“Thermodynamique cachée des particules” and the quantum potential

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ABSTRACT. According to de Broglie, temperature plays a basic rôle in quantum Hamilton-Jacobi theory. Here we show that a possible dependence on the temperature of the integration constants of the relativistic quantum Hamilton-Jacobi may lead to corrections to the dispersion relations. The change of the relativistic equations is simply described by means of a thermal coordinate.

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In Louis de Broglie view temperature plays a central rôle in quantum Hamilton-Jacobi theory. His investigation is summarized in the beautiful paper [1]. A related issue concerns possible deformations of the dispersion relations in the framework of the relativistic quantum Hamilton-Jacobi equation considered in [2]. The quantum versions of the Hamilton-Jacobi equations have been derived by first principles in [3, 4, 5]. Interestingly, such a formulation is strictly related to classical-quantum duality [6] and Legendre [7] dualities.

Consider the stationary Klein-Gordon equation $(-\hbar^2c^2\Delta + m^2c^4 - E^2)\psi = 0$. In one-dimensional space the associated quantum Hamilton-Jacobi equation is

$$(\partial_q S_0)^2 + m^2c^2 - \frac{E^2}{c^2} + \frac{\hbar^2}{2}\{S_0, q\} = 0 \ , \tag{1}$$

where $\{f,q\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$ denotes the Schwarzian derivative of f. The term $Q = \frac{\hbar^2}{4m}\{S_0, q\}$, is the quantum potential and $p = \partial_q \mathcal{S}_0$ the conjugate momentum. It is crucial that Q, which is always non-trivial [3, 4, 5], is quite different with respect to the one by de Broglie and Bohm. In [2] it has been shown that, upon averaging the mean speed on the period $[q, q + \frac{\pi}{k}]$, $k = \sqrt{E^2 - m^2c^4}/\hbar c$, one gets $v = c\frac{\sqrt{E^2-m^2c^4}}{E}/L_1$ with $L_1 \leq 1$ an integration constant of (1). Here we investigate a possible dependence of the initial conditions of (1) on the temperature T. More precisely, we will consider some analogies with thermal field theories suggesting a correction for the relativistic speed depending on the ratio m/T

$$v = c\frac{\sqrt{E^2 - m^2c^4}}{E}\alpha(m/T) \ . \tag{2}$$

Since the quantum potential plays the central rôle in our construction, it is worth stressing that the derivation and formulation of the quantum version of the Hamilton-Jacobi equations of [3, 4] is quite different from the one by de Broglie-Bohm. In particular, the quantum potential is basically different. Let us consider the case of non-relativistic quantum mechanics. de Broglie-Bohm made the identification $\psi = Re^{\frac{i}{\hbar}\mathcal{S}_0}$ with ψ the wave-function, not just a general complex solution of the Schrödinger equation, as it should be on mathematical grounds [3, 4]. This leads to the following trouble. Consider the case in which the wave-function is the eigenfunction of the Hamiltonian corresponding to a one-dimensional bound state, *e.g.* the harmonic oscillator. In this case ψ is proportional to a real function. It follows that $\mathcal{S}_0$ is a constant, so that the conjugate momentum is trivial. This would imply that at the quantum level the particle is at rest and starts moving in the classical limit. This is a well-known paradox observed by Einstein. In the derivation considered in [3, 4] such a paradox is naturally resolved by construction : $\{\mathcal{S}_0, q\}$ is well-defined if $\partial_q \mathcal{S}_0$ is never vanishing. In particular, the general solution of the non-relativistic quantum stationary Hamilton-Jacobi equation, which is formally equivalent to (1), with $(m^2c^4 - E^2)/2mc^2$ replaced by $V - E$, is

$$e^{\frac{2i}{\hbar}\mathcal{S}_0\{\delta\}} = e^{i\alpha}\frac{w + i\bar{\ell}}{w - i\ell} \ , \tag{3}$$

with $w = \psi^D/\psi \in \mathbb{R}$, where now ψ and ψ^D are two real linearly independent solutions of the stationary Schrödinger equation. $\delta = \{\alpha, \ell\}$, $\alpha \in \mathbb{R}$ and $\ell = \ell_1 + i\ell_2$ are integration constants. Note that $\ell_1 \neq 0$ even when $E = 0$, equivalent to having $\mathcal{S}_0 \neq cnst$, which is a necessary condition to define $\{\mathcal{S}_0, q\}$. This implies a non-trivial $\mathcal{S}_0$, even for a particle classically at rest. One has $\psi = {\mathcal{S}'_0}^{-1/2}\big(Ae^{\frac{i}{\hbar}\mathcal{S}_0} + Be^{-\frac{i}{\hbar}\mathcal{S}_0}\big)$, and $\psi \in \mathbb{R}$ implies $|A| = |B|$, so that $\mathcal{S}_0$ is non-trivial. The $\hbar \to 0$ limit is subtle and leads to the appearance of fundamental constants [3, 4, 5]. A basic observation in [2] is that averaging the period of the oscillating terms, which also appear in the non-relativistic case, leads to $\hbar$-independent solutions that, besides including the standard one, describe other solutions depending on the values of the integration constants (see also later). We note that the above decomposition of ψ, now called bipolar decomposition, is successfully used in studying molecular trajectories.

Another distinguished feature of the formulation introduced in [3, 4] is that energy quantization follows without any axiomatic interpretation of the wave-function. As a result, the quantum potential, intrinsically different from the one by de Broglie-Bohm, is always non-trivial. Like mc^2, it plays the rôle of intrinsic energy and it is at the basis of the quantum behavior. For example, besides energy quantization, it makes transparent its rôle in the tunnel's effect, where guarantees that the conjugate momentum always takes real values.

The general solution of (1) is (3), where now ψ and ψ^D are two real linearly independent solutions of the Klein-Gordon equation. Following Floyd [8], time parametrization is defined by Jacobi's theorem $t = \frac{\partial \mathcal{S}_0}{\partial E}$. Since $p = \partial_q \mathcal{S}_0$, it gives the group velocity $v = \partial E / \partial p$. Set $L = k\ell$, $L_1 = \Re L$, $L_2 = \Im L$. In the case $|L| = 1$ the mean speed reads

$$v = c\frac{\sqrt{E^2 - m^2c^4}}{E}\frac{1 + L_2\,\sin(2kq)}{L_1}\ .$$

The integration constants L_1 and L_2 may depend on particle's quantum numbers, energy and fundamental constants as well. As we said, here we will consider an intriguing dependence of L on the temperature. Such a possibility is related to a new way of considering the $\hbar \to 0$ limit which has been introduced in [2]. Let us first note that because of the $\hbar^{-1}$ term in $\sin(2kq)$, that typically is very strongly oscillating, the $\hbar \to 0$ limit is not well-defined. One possibility, considered in [3, 4], is that the integration constants may depend on E, $\hbar$ and other fundamental constants. In this way the term $\hbar^{-1}$ is cured by a suitable dependence on $\hbar$ of L, a procedure that leads to consider the Planck length (note that ℓ has the dimension of a length). The new way of getting the classical limit considered in [2] is based on the averaging of the oscillations. In particular

$$\langle v \rangle = \frac{k}{\pi} \int_q^{q+\frac{\pi}{k}} v(q')dq' = c\frac{\sqrt{E^2 - m^2c^4}}{E}\frac{1}{L_1} \ ,$$

which is reminiscent of the Dirac's averaging of the oscillating part of the free electron's speed $\frac{i}{2}c\hbar\dot{\alpha}_1^0 e^{-2iHt/\hbar}H^{-1}$ (see Dirac's treatment of the free electron in his book).

Note that a possible dependence of L on the temperature should not be a surprise. Actually, as observed by Dirac in computing the speed of the free electron, the uncertainty principle implies that it does not make sense considering scales which are much shorter than the Compton wavelength. On other hand, averaging on the period may lead to the breaking of the Lorentz group. Furthermore, it is natural to expect that such an averaging leads to a dependence on the degrees of freedom of the averaged space domain, which in general is not the empty space. In doing this we should remind the analogy with spontaneous symmetry breaking of Lorentz group in thermal QFT, we will shortly discuss later. Also note that the quantum potential is strictly related to the Fisher information and Shannon entropy [9]. It is then natural to expect that also here the temperature plays the central rôle.

Remarkably, dependence of L on the temperature is allowed just as a consequence of a basic physical property such as linearity of quantum mechanics. In the case of the Klein-Gordon equation, as in the case of the Schrödinger equation, such a property is equivalent to the invariance of the Schwarzian derivative under Möbius transformations. In this respect, note that a basic identity at the basis of [3, 4, 5] is that $(\partial_q \mathcal{S}_0)^2$ can be expressed as the difference of two Schwarzian derivatives $\left(\frac{\partial \mathcal{S}_0}{\partial q}\right)^2 = \frac{\beta^2}{2}\left(\{e^{\frac{2i}{\beta}\mathcal{S}_0}, q\} - \{\mathcal{S}_0, q\}\right)$, that forces us introducing the dimensional constant β, that is $\hbar$. We also note the appearance of the imaginary factor. It follows that Eq.(1) is equivalent to the Schwarzian equation $\{e^{\frac{2i}{\hbar}\mathcal{S}_0}, q\} = \frac{2}{\hbar^2 c^2}(E^2 - m^2c^4)$, which is invariant under a Möbius transformation of $e^{\frac{2i}{\hbar}\mathcal{S}_0}$ so that we can also assume that α and ℓ depend on the temperature so that Eq.(3) has the form

$$e^{\frac{2i}{\hbar}\mathcal{S}_0\{\delta(T)\}} = e^{i\alpha(T)}\frac{w + i\bar{\ell}(T)}{w - i\ell(T)} \ . \qquad (4)$$

This provides an intriguing possibility in the case of non-relativistic quantum mechanics. Actually, possible dependence on the temperature provides a sort of dynamics non detected by the wave-function and may open a new view on its collapse. As the temperature changes, $e^{\frac{2i}{\hbar}\mathcal{S}_0\{\delta(T)\}}$ moves on the boundary of the Poincaré disk (discrete jumps by Möbius transformations are also allowed) with $\{e^{\frac{2i}{\hbar}\mathcal{S}_0}, q\}$ remaining invariant. This happens also when $\{e^{\frac{2i}{\hbar}\mathcal{S}_0}, q\}$ is considered in the case of the relativistic quantum Hamilton-Jacobi equation (1). In particular, $\{e^{\frac{2i}{\hbar}\mathcal{S}_0}, q\} = \frac{2}{\hbar^2 c^2}(E^2 - m^2c^4)$ is invariant under variations of both the coordinate and the temperature.

Eq.(4) may be related to $t - T$ duality and non-commutative geometry (see [10] and references therein). In this respect, it is worth recalling de Broglie view [1]. In particular, in the framework of relativistic thermodynamics, he considered $t_m = h/mc^2$ as a particle internal time and identified it with an internal temperature $KT_m = mc^2$. de Broglie also suggested that, in the case of non-relativistic quantum mechanics, the collapse of the wave-function is related to a sort of particle's Brownian interaction with the environment.

Time-temperature duality is a basic feature of QFT, related to the analogy with classical statistical mechanics where the inverse temperature plays the rôle of imaginary time. Time and temperature also mix in performing analytic continuation along complex paths in the path-integral. Another well-known analogy concerns the transition amplitude for a particle for the time it that coincides with the classical partition function for a string of length t at $\beta = 1/\hbar$ [11]. It is widely believed that such a dualities are deeply related to the properties of space-time and should emerge in the string context.

In [10] it has been considered a string model where time-temperature and classical-quantum dualities, emerge naturally. It turns out that the solitonic sector of compactified strings have a dual description as quantum statistical partition function on higher dimensional spaces, built in terms of the Jacobian torus of the string worldsheet and of the compactified space. More precisely, in [10] it has been shown that in the case of compactification on a circle

$$\sum_{m,n\in\mathbb{Z}^g} e^{-\beta S_{m,n}} = \mathrm{tr}\ e^{-\beta H} , \tag{5}$$

where $\beta = 2R^2/\alpha'$, with R the compactification radius, and the Hamiltonian H is $\Delta_{J_\Omega}/2\pi$, with Δ_{J_Ω} the Laplacian on the Jacobian torus J_Ω

of the worldsheet. Eq.(5) is just the direct consequence of the stronger identity $H\Psi_{m,n} = S_{m,n}\Psi_{m,n}$, that is the set $\{S_{m,n}|m,n \in \mathbb{Z}^g\}$ *coincides* with the spectrum of H, a result deeply related to the theory of Riemann surfaces [12].

Temperature-time duality naturally emerges as a consequence of the complexified version of T-duality, a fundamental feature of string theory. By (5) the standard T-duality corresponds to the invariance, up to a multiplicative term given by powers of β, of the partition function under inversion of the temperature $\beta \longrightarrow \frac{1}{\beta}$. Complexification of β has basic motivations which interplay between physics and geometry. Set $\omega(A) = \text{tr}\left(Ae^{-\beta H}\right)/\,\text{tr}\; e^{-\beta H}$. Using the invariance of the trace under cyclic permutations we have

$$\omega(A(t)B) = \omega(BA(t+i\beta)) \; . \tag{6}$$

In such a context the complexification of β naturally appears in globally conformal invariant QFT [13]. It is worth noticing that both in [13] and in the BC system [14] the KMS (Kubo-Martin-Schwinger) states [15] play a central rôle. In particular, in the limit of 0-temperature the KMS states may be used to define the concept of point in noncommutative space, a basic issue in string theory [16].

A nice feature of the possible temperature dependence of the integration constants of (1) is that it does not arise as a consequence of a specific interaction, rather, as shown above, it is just due to the linearity of the Klein-Gordon equation, or, equivalently, of the Möbius symmetry of the Schwarzian derivative.

The above remarks suggest asking whether there exists an $L(T)$ such that the resulting expression for the velocity agrees with the experimental data. Let us then go back to the KMS states and note that the Fourier transformed form of (6) [17]

$$FT[\omega(AB)](p) \equiv \int d^dx e^{ip(x-y)}\omega(A(x)B(y)) = e^{-\beta p_0}FT[\omega(BA)](-p) \; , \tag{7}$$

shows that $FT[\omega(AB)](p)$ and $e^{-\beta p_0}FT[\omega(BA)](-p)$ cannot be both Lorentz invariant. Spontaneous breakdown of Lorentz boost symmetry at $T \neq 0$ is nicely seen considering the $(1,1)$ component of the 2×2 matrix propagator of a scalar field that at the tree level is [17]

$$FT[\omega(\varphi\varphi)](p) = \frac{1}{p^2 - m^2 + i\epsilon} + 2\pi\delta(p^2 - m^2)\frac{e^{-\beta|p_0|}}{1 - e^{-\beta|p_0|}} \; ,$$

which breaks Lorentz $\omega([M_{01}, T\varphi(x)\varphi(y)]) \neq 0$.

Even if we are considering a breaking of the Lorentz group from the quantum potential, which is a highly nonlocal term, it is clear that the analogy with thermal QFT suggests considering in our approach $e^{-\frac{mc^2}{KT}} = e^{-\frac{T_m}{T}}$, as order parameter of the Lorentz symmetry breaking. The crucial property of the KMS states is (6), or equivalently (7), showing that time evolution is invariant, upon commutation, under an imaginary shift of the time which is inversely proportional to the temperature. In the case of the quantum Hamilton-Jacobi equation, time is generated by Jacobi's theorem. Then one gets t as a function of q and a shift of time corresponds to a shift of q. On the other hand, this corresponds to a jump of $e^{\frac{2i}{\hbar}\mathcal{S}_0}$ on the boundary of the Poincaré disk, and may be compensated by changing $L(T)$ that, depending on its functional structure, may correspond to a change of T.

In the present formulation the dependence on m/T as order parameter of the breaking of Lorentz symmetry comes out as an effect due to the quantum potential and it is not seen as due to some particle interaction. Remarkably, even in this case there is an analogy with the thermal breakdown of symmetries in thermal field theory since the spontaneous breakdown of symmetries occurs even in cases with no interactions [18].

Let us now show that the modified relativistic dispersion relation induced by (2) is

$$E^2 = p^2c^2\alpha^2(m/T) + m^2c^4 \ . \tag{8}$$

Replacing $(m^2c^4 - E^2)/2mc^2$ by $V - E$, (1) is formally equivalent to the non-relativistic quantum stationary Hamilton-Jacobi equation. It follows that, as observed in [3, 4], Eq.(1) can be seen as a deformation of the classical relativistic Hamilton-Jacobi equation by a “conformal factor”. Actually, noting that $\{\mathcal{S}_0, q\} = -(\partial_q\mathcal{S}_0)^2\{q, \mathtt{s}\}$, $\mathtt{s} = \mathcal{S}_0(q)$, we see that (1) is equivalent to

$$E^2 = \left(\frac{\partial\mathcal{S}_0}{\partial q}\right)^2 c^2\left[1 - \hbar^2\mathcal{U}(\mathcal{S}_0)\right] + m^2c^4 \ , \tag{9}$$

with $\mathcal{U}(\mathcal{S}_0) = \{q, \mathtt{s}\}/2$, the canonical potential introduced in the framework of p–q duality [3, 4]. Eq.(9) can be expressed in the form $E^2 = (\partial_{\hat{q}}\mathcal{S}_0)^2c^2 + m^2c^4$ where $d\hat{q} = dq/\sqrt{1 - \hat{\beta}^2(q)}$ with $\hat{\beta}^2(q) = \hbar^2\{q, \mathtt{s}\}/2$. Integrating

$$\hat{q} = c\int^q dq' \frac{\partial_{q'}\mathcal{S}_0}{\sqrt{E^2 - m^2c^4}} = c\int^{\mathcal{S}_0(q)} \frac{d\mathtt{s}}{\sqrt{E^2 - m^2c^4}} \ . \tag{10}$$

Comparison with the dispersion relation (8) shows that the ansatz (2) is equivalent to the thermal deformation of the coordinate

$$\lim_{\text{“}\hbar\longrightarrow 0\text{”}} \hat{q} = q_T = \frac{q}{\alpha(m/T)} ,$$

so that the relativistic quantum Hamilton-Jacobi equation (1) becomes

$$E^2 = \left(\frac{\partial \mathcal{S}_0}{\partial q_T}\right)^2 c^2 + m^2 c^4 ,$$

which is Eq.(8). Eq.(10) is related to the relativistic classical action whose modification, leading to (8), is

$$L = -\frac{mc^2}{\gamma_T} ,$$

with $\gamma_T = 1/\sqrt{1 - \frac{v^2}{c^2\alpha^2(m/T)}}$, we have

$$p = \frac{\partial L}{\partial v} = \frac{mv\gamma_T}{\alpha^2(m/T)} ,$$

and

$$E = pv - L = mc^2\gamma_T ,$$

so that we get (8) and by $v = \partial_p E$,

$$v = \frac{pc^2}{E}\alpha^2(m/T) ,$$

which is, by (8), Eq.(2).

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