A lepton Dirac equation with additional mass term and a wave equation for a fourth neutrino

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ABSTRACT. From the relativistic wave equation for the electron we study a modified wave equation obtained by adding a second mass term. It is compatible with the electric gauge invariance. It does not modify the structure of plane waves, nor energy levels in the case of the H atom, calculated here with the H. Krüger's method. It allows to account for a desintegrating particle with spin $\frac{1}{2}$. We propose a wave equation for a fourth kind of neutrino.

Résumé : A partir de l'équation relativiste de l'électron, on étudie une modification de l'équation d'onde obtenue en ajoutant un second terme de masse. Celui-ci est compatible avec l'invariance de jauge électrique. Il ne modifie pas la structure des ondes planes, ni les niveaux d'énergie dans le cas de l'atome d'hydrogène, calculés ici par la méthode de H. Krüger. Il permet de prendre en compte la désintégration d'autres particules de spin $\frac{1}{2}$. *On propose une équation d'onde pour une quatrième sorte de neutrino.*

1 - Introduction

The Dirac equation [1] for the electron reads

$$0 = [\gamma^\mu(\partial_\mu + iqA_\mu) + im]\psi \; ; \quad q = \frac{e}{\hbar c} \; ; \quad m = \frac{m_0 c}{\hbar} \tag{1.1}$$

with the usual summation over repeated up and down index. A^μ are components of the electromagnetic potential space-time vector, e is the charge of the electron and m_0 its proper mass. We use the following matrices :

$$\gamma_0 = \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \; ; \; I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; ; \; -\gamma_j = \gamma^j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix} \tag{1.2}$$

where σ_j, $j = 1,\ 2,\ 3$ are Pauli matrices. We also use Weyl spinors ξ and η :

$$\psi = \begin{pmatrix}\xi \\ \eta\end{pmatrix} \ ; \ \xi = \begin{pmatrix}\xi_1 \\ \xi_2\end{pmatrix} \ ; \ \eta = \begin{pmatrix}\eta_1 \\ \eta_2\end{pmatrix}. \tag{1.3}$$

We use here only Pauli matrices and the $Cl_3 = M_2(\mathbb{C})$ Clifford algebra they generate [2]. In this frame the Dirac wave reads

$$\phi = \sqrt{2}\begin{pmatrix}\xi_1 & -\eta_2^* \\ \xi_2 & \eta_1^*\end{pmatrix}, \quad \widehat{\phi} = \sqrt{2}\begin{pmatrix}\eta_1 & -\xi_2^* \\ \eta_2 & \xi_1^*\end{pmatrix} \tag{1.4}$$

while the Dirac equation reads

$$0 = \nabla\widehat{\phi}\sigma_{21} + qA\widehat{\phi} + m\phi \tag{1.5}$$

$$\nabla = \sigma^\mu\partial_\mu \ , \quad \partial_\mu = \frac{\partial}{\partial x^\mu} \ ; \ A = \sigma^\mu A_\mu$$

$$\sigma^0 = I \ ; \quad \sigma^j = -\sigma_j \ , \quad \sigma_{21} = \sigma_2\sigma_1 = -i\sigma_3. \tag{1.6}$$

We previously studied three kinds of modification-extension of the Dirac equation. The first one [2][3][5] changed the mass term by introducing the Yvon-Takabayasi β angle

$$0 = \nabla\widehat{\phi}\sigma_{21} + qA\widehat{\phi} + me^{-i\beta}\phi \ ; \quad \rho e^{i\beta} = \det(\phi). \tag{1.7}$$

This wave equation is equivalent to its invariant form [3]

$$0 = \overline{\phi}(\nabla\widehat{\phi})\sigma_{21} + \overline{\phi}qA\widehat{\phi} + m\rho \ ; \quad \overline{\phi} = \widehat{\phi}^\dagger \tag{1.8}$$

obtained from (1.7) by multiplying by $\overline{\phi}$ on the left side and by identities

$$\phi\overline{\phi} = \overline{\phi}\phi = \det(\phi) \ ; \quad \widehat{\phi}\phi^\dagger = \phi^\dagger\widehat{\phi} = \det(\widehat{\phi}) = \rho e^{-i\beta}. \tag{1.9}$$

The wave equation (1.7) is homogeneous and non-linear. It solves the problem of negative energies for plane waves. It gives in the case of the H atom a countable set of solutions corresponding to the bound states of the hydrogen atom, with exactly the good number of states and the energy levels of the linear theory [2][3][6]. These solutions are very close to the solutions of the linear equation such as the Yvon-Takabayasi angle is everywhere defined and small. But the linear combinations of these solutions will have no such luck as to be solutions, or even approximations of solutions in the non-linear case, because the determinant giving the Yvon-Takabayasi angle is quadratic. The solutions labelled by the quantum numbers j, κ, λ, n, are plausibly the only solutions of the non-linear equation, in the case of the bound states of the hydrogen atom. The homogeneous non-linear equation is the only example we know of a non-linear wave equation for which it can exist quantum energy levels, with exactly the right number of energy levels and the right energy levels.

Then the homogeneous non-linear equation has many advantages. Introducing the Yvon-Takabayasi angle into the wave equation allows to suppress this angle everywhere it gives complicated results. The impulse-energy of the wave is the same as the impulse-energy of the electron-particle. Electromagnetic forces acting on the wave are identical to those acting on a classical electromagnetic fluid, as we prove it into appendix B of [2]. The homogeneous non-linear equation enables us more easily to see that the invariance group of the electromagnetism is greater than expected : Let R be a transformation of the space-time into itself such as

$$x' = R(x) = MxM^\dagger \tag{1.10}$$

$$x = x^0 + \vec{x} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \tag{1.11}$$

$$x' = x'^0 + \vec{x'} = \begin{pmatrix} x'^0 + x'^3 & x'^1 - ix'^2 \\ x'^1 + ix'^2 & x'^0 - x'^3 \end{pmatrix} \tag{1.12}$$

where M si any invertible element in Cl_3 and

$$\det(M) = re^{i\theta} \ ; \quad r \neq 0 \ ; \quad \phi' = M\phi \ ; \quad \widehat{\phi}' = \widehat{M}\widehat{\phi}. \tag{1.13}$$

R is a Lorentz dilation, composed of a Lorentz rotation conserving space and time orientation and of a pure homothety with ratio r. The wave equation (1.7) is form invariant and its invariant form (1.8) does not change under the dilation defined by (1.10) (1.13) if and only if

$$q'A' = \overline{M}qA\widehat{M} \ ; \quad m'\rho' = m\rho \ ; \quad m = rm', \tag{1.14}$$

because we get for any M the identity

$$\nabla = \overline{M}\nabla'\widehat{M} \ , \quad \nabla' = \sigma^\mu \partial'_\mu \ , \quad \partial'_\mu = \frac{\partial}{\partial x'^\mu}. \tag{1.15}$$

With (1.10) to (1.15) the equation (1.8) is invariant under the group $(Cl_3^*, \times)$ where Cl_3^* is the subset of invertible elements in Cl_3 [2]. The invariant equation (1.8) **includes** the Lagrangian density $\mathcal{L}$ which gives (1.7) because naming $< M >$ the (real) scalar part of M we get [3]

$$\mathcal{L} =< \overline{\phi}(\nabla\widehat{\phi})\sigma_{21} + \overline{\phi}qA\widehat{\phi} + m\rho > . \tag{1.16}$$

The second extension of the Dirac theory replaces σ_{21} by σ_{32} or σ_{13} and interprets this as giving the wave equation of the two other kinds of charged leptons, muons with mass m' and tauons with mass m'', with two other Lagrangian densities :

$$\mathcal{L} =< \overline{\phi}(\nabla\widehat{\phi})\sigma_{32} + \overline{\phi}qA\widehat{\phi} + m'\rho > \tag{1.17}$$

$$\mathcal{L} =< \overline{\phi}(\nabla\widehat{\phi})\sigma_{13} + \overline{\phi}qA\widehat{\phi} + m''\rho > . \tag{1.18}$$

Third modification of the Dirac theory : we get out of the frame of the Lagrangian formalism if we modify the wave equation (1.8) without changing its scalar part which is the Lagrangian density giving the wave equation before modification. This modification can be gauge invariant if we add to (1.8) a mass term commuting with σ_{21}. There are two possibilities :

$$0 = \overline{\phi}(\nabla\widehat{\phi})\sigma_{21} + \overline{\phi}qA\widehat{\phi} + m\rho(1 + \xi\sigma_3) \tag{1.19}$$

$$0 = \overline{\phi}(\nabla\widehat{\phi})\sigma_{21} + \overline{\phi}qA\widehat{\phi} + m\rho(1 + i\zeta\sigma_3). \tag{1.20}$$

where ξ and ζ are fixed real terms. This third modification is now the main purpose of our study.

2 - Plane waves and gauge invariance

Equations (1.19) (1.20) are respectively equivalent to

$$0 = \nabla\widehat{\phi}\sigma_{21} + qA\widehat{\phi} + me^{-i\beta}\phi(1 + \xi\sigma_3) \tag{2.1}$$

$$0 = \nabla\widehat{\phi}\sigma_{21} + qA\widehat{\phi} + me^{-i\beta}\phi(1 + i\zeta\sigma_3). \tag{2.2}$$

They simplify in the case $A = 0$ and are respectively equivalent to

$$0 = \nabla\widehat{\phi}\sigma_{21} + me^{-i\beta}\phi(1 + \xi\sigma_3) \tag{2.3}$$

$$0 = \nabla\widehat{\phi}\sigma_{21} + me^{-i\beta}\phi(1 + i\zeta\sigma_3). \tag{2.4}$$

Usual plane waves with a phase φ and a reduced speed v read here

$$\phi = \phi_0 e^{\varphi\sigma_{21}} \ ; \quad \varphi = mv_\mu x^\mu \ ; \quad v = v_\mu\sigma^\mu \tag{2.5}$$

where ϕ_0 is a fixed term. We get

$$\nabla\widehat{\phi} = \sigma^\mu \partial_\mu(\widehat{\phi}_0 e^{\varphi\sigma_{21}}) = mv\widehat{\phi}\sigma_{21}. \qquad (2.6)$$

Then (2.1) is equivalent to

$$v\widehat{\phi} = e^{-i\beta}\phi(1+\xi\sigma_3) \qquad (2.7)$$

which is equivalent, with $\widehat{\sigma}_j = -\sigma_j$, to

$$\widehat{v}\phi = e^{i\beta}\widehat{\phi}(1-\xi\sigma_3). \qquad (2.8)$$

Using both (2.7) and (2.8) we get

$$\begin{aligned}(v\cdot v)\phi &= v\widehat{v}\phi = v[e^{i\beta}\widehat{\phi}(1-\xi\sigma_3)] = e^{i\beta}(v\widehat{\phi})(1-\xi\sigma_3)\\ &= e^{i\beta}[e^{-i\beta}\phi(1+\xi\sigma_3)](1-\xi\sigma_3) = \phi(1-\xi^2). \qquad (2.9)\end{aligned}$$

We then get

$$\begin{aligned}\| v \|^2 &= v\cdot v = 1-\xi^2\\ \| mv \| &= m\sqrt{1-\xi^2} = \frac{m_0 c}{\hbar}\sqrt{1-\xi^2}. \qquad (2.10)\end{aligned}$$

We let

$$m_0' = m_0\sqrt{1-\xi^2}\ ;\quad mv = \frac{m_0' c}{\hbar}v' \qquad (2.11)$$

which gives

$$1 = \| v' \| = \sqrt{v_0'^2 - \vec{v}\,'^2}. \qquad (2.12)$$

It is then this reduced speed v' which is linked with the usual $\vec{\mathrm{v}}$ velocity by the relativistic formulas

$$v'_0 = \frac{1}{\sqrt{1-\frac{\vec{\mathrm{v}}^{\,2}}{c^2}}}\ ,\quad v'_j = \frac{\frac{\mathrm{v}_j}{c}}{\sqrt{1-\frac{\vec{\mathrm{v}}^{\,2}}{c^2}}}. \qquad (2.13)$$

This tells us : The true proper mass of the particle is m'_0.

With (2.2) we get instead of (2.7)

$$v\widehat{\phi} = e^{-i\beta}\phi(1+i\zeta\sigma_3) \qquad (2.14)$$

which is equivalent to

$$\widehat{v}\phi = e^{i\beta}\widehat{\phi}(1 + i\zeta\sigma_3) \tag{2.15}$$

or, taking adjoints, to

$$\phi^\dagger \overline{v} = e^{-i\beta}(1 - i\zeta\sigma_3)\overline{\phi}. \tag{2.16}$$

We then get

$$\begin{aligned} v\widehat{\phi}\phi^\dagger\overline{v} &= e^{-i\beta}\phi(1 + i\zeta\sigma_3)e^{-i\beta}(1 - i\zeta\sigma_3)\overline{\phi} \\ v\rho e^{-i\beta}\overline{v} &= e^{-2i\beta}\phi(1 + \zeta^2)\overline{\phi} \\ \rho v\overline{v} &= e^{-i\beta}(1 + \zeta^2)\phi\overline{\phi} = e^{-i\beta}(1 + \zeta^2)\rho e^{i\beta} \\ \parallel v \parallel &= v\widehat{\overline{v}} = v\overline{v} = 1 + \zeta^2. \end{aligned} \tag{2.17}$$

We then let instead of (2.11)

$$m'_0 = m_0\sqrt{1 + \zeta^2}\ ; \quad mv = \frac{m'_0 c}{\hbar}v' \tag{2.18}$$

which gives again (2.12) and (2.13) : The true proper mass of the particle is m'_0.

As the Dirac equation or the homogeneous non-linear equation, the equation (2.1) is gauge invariant under the gauge transformation :

$$\phi \mapsto \phi' = \phi e^{ia\sigma_3}\ ; \quad A_\mu \mapsto A_\mu{}' = A_\mu - \frac{1}{q}\partial_\mu a. \tag{2.19}$$

This equation is not obtained from a Lagrangian density, we cannot use the Noether's theorem to get an associated conservative current.

3 - The hydrogen atom

To solve the equation (2.1) we use a method separating the variables in spherical coordinates :

$$x^1 = r\sin\theta\cos\varphi\ ;\ x^2 = r\sin\theta\sin\varphi\ ;\ x^3 = r\cos\theta. \tag{3.1}$$

We use the following notations :

$$\begin{aligned} &i_1 = \sigma_{23} = i\sigma_1\ ;\ i_2 = \sigma_{31} = i\sigma_2\ ;\ i_3 = \sigma_{12} = i\sigma_3 \\ &S = \exp(-\frac{\varphi}{2}i_3)\exp(-\frac{\theta}{2}i_2)\ ;\ \Omega = r^{-1}(\sin\theta)^{-1/2}S \\ &\nabla' = \partial_0 - (\sigma_3\partial_r + \frac{1}{r}\sigma_1\partial_\theta + \frac{1}{r\sin\theta}\sigma_2\partial_\varphi). \end{aligned} \tag{3.2}$$

H. Krüger [4] got the identity

$$\Omega^{-1}\nabla = \nabla'\Omega^{-1}. \tag{3.3}$$

To separate the temporal variable $x^0 = ct$ and the angular variable φ from the radial variable r and the angular variable θ we let

$$\phi = \Omega X e^{(\lambda\varphi - Ex^0 + \delta)i_3} \tag{3.4}$$

where X is a function of r and θ with value into Cl_3, E= $\hbar c E$ is the energy of the particle, δ is a fixed arbitrary phase and λ is a real constant. We then get

$$\Omega^{-1}\phi = X e^{(\lambda\varphi - Ex^0 + \delta)i_3} \ ; \ \Omega^{-1}\widehat{\phi} = \widehat{X} e^{(\lambda\varphi - Ex^0 + \delta)i_3}. \tag{3.5}$$

We also get

$$\begin{aligned} \rho e^{i\beta} &= \det(\phi) = \det(\Omega)\det(X)\det[e^{(\lambda\varphi - Ex^0 + \delta)i_3}] \\ \det(\Omega) &= r^{-2}(\sin\theta)^{-1} \ ; \ \det[e^{(\lambda\varphi - Ex^0 + \delta)i_3}] = 1 \\ \rho e^{i\beta} &= \frac{\det(X)}{r^2\sin\theta}. \end{aligned} \tag{3.6}$$

We let

$$\rho_X e^{i\beta_X} = \det(X) \tag{3.7}$$

which gives

$$\rho e^{i\beta} = \frac{\rho_X}{r^2\sin\theta} e^{i\beta_X} \ ; \ \rho = \frac{\rho_X}{r^2\sin\theta} \ ; \ \beta = \beta_X. \tag{3.8}$$

The Yvon-Takabayasi β angle is a function of only r and θ.

For the hydrogen atom we have

$$qA = qA^0 = -\frac{\alpha}{r} \tag{3.9}$$

where α is the fine structure constant. The equation (2.1) gives with (3.4)

$$(E + \frac{\alpha}{r})\widehat{X}i_3 + \sigma_3\partial_r\widehat{X} + \frac{1}{r}\sigma_1\partial_\theta\widehat{X} + \frac{\lambda}{r\sin\theta}\sigma_2\widehat{X}i_3 = me^{-i\beta}Xi_3(1 + \xi\sigma_3) \tag{3.10}$$

while (2.2) gives with (3.4)

$$(E+\frac{\alpha}{r})\widehat{X}i_3+\sigma_3\partial_r\widehat{X}+\frac{1}{r}\sigma_1\partial_\theta\widehat{X}+\frac{\lambda}{r\sin\theta}\sigma_2\widehat{X}i_3=me^{-i\beta}Xi_3(1+i\zeta\sigma_3). \tag{3.11}$$

We let now

$$X=\begin{pmatrix}a & -b^*\\ c & d^*\end{pmatrix}\ ;\ \widehat{X}=\begin{pmatrix}d & -c^*\\ b & a^*\end{pmatrix} \tag{3.12}$$

where a, b, c, d are functions of r and θ with value into $\mathbb{C}$, b^* is the conjugated of b. Putting these matrices into (3.10) we get the equivalent system :

$$\begin{aligned}
&i(E+\frac{\alpha}{r})d+\partial_r d+\frac{1}{r}(\partial_\theta+\frac{\lambda}{\sin\theta})b=ime^{-i\beta}(1+\xi)a\\
&i(E+\frac{\alpha}{r})a-\partial_r a-\frac{1}{r}(\partial_\theta+\frac{\lambda}{\sin\theta})c=ime^{i\beta}(1-\xi)d\\
&i(E+\frac{\alpha}{r})b-\partial_r b+\frac{1}{r}(\partial_\theta-\frac{\lambda}{\sin\theta})d=ime^{-i\beta}(1+\xi)c\\
&i(E+\frac{\alpha}{r})c+\partial_r c-\frac{1}{r}(\partial_\theta-\frac{\lambda}{\sin\theta})a=ime^{i\beta}(1-\xi)b.
\end{aligned} \tag{3.13}$$

This system is reduced, if β is null or negligible, to

$$\begin{aligned}
&i(E+\frac{\alpha}{r})d+\partial_r d+\frac{1}{r}(\partial_\theta+\frac{\lambda}{\sin\theta})b=im(1+\xi)a\\
&i(E+\frac{\alpha}{r})a-\partial_r a-\frac{1}{r}(\partial_\theta+\frac{\lambda}{\sin\theta})c=im(1-\xi)d\\
&i(E+\frac{\alpha}{r})b-\partial_r b+\frac{1}{r}(\partial_\theta-\frac{\lambda}{\sin\theta})d=im(1+\xi)c\\
&i(E+\frac{\alpha}{r})c+\partial_r c-\frac{1}{r}(\partial_\theta-\frac{\lambda}{\sin\theta})a=im(1-\xi)b.
\end{aligned} \tag{3.14}$$

As there are only two angular operators, we can let

$$a=AU\ ;\ \ b=BV\ ;\ \ c=CV\ ;\ \ d=DU \tag{3.15}$$

where A, B, C, D are functions of r and U and V are functions of θ. If κ exists satisfying

$$U'-\frac{\lambda}{\sin\theta}U=-\kappa V\ ;\ \ V'+\frac{\lambda}{\sin\theta}V=\kappa U \tag{3.16}$$

then (3.14) is equivalent to

$$\begin{aligned} i(E+\frac{\alpha}{r})D+D'+\frac{\kappa}{r}B &= im(1+\xi)A \\ i(E+\frac{\alpha}{r})A-A'-\frac{\kappa}{r}C &= im(1-\xi)D \\ i(E+\frac{\alpha}{r})B-B'-\frac{\kappa}{r}D &= im(1+\xi)C \\ i(E+\frac{\alpha}{r})C+C'+\frac{\kappa}{r}A &= im(1-\xi)B. \end{aligned} \tag{3.17}$$

The resolution of the angular system (3.16) gives integrable functions if and only if κ is a not null integer, the total angular momentum j and the magnetic quantum number λ [2] satisfy

$$j=|\kappa|-\frac{1}{2}\ ;\quad \lambda=-j,\ -j+1,\cdots j-1,\ j. \tag{3.18}$$

To solve the radial system (3.17) we let

$$\begin{aligned} x=mr\ ;\ \epsilon=\frac{E}{m}\ ;\ a(x)=A(r)=A(\frac{x}{m}) \\ b(x)=B(r)\ ;\ c(x)=C(r)\ ;\ d(x)=D(r) \end{aligned} \tag{3.19}$$

which allows to get instead of (3.17)

$$\begin{aligned} i(\epsilon+\frac{\alpha}{x})d+d'+\frac{\kappa}{x}b &= i(1+\xi)a \\ i(\epsilon+\frac{\alpha}{x})a-a'-\frac{\kappa}{x}c &= i(1-\xi)d \\ i(\epsilon+\frac{\alpha}{x})b-b'-\frac{\kappa}{x}d &= i(1+\xi)c \\ i(\epsilon+\frac{\alpha}{x})c+c'+\frac{\kappa}{x}a &= i(1-\xi)b. \end{aligned} \tag{3.20}$$

By subtracting and adding the first and fourth equation, next the second and third, we get

$$\begin{aligned} i(\epsilon+\frac{\alpha}{x})(d-c)+(d-c)'+\frac{\kappa}{x}(b-a) &= i(a-b)+i\xi(a+b) \\ i(\epsilon+\frac{\alpha}{x})(d+c)+(d+c)'+\frac{\kappa}{x}(b+a) &= i(a+b)+i\xi(a-b) \\ i(\epsilon+\frac{\alpha}{x})(a-b)-(a-b)'+\frac{\kappa}{x}(d-c) &= i(d-c)-i\xi(d+c) \\ i(\epsilon+\frac{\alpha}{x})(a+b)-(a+b)'-\frac{\kappa}{x}(c+d) &= i(d+c)-i\xi(d-c). \end{aligned} \tag{3.21}$$

We let

$$\begin{aligned} &a-b=F_-+iG_- \ ; \ a+b=F_++iG_+ \\ &d-c=F_--iG_- \ ; \ c+d=F_+-iG_+ \ ; \ i\xi=\zeta. \end{aligned} \tag{3.22}$$

This gives

$$\begin{aligned} &i(\epsilon+\frac{\alpha}{x})(F_--iG_-)+(F_--iG_-)'-\frac{\kappa}{x}(F_-+iG_-) \\ &=i(F_-+iG_-)+\zeta(F_++iG_+) \\ &i(\epsilon+\frac{\alpha}{x})(F_+-iG_+)+(F_+-iG_+)'+\frac{\kappa}{x}(F_++iG_+) \\ &=i(F_++iG_+)+\zeta(F_-+iG_-) \\ &i(\epsilon+\frac{\alpha}{x})(F_-+iG_-)-(F_-+iG_-)'+\frac{\kappa}{x}(F_--iG_-) \\ &=i(F_--iG_-)-\zeta(F_+-iG_+) \\ &i(\epsilon+\frac{\alpha}{x})(F_++iG_+)-(F_++iG_+)'-\frac{\kappa}{x}(F_+-iG_+) \\ &=i(F_+-iG_+)-\zeta(F_--iG_-). \end{aligned} \tag{3.23}$$

Adding and subtracting we get

$$\begin{aligned} (-1+\epsilon+\frac{\alpha}{x})F_--G_-'-\frac{\kappa}{x}G_-&=\zeta G_+ \\ (1+\epsilon+\frac{\alpha}{x})G_-+F_-'-\frac{\kappa}{x}F_-&=\zeta F_+ \\ (-1+\epsilon+\frac{\alpha}{x})F_+-G_+'+\frac{\kappa}{x}G_+&=\zeta G_- \\ (1+\epsilon+\frac{\alpha}{x})G_++F_+'+\frac{\kappa}{x}F_+&=\zeta F_-. \end{aligned} \tag{3.24}$$

This system is made of two systems with opposite signs of κ in the case $\zeta=0$, which is the case of the Dirac equation. To solve the system in the general case we use

$$\begin{aligned} &F_+=e^{-\Lambda x}\sum_{n=0}^{\infty}a_nx^{s+n} \ ; \ G_+=e^{-\Lambda x}\sum_{n=0}^{\infty}b_nx^{s+n} \\ &F_-=e^{-\Lambda x}\sum_{n=0}^{\infty}c_nx^{s+n} \ ; \ G_-=e^{-\Lambda x}\sum_{n=0}^{\infty}d_nx^{s+n}. \end{aligned} \tag{3.25}$$

Putting these functions into (3.24) we get, from $e^{-\Lambda x}x^{s-1}$ terms :

$$\begin{aligned}\alpha d_0 + (s-\kappa)c_0 &= 0\\ \alpha c_0 - (s+\kappa)d_0 &= 0\\ \alpha a_0 - (s-\kappa)b_0 &= 0\\ \alpha b_0 + (s+\kappa)a_0 &= 0.\end{aligned} \tag{3.26}$$

This system has a not null solution only if determinants are null, which gives

$$s^2 = \kappa^2 - \alpha^2 \tag{3.27}$$

and the convergence at the origin implies

$$s = \sqrt{\kappa^2 - \alpha^2} \tag{3.28}$$

which is always defined as κ is a not null integer. Coefficients of $e^{-\Lambda x}x^{s+n-1}$ give :

$$\begin{aligned}(1+\epsilon)d_{n-1} - \Lambda c_{n-1} - \zeta a_{n-1} + \alpha d_n + (s+n-\kappa)c_n &= 0\\ (-1+\epsilon)c_{n-1} + \Lambda d_{n-1} - \zeta b_{n-1} + \alpha c_n - (s+n+\kappa)d_n &= 0\\ (-1+\epsilon)a_{n-1} + \Lambda b_{n-1} - \zeta d_{n-1} + \alpha a_n - (s+n-\kappa)b_n &= 0\\ (1+\epsilon)b_{n-1} - \Lambda a_{n-1} - \zeta c_{n-1} + \alpha b_n + (s+n+\kappa)a_n &= 0.\end{aligned} \tag{3.29}$$

The integrability of radial functions implies that series in (3.25) are polynomials with degree n. We begin with the case $n > 0$. We then get

$$\begin{aligned}(1+\epsilon)d_n - \Lambda c_n = \zeta a_n \ &; \ (-1+\epsilon)c_n + \Lambda d_n = \zeta b_n\\ (-1+\epsilon)a_n + \Lambda b_n = \zeta d_n \ &; \ (1+\epsilon)b_n - \Lambda a_n = \zeta c_n.\end{aligned} \tag{3.30}$$

If $\zeta \neq 0$ we get

$$c_n = \frac{1+\epsilon}{\zeta}b_n - \frac{\Lambda}{\zeta}a_n \ ; \ d_n = \frac{-1+\epsilon}{\zeta}a_n + \frac{\Lambda}{\zeta}b_n$$

$$(1+\epsilon)[\frac{-1+\epsilon}{\zeta}a_n + \frac{\Lambda}{\zeta}b_n] - \Lambda[\frac{1+\epsilon}{\zeta}b_n - \frac{\Lambda}{\zeta}a_n] = \zeta a_n \tag{3.31}$$

$$(-1+\epsilon)[\frac{1+\epsilon}{\zeta}b_n - \frac{\Lambda}{\zeta}a_n] + \Lambda[\frac{-1+\epsilon}{\zeta}a_n + \frac{\Lambda}{\zeta}b_n] = \zeta b_n. \tag{3.32}$$

This gives

$$(-1+\epsilon^2+\Lambda^2)a_n = \zeta^2 a_n \ ; \ (-1+\epsilon^2+\Lambda^2)b_n = \zeta^2 b_n. \tag{3.33}$$

So we must have

$$\Lambda^2 = 1 + \zeta^2 - \epsilon^2 \ ; \ \Lambda = \sqrt{1 + \zeta^2 - \epsilon^2}. \tag{3.34}$$

>From (3.29) we calculate terms with index n :

$$\begin{aligned}
n(2s+n)d_n &= \alpha\zeta a_{n-1} - \zeta(s+n-\kappa)b_{n-1} \\
&\quad + [\alpha\Lambda - (s+n-\kappa)(1-\epsilon)]c_{n-1} \\
&\quad + [\Lambda(s+n-\kappa) - \alpha(1+\epsilon)]d_{n-1} \\
n(2s+n)c_n &= \alpha\zeta b_{n-1} + \zeta(s+n+\kappa)a_{n-1} \\
&\quad - [\alpha\Lambda + (s+n+\kappa)(1+\epsilon)]d_{n-1} \\
&\quad + [\Lambda(s+n+\kappa) + \alpha(1-\epsilon)]c_{n-1} \\
n(2s+n)a_n &= \alpha\zeta d_{n-1} + \zeta(s+n-\kappa)c_{n-1} \\
&\quad - [\alpha\Lambda + (s+n-\kappa)(1+\epsilon)]b_{n-1} \\
&\quad + [\Lambda(s+n-\kappa) + \alpha(1-\epsilon)]a_{n-1} \\
n(2s+n)b_n &= \alpha\zeta c_{n-1} - \zeta(s+n+\kappa)d_{n-1} \\
&\quad + [\alpha\Lambda - (s+n+\kappa)(1-\epsilon)]a_{n-1} \\
&\quad + [\Lambda(s+n+\kappa) - \alpha(1+\epsilon)]b_{n-1}.
\end{aligned} \tag{3.35}$$

If n is the degree of radial polynomials we can also use (3.30) and we get with (3.34) and (3.29)

$$\begin{aligned}
0 &= 2[\Lambda(s+n) - \alpha\epsilon][-\zeta c_{n-1} - \Lambda a_{n-1} + (1+\epsilon)b_{n-1}] \\
0 &= 2[\Lambda(s+n) - \alpha\epsilon][(s+n+\kappa)a_n + \alpha b_n]
\end{aligned} \tag{3.36}$$

and also

$$\begin{aligned}
0 &= 2[\Lambda(s+n) - \alpha\epsilon][-\zeta d_{n-1} + \Lambda b_{n-1} + (-1+\epsilon)a_{n-1}] \\
0 &= 2[\Lambda(s+n) - \alpha\epsilon][(s+n-\kappa)b_n - \alpha a_n].
\end{aligned} \tag{3.37}$$

We cannot get

$$\begin{aligned}
0 &= (s+n+\kappa)a_n + \alpha b_n \\
0 &= -\alpha a_n + (s+n-\kappa)b_n
\end{aligned} \tag{3.38}$$

with a_n or b_n not null, because the determinant of this system is

$$(s+n)^2 - \kappa^2 + \alpha^2 = n(2s+n) > 0 \text{ if n} > 0. \tag{3.39}$$

Therefore, in the case $n > 0$, we get

$$\Lambda(s+n) = \alpha\epsilon \tag{3.40}$$

$$\alpha^2\epsilon^2 = (1+\zeta^2-\epsilon^2)(s+n)^2$$

$$\epsilon^2 = \frac{1+\zeta^2}{1+\dfrac{\alpha^2}{(s+n)^2}}$$

$$\epsilon = \frac{\sqrt{1+\zeta^2}}{\sqrt{1+\dfrac{\alpha^2}{(s+n)^2}}}. \tag{3.41}$$

This gives also

$$\epsilon = \frac{\sqrt{1-\xi^2}}{\sqrt{1+\dfrac{\alpha^2}{(s+n)^2}}}. \tag{3.42}$$

Then the energy E of the particle satisfies with (2.11) :

$$\mathrm{E} = \frac{\mathrm{m}_0\mathrm{c}^2\sqrt{1-\xi^2}}{\sqrt{1+\dfrac{\alpha^2}{(\mathrm{s}+\mathrm{n})^2}}} = \frac{\mathrm{m}_0'\mathrm{c}^2}{\sqrt{1+\dfrac{\alpha^2}{(\mathrm{s}+\mathrm{n})^2}}}. \tag{3.43}$$

A particle following the Dirac equation with a proper mass m_0' has therefore the same bound states as a particle with a proper mass m_0 following (2.1).

4 - Case of radial polynomials with degree 0

The number of the bound states with a principal quantum number $\mathbf{n} = n + |\kappa|$ is $2\mathbf{n}^2$. This number is obtained from the Dirac theory as a sum

$$2\mathbf{n}^2 = \mathbf{n}(\mathbf{n}+1) + \mathbf{n}(\mathbf{n}-1) \tag{4.1}$$

because there is, in the case $n = 0$ only one possible sign for κ. It is necessary to get also this result from (2.1). We let here in the place of (3.25)

$$F_+ = e^{-\Lambda x}a_0x^s \ ; \ G_+ = e^{-\Lambda x}b_0x^s$$

$$F_- = e^{-\Lambda x}c_0x^s \ ; \ G_- = e^{-\Lambda x}d_0x^s. \tag{4.2}$$

The radial system (3.24) is then equivalent to

$$0 = \alpha d_0 + (s - \kappa)c_0 \tag{4.3}$$
$$0 = \alpha c_0 - (s + \kappa)d_0 \tag{4.4}$$
$$0 = \alpha a_0 - (s - \kappa)b_0 \tag{4.5}$$
$$0 = \alpha b_0 + (s + \kappa)a_0 \tag{4.6}$$
$$\zeta a_0 = (1 + \epsilon)d_0 - \Lambda c_0 \tag{4.7}$$
$$\zeta b_0 = (-1 + \epsilon)c_0 + \Lambda d_0 \tag{4.8}$$
$$\zeta d_0 = (-1 + \epsilon)a_0 + \Lambda b_0 \tag{4.9}$$
$$\zeta c_0 = (1 + \epsilon)b_0 - \Lambda a_0. \tag{4.10}$$

Equations (4.3) to (4.6) have not null solutions only if $s = \sqrt{\kappa^2 - \alpha^2}$ and in this case we get

$$s = |\kappa|\sqrt{1 - \frac{\alpha^2}{\kappa^2}} > 0 \tag{4.11}$$
$$d_0 = \frac{\kappa - s}{\alpha}c_0 \ ; \ b_0 = \frac{s - \kappa}{\alpha}a_0. \tag{4.12}$$

Putting these relations into (4.7) - (4.10) we get

$$(1 + \epsilon)\frac{\kappa - s}{\alpha}c_0 - \Lambda c_0 = \zeta a_0 \tag{4.13}$$
$$(-1 + \epsilon)c_0 + \Lambda\frac{\kappa - s}{\alpha}c_0 = \zeta\frac{s - \kappa}{\alpha}a_0 \tag{4.14}$$
$$(-1 + \epsilon)a_0 + \Lambda\frac{s - \kappa}{\alpha}a_0 = \zeta\frac{\kappa - s}{\alpha}c_0 \tag{4.15}$$
$$(1 + \epsilon)\frac{s - \kappa}{\alpha}a_0 - \Lambda a_0 = \zeta c_0. \tag{4.16}$$

With (4.14) and (4.15) we get

$$\frac{\alpha}{s - \kappa}(-1 + \epsilon)c_0 - \Lambda c_0 = \zeta a_0 \tag{4.17}$$
$$\frac{\alpha}{\kappa - s}(-1 + \epsilon)a_0 - \Lambda a_0 = \zeta c_0. \tag{4.18}$$

Comparison between (4.13) and (4.17), or between (4.16) and (4.18) gives

$$\begin{aligned}(1+\epsilon)\frac{\kappa-s}{\alpha} &= \frac{\alpha}{s-\kappa}(-1+\epsilon)\\ (1+\epsilon)(\kappa-s)^2 &= \alpha^2(1-\epsilon)\\ \alpha^2-(\kappa-s)^2 &= [\alpha^2+(\kappa-s)^2]\epsilon\\ \epsilon &= \frac{\alpha^2-(\kappa-s)^2}{\alpha^2+(\kappa-s)^2}. \end{aligned} \tag{4.19}$$

But we have

$$\begin{aligned}(\kappa-s)^2 &= \kappa^2-2\kappa s+s^2 = \kappa^2-2\kappa s+\kappa^2-\alpha^2\\ &= 2\kappa(\kappa-s)-\alpha^2\\ \alpha^2-(\kappa-s)^2 &= \alpha^2-[2\kappa(\kappa-s)-\alpha^2] = 2[\alpha^2+\kappa(s-\kappa)] \qquad (4.20)\\ \epsilon &= \frac{2[\alpha^2+\kappa(s-\kappa)]}{2\kappa(\kappa-s)} = \frac{\alpha^2+\kappa s-\kappa^2}{\kappa^2-\kappa s}\\ &= \frac{\kappa s-s^2}{\kappa^2-\kappa s} = \frac{s(\kappa-s)}{\kappa(\kappa-s)} = \frac{s}{\kappa} \qquad (4.21)\end{aligned}$$

We can remark that the energy levels with $n=0$ are independent of ξ. As ϵ and s are positive, κ must also be positive : We get again the true result on the number of states $2\mathbf{n}^2$ for the principal quantum number $\mathbf{n}$. This result does not depend on ξ or ζ, then it is available for each linear equation considered here.

We must nevertheless point out a difference with the equation (1.7) about the Yvon-Takabayasi angle. It is not null in the plane $z=0$. So the non-linear equation and its linear approximation may have solutions with a small difference of the energy levels.

5 - Resolution in the Coulombian case with σ_1 and σ_2

We start now from the wave equation (2.1) where we replace σ_3 by σ_1 or σ_2. Wave equations are now

$$0 = \nabla\widehat{\phi}\sigma_{32}+qA\widehat{\phi}+me^{-i\beta}\phi(1+\xi\sigma_1) \tag{5.1}$$
$$0 = \nabla\widehat{\phi}\sigma_{13}+qA\widehat{\phi}+me^{-i\beta}\phi(1+\xi\sigma_2). \tag{5.2}$$

And the linear approximations of these wave equations, in the case of a β angle null or very small, are respectively

$$0 = \nabla\widehat{\phi}\sigma_{32} + qA\widehat{\phi} + m\phi(1 + \xi\sigma_1) \tag{5.3}$$

$$0 = \nabla\widehat{\phi}\sigma_{13} + qA\widehat{\phi} + m\phi(1 + \xi\sigma_2). \tag{5.4}$$

For (5.1) and (5.3) we let

$$\phi = \phi_1 e^{\frac{\pi}{4}i\sigma_2}. \tag{5.5}$$

We then get

$$\widehat{\phi} = \widehat{\phi}_1 e^{\frac{\pi}{4}i\sigma_2} \ ; \ \overline{\phi} = e^{-\frac{\pi}{4}i\sigma_2}\overline{\phi}_1$$

$$\rho e^{i\beta} = \phi\overline{\phi} = \phi_1 e^{\frac{\pi}{4}i\sigma_2} e^{-\frac{\pi}{4}i\sigma_2}\overline{\phi}_1 = \phi_1\overline{\phi}_1 = \det(\phi_1). \tag{5.6}$$

Then β is the same for ϕ or ϕ_1. Multiplying respectively (5.1) or (5.3) by $e^{-\frac{\pi}{4}i\sigma_2}$ on the right, and using

$$e^{\frac{\pi}{4}i\sigma_2}\sigma_{32}e^{-\frac{\pi}{4}i\sigma_2} = e^{\frac{\pi}{2}i\sigma_2}\sigma_{32} = i\sigma_2\sigma_{32} = -i\sigma_3 = \sigma_{21} \tag{5.7}$$

$$e^{\frac{\pi}{4}i\sigma_2}\sigma_1 e^{-\frac{\pi}{4}i\sigma_2} = e^{\frac{\pi}{2}i\sigma_2}\sigma_1 = i\sigma_2\sigma_1 = i(-i\sigma_3) = \sigma_3 \tag{5.8}$$

we get

$$0 = \nabla\widehat{\phi}_1\sigma_{21} + qA\widehat{\phi}_1 + me^{-i\beta}\phi_1(1 + \xi\sigma_3) \tag{5.9}$$

$$0 = \nabla\widehat{\phi}_1\sigma_{21} + qA\widehat{\phi}_1 + m\phi_1(1 + \xi\sigma_3). \tag{5.10}$$

Therefore we return to wave equations previously studied, and we get the same results.

For (5.2) and (5.4) we let

$$\phi = \phi_2 e^{-\frac{\pi}{4}i\sigma_1}. \tag{5.11}$$

We then get

$$\widehat{\phi} = \widehat{\phi}_2 e^{-\frac{\pi}{4}i\sigma_1} \ ; \ \overline{\phi} = e^{\frac{\pi}{4}i\sigma_1}\overline{\phi}_2$$

$$\rho e^{i\beta} = \phi\overline{\phi} = \phi_2 e^{-\frac{\pi}{4}i\sigma_1} e^{\frac{\pi}{4}i\sigma_1}\overline{\phi}_2 = \phi_2\overline{\phi}_2 = \det(\phi_2). \tag{5.12}$$

Then β is the same for ϕ or ϕ_2. Multiplying respectively (5.2) or (5.4) by $e^{\frac{\pi}{4}i\sigma_1}$ on the right, and using

$$e^{-\frac{\pi}{4}i\sigma_1}\sigma_{13}e^{\frac{\pi}{4}i\sigma_1} = e^{-\frac{\pi}{2}i\sigma_1}\sigma_{13} = -i\sigma_1\sigma_{13} = -i\sigma_3 = \sigma_{21} \tag{5.13}$$

$$e^{-\frac{\pi}{4}i\sigma_1}\sigma_2 e^{\frac{\pi}{4}i\sigma_1} = e^{-\frac{\pi}{2}i\sigma_1}\sigma_2 = -i\sigma_1\sigma_2 = -i(i\sigma_3) = \sigma_3 \tag{5.14}$$

we get

$$0 = \nabla\widehat{\phi}_2\sigma_{21} + qA\widehat{\phi}_2 + me^{-i\beta}\phi_2(1+\xi\sigma_3) \quad (5.15)$$

$$0 = \nabla\widehat{\phi}_2\sigma_{21} + qA\widehat{\phi}_2 + m\phi_2(1+\xi\sigma_3). \quad (5.16)$$

Therefore we return again to wave equations previously studied, and we get the same results.

6 - Conservation or not of the current of probability.

In the frame of the Clifford algebra Cl_3 the density of probability $\psi^\dagger\psi$, which is the time component of a space-time vector, reads D_0^0, time component of the space-time vector D_0, one of four space-time vectors $D_\mu = \phi\sigma_\mu\phi^\dagger$ forming a mobile orthogonal basis of the space-time [2] :

$$D_0 \cdot D_0 = \rho^2 \ ; \ D_j \cdot D_j = -\rho^2 \ , \ j = 1,\ 2,\ 3$$
$$D_\mu \cdot D_\nu = 0 \ , \ \mu \neq \nu. \quad (6.1)$$

The wave ϕ allows to consider a second mobile orthogonal basis :

$$\overline{D}_\mu = \overline{\phi}\sigma_\mu\widehat{\phi} \ ; \ \overline{D}_0 \cdot \overline{D}_0 = \rho^2 \ ; \ \overline{D}_j \cdot \overline{D}_j = -\rho^2 \ , \ j = 1,\ 2,\ 3$$
$$\overline{D}_\mu \cdot \overline{D}_\nu = 0 \ , \ \mu \neq \nu. \quad (6.2)$$

Components $\overline{D}_\mu^\nu$ satisfy, with $j,\ k = 1,\ 2,\ 3$:

$$\overline{D}_0^0 = D_0^0 \ ; \ \overline{D}_j^0 = -D_0^j \ ; \ \overline{D}_0^j = -D_j^0 \ ; \ \overline{D}_j^k = D_k^j. \quad (6.3)$$

The differential term of the invariant form of the wave equation includes

$$\overline{\phi}(\nabla\widehat{\phi}) = v^1 + iv^2 \ ; \ v^1 = \frac{1}{2}[\overline{\phi}(\nabla\widehat{\phi}) + (\overline{\phi}\nabla)\widehat{\phi}]$$
$$iv^2 = \frac{1}{2}[\overline{\phi}(\nabla\widehat{\phi}) - (\overline{\phi}\nabla)\widehat{\phi}]. \quad (6.4)$$

We can see v^1 as a space-time vector and iv^2 as a space-time pseudo-vector since

$$\overline{\phi} = \widehat{\phi}^\dagger \ ; \ \nabla^\dagger = \nabla \ ; \ [\overline{\phi}(\nabla\widehat{\phi})]^\dagger = (\overline{\phi}\nabla)\widehat{\phi}$$
$$v^{1\dagger} = v^1 \ ; \ (iv^2)^\dagger = -iv^2. \quad (6.5)$$

We can then let

$$\begin{aligned} v^1 &= v^1_\mu \sigma^\mu = v^1_0 - v^1_1\sigma_1 - v^1_2\sigma_2 - v^1_3\sigma_3 \\ v^2 &= v^2_\mu \sigma^\mu = v^2_0 - v^2_1\sigma_1 - v^2_2\sigma_2 - v^2_3\sigma_3. \end{aligned} \quad (6.6)$$

We get

$$\overline{\phi}(\nabla\widehat{\phi}) = v^1_0 - v^1_1\sigma_1 - v^1_2\sigma_2 - v^1_3\sigma_3 + iv^2_0 - v^2_1 i\sigma_1 - v^2_2 i\sigma_2 - v^2_3 i\sigma_3 \quad (6.7)$$

$$\overline{\phi}(\nabla\widehat{\phi})\sigma_{21} = -v^2_3 - v^1_2\sigma_1 + v^1_1\sigma_2 + v^2_0\sigma_3 - v^2_2 i\sigma_1 + v^2_1 i\sigma_2 - v^1_0 i\sigma_3 + v^1_3 i.$$

We get also

$$\begin{aligned} 2v^1 &= \overline{\phi}(\nabla\widehat{\phi}) + (\overline{\phi}\nabla)\widehat{\phi} = \overline{\phi}(\sigma^\mu\partial_\mu\widehat{\phi}) + (\partial_\mu\overline{\phi}\sigma^\mu)\widehat{\phi} = \partial_\mu(\overline{\phi}\sigma^\mu\widehat{\phi}) \\ \overline{\phi}\sigma_0\widehat{\phi} &= \overline{D}_0 = \overline{D}^\mu_0\sigma_\mu = D^0_0\sigma_0 - D^0_1\sigma_1 - D^0_2\sigma_2 - D^0_3\sigma_3 \\ \partial_0(\overline{\phi}\sigma^0\widehat{\phi}) &= \partial_0(\overline{\phi}\sigma_0\widehat{\phi}) = \partial_0 D^0_0\sigma_0 - \partial_0 D^0_1\sigma_1 - \partial_0 D^0_2\sigma_2 - \partial_0 D^0_3\sigma_3. \end{aligned} \quad (6.8)$$

We get also for $j = 1,\ 2,\ 3$:

$$\begin{aligned} \overline{\phi}\sigma^j\widehat{\phi} &= -\overline{\phi}\sigma_j\widehat{\phi} = -\overline{D}_j = -\overline{D}^\mu_j\sigma_\mu \\ &= D^j_0\sigma_0 - D^j_1\sigma_1 - D^j_2\sigma_2 - D^j_3\sigma_3 \\ \partial_j(\overline{\phi}\sigma^j\widehat{\phi}) &= \partial_j D^j_0\sigma_0 - \partial_j D^j_1\sigma_1 - \partial_j D^j_2\sigma_2 - \partial_j D^j_3\sigma_3. \end{aligned} \quad (6.9)$$

This gives

$$\begin{aligned} 2v^1 &= 2(v^1_0\sigma_0 - v^1_1\sigma_1 - v^1_2\sigma_2 - v^1_3\sigma_3) \\ &= \partial_\mu(\overline{\phi}\sigma^\mu\widehat{\phi}) = \partial_0(\overline{\phi}\sigma^0\widehat{\phi}) + \partial_j(\overline{\phi}\sigma^j\widehat{\phi}) \\ &= \partial_0 D^0_0\sigma_0 - \partial_0 D^0_1\sigma_1 - \partial_0 D^0_2\sigma_2 - \partial_0 D^0_3\sigma_3 \\ &+ \partial_j D^j_0\sigma_0 - \partial_j D^j_1\sigma_1 - \partial_j D^j_2\sigma_2 - \partial_j D^j_3\sigma_3 \\ &= (\partial_\mu D^\mu_0)\sigma_0 - (\partial_\mu D^\mu_1)\sigma_1 - (\partial_\mu D^\mu_2)\sigma_2 - (\partial_\mu D^\mu_3)\sigma_3 \end{aligned} \quad (6.10)$$

and we get, for $\nu = 0,\ 1,\ 2,\ 3$:

$$2v^1_\nu = \partial_\mu D^\mu_\nu = \nabla\cdot D_\nu. \quad (6.11)$$

The calculation of the gauge term $\overline{\phi}qA\widehat{\phi}$ is similar and we get

$$\overline{\phi}qA\widehat{\phi} = q[(A\cdot D_0)\sigma_0 - (A\cdot D_1)\sigma_1 - (A\cdot D_2)\sigma_2 - (A\cdot D_3)\sigma_3]. \quad (6.12)$$

The wave equation (1.19) is equivalent to the system of its components in the basis $(1,\ \sigma_1,\ \sigma_2,\ \sigma_3, i\sigma_1,\ i\sigma_2,\ i\sigma_3,\ i)$:

$$0 = -v_3^2 + q(A \cdot D_0) + m\rho \tag{6.13}$$
$$0 = -v_2^1 - q(A \cdot D_1) \tag{6.14}$$
$$0 = v_1^1 - q(A \cdot D_2) \tag{6.15}$$
$$0 = v_0^2 - q(A \cdot D_3) + m\xi\rho \tag{6.16}$$
$$0 = -v_2^2 \tag{6.17}$$
$$0 = v_1^2 \tag{6.18}$$
$$0 = -v_0^1 \tag{6.19}$$
$$0 = v_3^1. \tag{6.20}$$

There are, as with the homogeneous non-linear wave equation (1.7), two conservative currents, the current D_0 which has as time component the probability density $D_0^0 = \psi^\dagger\psi$, and the current D_3. To see this, we use (6.11) and the system (6.13) to (6.20) becomes

$$0 = -v_3^2 + q(A \cdot D_0) + m\rho \tag{6.21}$$
$$0 = -\frac{1}{2}(\nabla \cdot D_2) - q(A \cdot D_1) \tag{6.22}$$
$$0 = \frac{1}{2}(\nabla \cdot D_1) - q(A \cdot D_2) \tag{6.23}$$
$$0 = v_0^2 - q(A \cdot D_3) + m\xi\rho \tag{6.24}$$
$$0 = -v_2^2 \tag{6.25}$$
$$0 = v_1^2 \tag{6.26}$$
$$0 = -\frac{1}{2}(\nabla \cdot D_0) \tag{6.27}$$
$$0 = \frac{1}{2}(\nabla \cdot D_3). \tag{6.28}$$

Equation (6.27) is the law of conservation of the current of probability, and (6.28) is the law of conservation of the current D_3, linked to the chiral gauge $\phi \mapsto e^{ia}\phi$ [2].

The same calculation, from the wave equation (1.20) gives instead of

(6.21) to (6.28) the system

$$0 = -v_3^2 + q(A \cdot D_0) + m\rho \tag{6.29}$$
$$0 = -\frac{1}{2}(\nabla \cdot D_2) - q(A \cdot D_1) \tag{6.30}$$
$$0 = \frac{1}{2}(\nabla \cdot D_1) - q(A \cdot D_2) \tag{6.31}$$
$$0 = v_0^2 - q(A \cdot D_3) \tag{6.32}$$
$$0 = -v_2^2 \tag{6.33}$$
$$0 = v_1^2 \tag{6.34}$$
$$0 = -\frac{1}{2}(\nabla \cdot D_0) + m\rho\zeta \tag{6.35}$$
$$0 = \frac{1}{2}(\nabla \cdot D_3). \tag{6.36}$$

The current D_3 is still conservative, but the current of probability D_0 is no more conservative because (6.35) implies

$$\partial_\mu D_0^\mu = 2m\rho\zeta. \tag{6.37}$$

In the simple case where D_0^0 does not vary in the space and $D_0^j = 0$, for instance with a plane wave and a null velocity we get $\rho = D_0^0$ and (6.37) gives

$$\partial_0 D_0^0 = 2m\zeta D_0^0 \ ; \ \partial_0[\ln(D_0^0)] = 2m\zeta$$
$$D_0^0 = ke^{2m\zeta x^0}. \tag{6.38}$$

The density of probability in the case $\zeta < 0$ is a declining exponential function. This is usually interpreted as a radioactive decline. The corresponding particle has then a half-life :

$$T = \frac{\ln(2)}{2mc|\zeta|}. \tag{6.39}$$

If ζ is small, we then get $m_0 \approx m'_0$ and

$$\zeta \approx -\frac{\ln(2)\hbar}{2m'_0c^2T}. \tag{6.40}$$

For a muon the lifetime is $T = 2.2 \times 10^{-6}$s, the proper mass is $m'_0 = 1.88 \times 10^{-28}$kg, so $\zeta \approx -1 \times 10^{-18}$, and for a tau the lifetime is $T =$

2.9×10^{-13}s, the proper mass is $m_0' = 3.16\times10^{-27}$kg, so $\zeta \approx -4.4\times10^{-13}$ The difference between m_0 and m_0' is too small to be seen in experiments.

As the proper mass is inversely proportional to ζ in (6.40), this parameter is very small, but it can be greater for neutrinos, their proper masses being much smaller.

7 - A fourth possibility

We previously associate each kind of neutrinos to one of the three σ_j terms. But the Cl_3 algebra contains four independent terms with square -1 : σ_{12}, σ_{23}, σ_{31}, and $i = \sigma_{123}$ Then there must be a fourth wave equation on the model of (1.7) :

$$0 = \nabla\widehat{\phi}i + me^{-i\beta}\phi \ ; \quad \rho e^{i\beta} = \det(\phi). \tag{7.1}$$

Usual plane waves with a phase φ and a reduced speed v read here

$$\phi = \phi_0 e^{i\varphi} \ ; \quad \varphi = mv_\mu x^\mu \ ; \quad v = v_\mu \sigma^\mu \tag{7.2}$$

where ϕ_0 is a fixed term. We get

$$\nabla\widehat{\phi} = \sigma^\mu\partial_\mu(\widehat{\phi}_0 e^{-i\varphi}) = -ime^{-i\varphi}v\widehat{\phi}_0. \tag{7.3}$$

We get also, with $\det(\phi_0) = \rho_0 e^{i\beta_0}$

$$\phi\overline{\phi} = \rho e^{i\beta} = \phi_0 e^{i\varphi}\overline{\phi}_0 e^{i\varphi} = \phi_0\overline{\phi}_0 e^{2i\varphi} = \rho_0 e^{i(\beta_0+2\varphi)} \tag{7.4}$$

and we get

$$\rho = \rho_0 \ ; \ \beta = \beta_0 + 2\varphi. \tag{7.5}$$

Then (7.1) is equivalent to

$$0 = -imve^{-i\varphi}\widehat{\phi}_0 i + me^{-i(\beta_0+2\varphi)}e^{i\varphi}\phi_0 \tag{7.6}$$

$$v\widehat{\phi}_0 = -e^{-i\beta_0}\phi_0 \tag{7.7}$$

$$\widehat{v}\phi_0 = -e^{i\beta_0}\widehat{\phi}_0. \tag{7.8}$$

With both (7.7) and (7.8) we get

$$\begin{aligned}(v\cdot v)\phi_0 &= v\widehat{v}\phi_0 = -v(-\widehat{v}\phi_0) = -ve^{i\beta_0}\widehat{\phi}_0 = -e^{i\beta_0}v\widehat{\phi}_0 \\ &= -e^{i\beta_0}(-e^{-i\beta_0}\phi_0) = \phi_0 \end{aligned} \tag{7.9}$$

$$v\cdot v = 1 \text{ if } \det(\phi_0) \neq 0. \tag{7.10}$$

and we get usual relations (2.13) between reduced speed and velocity.

This wave equation is very different from the three ones with a σ_{ij} term, there is no linear approximation, the Yvon-Takabayasi angle being used and not small. The invariant form of this wave equation is

$$0 = \overline{\phi}(\nabla\widehat{\phi})i + m\rho. \tag{7.11}$$

With (1.10), (1.13) and $m = rm'$ this equation is invariant. Therefore it is relativistic invariant. We can add a gauge term $\overline{\phi}qB\widehat{\phi}$, we get

$$0 = \overline{\phi}(\nabla\widehat{\phi})i + \overline{\phi}qB\widehat{\phi} + m\rho. \tag{7.12}$$

But this equation is very different from (1.8) : for instance the second order equation contains a $(\Box - m^2)\phi$ term instead of the $(\Box + m^2)\phi$ term in the second order equation issued from the Dirac equation. More, the four D_μ currents are conservative, not only the D_0 and D_3 current : With (6.4), (7.12) is equivalent to

$$0 = (v^1 + iv^2)i + \overline{\phi}qB\widehat{\phi} + m\rho \tag{7.13}$$

or to the vectorial system

$$v^1 = 0 \tag{7.14}$$

$$v^2 = \overline{\phi}qB\widehat{\phi} + m\rho \tag{7.15}$$

$>$From equations (6.6) and (6.11), (7.14) is equivalent to

$$\nabla \cdot D_\mu = 0 \ ; \ \mu = 0,\ 1,\ 2,\ 3. \tag{7.16}$$

As D_0 is conservative this wave has no radioactive decline, is stable.

The generator i of the gauge is the generator of the Lochak's theory of the monopole [7] . So a charged lepton corresponding to the neutral wave following (7.1) should be very different from an electron, a muon or a tauon, and perhaps similar to the Lochak's magnetic monopole if B is a Cabibbo-Ferrari potential.

As the β angle is the angle of the $U(1)$ group which is the kernel of the homomorphism from Cl_3^* into the D^* group of Lorentz dilations [2][3] we get only the identity of D^*, no moving center of dilation, no moving particle [8].

The coupling between a charged lepton and its neutrino may not exist here or may be very different and with (7.1) we may have no electroweak interaction. A fourth neutrino interacting only by gravitation is expected today.

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