}{2}\,\mathrm{m}_{eff}(\mathrm{x(t)})\;\frac{\mathrm{dx}}{\mathrm{dt}}^2+\Phi(\mathrm{x(t)})+\phi(\mathrm{x(t)})+\frac{\mathrm{E}_{nf}-\mathrm{E}_{ni}}{2}=\mathrm{E(t)} \tag{62}$$

where the ergodic assumption requires that

$$\mathrm{P(E)}d\mathrm{E}=\frac{1}{c_n\mathrm{h}}\mathrm{T(E)}d\mathrm{E}=\frac{d\mathrm{t}}{\tau} \tag{63}$$

So once P(E) is known, we can integrate to find E(t). Here one complete cycle of fluctuation lasts a time 2τ so that, essentially

$$\frac{1}{c_n \mathrm{h}} \int_{\mathrm{E}_{ni}}^{\mathrm{E}_{nf}} \mathrm{T(E)} d\mathrm{E} = \int_{\mathrm{t}_i}^{\mathrm{t}_i+\tau} \frac{d\mathrm{t}}{\tau} \tag{64}$$

For all the physical quantities Q(E) defined in the E-states ensemble we have therefore

$$\int_{\mathrm{E}_{ni}}^{\mathrm{E}_{nf}} \mathrm{Q(E)P(E)} d\mathrm{E} = \frac{1}{\tau} \int_{\mathrm{t}_i}^{\mathrm{t}_i+\tau} \mathrm{Q(E(t))} d\mathrm{t} \tag{65}$$

This insures ergodicity. We do not give solved examples in this paper, but only note that if for each state n we take τ not shorter than a few mean periods, then standard techniques can be used - starting from the stationary solutions $\mathrm{v(x,x_0(E))}$ - to visualize a phase-space imagine of the trajectories available to each particle.

4.3 Thermodynamic framework

In a previous paper [59] we introduced a peculiar interpretation of the Bose-Einstein distribution (1), as enlightened by a basic balance principle : in statistical equilibrium, the fluctuation entropy - i.e. the amount of entropy (let us call it here $\Delta\Sigma_{pert}$) introduced by the perturbation into the classical system must be equal to the entropy difference $\Delta\Sigma_c$"receivable" by the last. We resume here the fundamental concept, adding some new knowledge for the anharmonic case [60]. First of all, we attribute to the (vacuum-oscillator) interaction a lifetime

$$\tau(\mathrm{U}_c) = \frac{\hbar}{k \int \frac{d\mathrm{U}_c}{c_\mathrm{v}(\mathrm{U}_c)}} = \frac{< c_\mathrm{v} > \hbar}{k\,(\mathrm{U}_c - \mathrm{U}_{c0})} \tag{66}$$

To calculate this quantity, we can use in a first approximation the unperturbed, classical oscillators characteristic functions : therefore U_c is the classical thermodynamic energy ; $c_\mathrm{v}(\mathrm{U}_c)$ is the classical constant-volume specific-heat and $1/< c_\mathrm{v} >$ an equivalent integral mean just defined by the same eq. (66). In the same equation U_{c0} (a constant with respect to the U_c fluctuation, but we may take it dependent on the fixed temperature T, see Part II of this work) is to cut off slow motions, since anharmonic oscillators may have too long periods for meaningful interaction ; k is the Boltzmann constant. We define the fluctuation entropy

([12]) as

$$\Delta\Sigma_{pert} = \frac{1}{\hbar}\int_{\mathrm{U}_i(\mathrm{T})}^{\mathrm{U}_f(\mathrm{T})} \tau(\mathrm{U}_c)d\mathrm{U}_c = \frac{<< c_{\mathrm{v}} >>_{if}}{k} \ln \frac{\mathrm{U}_f(\mathrm{T}) - \mathrm{U}_{c0}(\mathrm{T})}{\mathrm{U}_i(\mathrm{T}) - \mathrm{U}_{c0}(\mathrm{T})} \quad (67)$$

Now we equate this last to the amount of entropy $\Delta\Sigma_c$. This one can be simply defined as ([13])

$$\Delta\Sigma_c = \frac{\mathrm{U}_f(\mathrm{T}) - \mathrm{U}_i(\mathrm{T})}{k\mathrm{T}} = \frac{\Delta\mathrm{U}(\mathrm{T})}{k\mathrm{T}} \quad (68)$$

Equating (67) and (68), we take out U_i and get the energy distribution in the form

$$< \mathrm{U}_c >_{fluct} = \frac{\mathrm{U}_i(\mathrm{T}) + \mathrm{U}_f(\mathrm{T})}{2} = \mathrm{U}_{c0}(\mathrm{T}) + \frac{\Delta\mathrm{U}(\mathrm{T})}{\mathrm{Exp}\left[\frac{\Delta\mathrm{U}}{<<c_{\mathrm{v}}>>_{if}\mathrm{T}}\right] - 1} + \frac{\Delta\mathrm{U}(\mathrm{T})}{2} \quad (69)$$

The obvious conditions we have to impose to this expression are that it must be consistent with the mechanical part of the model, and coincident with the correspondent quantum-mechanical expectation $\mathrm{U}_{qm}\,(\mathrm{T})$. So we write

$$< \mathrm{U}_c >_{fluct} = \frac{< \mathrm{E}_{ni} > + < \mathrm{E}_{nf} >}{2} = \frac{\sum_{n=1}^{\infty} (\mathrm{E}_{ni} + \mathrm{E}_{nf})\,\mathrm{Exp}\left[-\frac{\mathrm{E}_n}{k\mathrm{T}}\right]}{2\sum_{n=1}^{\infty}\mathrm{Exp}\left[-\frac{\mathrm{E}_n}{k\mathrm{T}}\right]} =$$

$$= \mathrm{U}_{qm}\,(\mathrm{T}) = \frac{\sum_{n=1}^{\infty}\mathrm{E}_n\mathrm{Exp}\left[-\frac{\mathrm{E}_n}{k\mathrm{T}}\right]}{\sum_{n=1}^{\infty}\mathrm{Exp}\left[-\frac{\mathrm{E}_n}{k\mathrm{T}}\right]} \quad (70)$$

These equations can be used to find the spectrum of values E_{ni}, E_{nf} necessary to the mechanical model equations for every given case. We will see this in the next sections, dedicated to basic examples, to be found in Part II of the paper.

5 Conclusion

In this paper, we have stated theoretically that using an Hamiltonian (39) with the imaginary potential (40) brings the quantum-mechanical

[12] We apply here the integral rule found in [14]. It is a prescription able to insure detailed balancing in classically-calculated transition probability expressions.

[13] See a discussion in ref [59].

hydrodynamic equations to be solved consistently with a classical-like theorem of the form (59), provided we add to this frame : a) fluctuating energy statistical equations, with implied mass functions $m_{eff}(x)$ and flow functions $\nu_n(x)$, and b) the ergodic assumption (63). In eq. (59), as well as in the imaginary potential definition, the characteristic mass function $m_{eff}(x)$ takes a dominant role. It appears as a new physical actor able to reconciliate (at the basic level here presented at least) the quantum theory with the classical. We have given detailed expressions of the mass functions as well as of the other quantities involved by the model ; in the following Part II of this work we will exhibit in sampled graphs the solutions we obtained numerically for the two most important cases of the rectangular well and harmonic oscillator. Consistence of the global model with the main thermodynamic quantum properties has also been shown theoretically, and the resulting equations will be used in Part II for the practical purpose of finding the appropriate extreme energy values of the fluctuation intervals for the cases at hand. A view is given on the constitutive equation of the wave-function phase gradient in terms of characteristic momenta and the classical limit framework associated to the model will also be investigated in Part II. We note at last that due to the characteristic form of the HDF potential expressed in (10), where a second derivative of the squared velocity field plays the dominant role, we foretold in [42,58,61] the ability of this potential to supply energy during barrier jumping, so that basing on the same framework a classical-like interpretation of the tunnel effect can be expected with detailed calculations in future work.

References

[1] MEHRA J. and RECHENBERG H., The Historical Development of Quantum Theory, Springer-Verlag, NY (1982)
[2] MASLOV, V. P. and FEDORNIK M. V., Semiclassical approximations in Quantum Mechanics, D. Reidel Publ. Comp., Dordrecht (1981)
[3] WHEELER J. and ZUREK W. eds., Quantum theory and Measurements, Princ. Univ. Press, NJ (1983)
[4] GUTZWILLER M. C., J. Math. Phys. 12, 343 (1971)
[5] GUTZWILLER M. C., Chaos in Classical and Quantum Mechanics, Springer Verlag, N.Y. (1990)
[6] DU M. L. and DELOS, J. B. Phys. Rev. Lett. 58, 1731 (1987)
[7] BOGOMOLNYI E. G., Sov. Phys. JETP 69, 275 (1989)
[8] COURTNEY M., JIAO H., SPELLMEYER N. and KLEPPNER D., Phys. Rev. Lett. 73, 1340 (1994)
[9] GAO J. and DELOS J. B., Phys. Rev. A 56, 356 (1997)

[10] STOCKMANN H. J., Quantum Chaos, Cambridge University Press, (1999)
[11] KLEPPNER D. and DELOS J. B., Foundations of Physics, 31, 593 (2001)
[12] CONNORS R. D. and KEATING J. P., Foundations of Physics 31, 669 (2001)
[13] FERMI E., PASTA J. and ULAM S., Lect. Appl. Math. 15,143 (1974)
[14] MASTROCINQUE G., Nuovo Cimento B 91, 179 (1986)
[15] MASTROCINQUE G., Nuovo Cimento B 111, 19 (1996)
[16] FENYES I., Zeitschrift fur Physik CXXXII (1952)
[17] LUDWIG G., Die Grundlagen der Quantenmechanik, Berlin (1954)
[18] AHARONOV Y. and BOHM D., Phys. Rev. 115, 485 (1959)
[19] NELSON E., Phys. Rev. 150, 1079 (1966)
[20] LANZ L., LUGIATO A. and RAMELLA G., Articoli vari, in Physica (1969-71)
[21] GALGANI L. and SCOTTI A., Rivista del Nuovo Cimento II, 189 (1972)
[22] GALGANI L. and SCOTTI A., Phys. Rev. Lett. 28, 1173 (1972)
[23] BECK C., Nonlinearity 8, 423 (1995)
[24] HOLLE A., MAIN J., WIEBUSCH G., RATCH H. and WELGE K. H., Phys. Rev. Lett. 61, 161 (1988)
[25] CARATI A. and GALGANI L., Foundations of Physics 31, 69 (2001)
[26] CARATI A., DELZANNO P., GALGANI L. and SASSARINI J., Nonlinearity 8, 65 (1995)
[27] de BROGLIE L., Comptes Rendues Acad.., 183, 447 (1926)
[28] de BROGLIE L., Comptes Rendues Acad.., 185, 380 (1927)
[29] BOHM D., Phys. Rev. 85, 166/180 (1952)
[30] BOHM D. and VIGIER J. P., Phys. Rev. 96, 208 (1954)
[31] BOHM D., Causality and Chance in Modern Physics, Routledge and Kegan Paul plc, London (1957)
[32] BOHM D., Wholeness and the Implicate Order, Routledge and Kegan Paul plc, London (1980)
[33] HOLLAND P. R., The Quantum Theory of Motion, Cambridge University Press, London (1993)
[34] de BROGLIE L., Non Linear Wave Mechanics, Elsevier, Amsterdam (1960)
[35] BELINFANTE F. J., A Survey of Hidden Variables Theories, Pergamon Press, Oxford (1973)
[36] PHILIPPIDIS, C.,DEWDNEY, C. and HILEY, B. J., Nuovo Cimento 52B, 15 (1979)
[37] DEWDNEY C. and HOLLAND P. R., In Quantum Mechanics Versus Local Realism, Plenum Press, N. Y. (1988)
[38] BRANDT S., DAHMEN H. D., GJIONAJ E. and STROH T., Physics Letters A 249, 265 (1998)
[39] MASTROCINQUE G., Ann. de la Fond. L. de Broglie, 27 113 (2002)
[40] MASTROCINQUE G., Ann. de la Fond. L. de Broglie 27, 661 (2002)
[41] MASTROCINQUE G., Ann. de la Fond. L. de Broglie 28, 9 (2003)
[42] MASTROCINQUE G., Ann. de la Fond. L. de Broglie 28, 119 (2003)
[43] EINSTEIN A., Phys. Z. 10, 185 (1909)

[44] KAPITZA P. L., Usp. Fiz. Nauk 44, 7 (1951)
[45] KAPITZA P. L., Zh. Eksp. Teor. Fiz. 21, 588 (1951)
[46] JAMMER, M., Storia del concetto di massa nella fisica classica e moderna, Feltrinelli Ed., Milano 1974
[47] MIKHAILOV, V. F., Ann. de la Fond. L. de Broglie, 24, 161 (1999)
[48] COSTA DE BEAUREGARD, O., Ann. de la Fond. L. de Broglie, 25, 129 (2000)
[49] GALERIU, C., Ann. de la Fond. L. de Broglie, 28, 49 (2003)
[50] WOODWARD J.F., Found. of Phys. Lett. 5, 425 (1995)
[51] OUDET X. and LOCHAK G., Journal of Magnetism and Magnetic Materials, 65, 99 (1987)
[52] OUDET X., Ann. de la Fond. L. de Broglie, 25, 1 (2000)
[53] OUDET X., Ann. de la Fond. L. de Broglie, 29, 493 (2004)
[54] OUDET X., Ann. de la Fond. L. de Broglie, 34, 209 (2009)
[55] PRIGOGINE I., NICOLIS G. and BABLOYANTZ A., Thermodynamics of evolution, in Physics Today, XXV (1972)
[56] DE LA PENA-AUERBACH L., Stochastic Theory of Quantum Mechanics for particles with spin, J. of Math. Phys. XII (1971)
[57] PHILIPPIDIS C., BOHM D. and KAYE R. D., Nuovo Cimento 71B, 75 (1982)
[58] MASTROCINQUE G., Comm. to LXXXVII Congr.SIF Sez. VI, Milano, 24-29 Sept. (2001)
[59] MASTROCINQUE G., Nuovo Cimento B 105, 1315 (1990)
[60] MASTROCINQUE G., Comm. to XCIV Congr.SIF Sez. VI, Genova, 22-27 Sept. (2008)
[61] MASTROCINQUE G., Comm. to LXXXIX Congr.SIF Sez. VI, Parma, 17-22 Sept. (2003)

(Manuscrit reçu le 30 juin 2011)