

An alternative quantization procedure for the Hydrogen atom

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ABSTRACT. We introduce a (non-standard) laplacian form producing squared angular momentum values equal to $\hbar^2 [l(l+1) + 1/4]$. So we set up an "alternative" Hydrogen atom model, which can also be thought as a sort of precursor to the standard quantum atom. It shows the following properties : torus-like orbitals; appearance of a "zero-point" rotational quantum number (it adds zenithal altitude fluctuations with momentum $\hbar/2$); S-like wavefunctions made zero over nucleus; corrections to spectroscopic terms, linearly dependent on the operators \hat{l}_z and \hat{l}_{ns}^2 , identical to the standard ones; quantization procedure resolvable into a quantization in the constant azimuth plane, followed by rotation around the polar axis. Although we come to a subtlety concerning the azimuthal component, these features make the model suitable for decomposition in one-dimensional motions, whose quantum properties we could recently approach by a method based on ergodic statistics of classical-like time laws with variable mass. Comparing the present results with the standard quantum theory and the corresponding quasi-classical case, we trace a possible path to implement our one-dimensional calculations up to describe 2D and 3D motions; and to identify some ultimate differences, worth of further investigation, with those standard models.

RÉSUMÉ. On introduit en cet article une forme laplacienne (non-standard) amenant à des valeurs du moment cinétique carré égales à $\hbar^2 [l(l+1) + 1/4]$. On bâti ainsi un modèle d'atome d'Hydrogène "alternatif", que l'on peut aussi considérer comme une sorte de précurseur de l'atome quantique standard. Il montre les propriétés suivantes : orbitales avec un caractère toroïdale; apparition d'un nombre quantique rotationnel "de point zéro" (il ajoute des fluctuations d'altitude zénithale avec moment $\hbar/2$); fonctions d'onde de type S s'annulant sur le noyau; corrections aux termes spectroscopiques, linéairement dépen-

dantes de \hat{l}_z and \hat{l}_{ns}^2 , identiques en comparaison avec le modèle standard ; procédure de quantisation se décomposant en quantisation dans le plan à azimuth constant, suivie par rotation autour de l'axe polaire. Bien que l'on parvienne à une subtilité concernant la composante azimuthale, ces caractéristiques rendent le modèle apte à la résolution en mouvements uni-dimensionnels, dont les propriétés quantiques on a pu récemment approcher par une méthode fondée sur un moyennage statistique ergodique de lois temporelles de type classique mais à masse variable. En faisant la comparaison des résultats présents avec le modèle quantique et le cas quasi-classique correspondant, on trace un chemin possible pour développer nos modèles uni-dimensionnels jusqu'à pouvoir décrire des mouvements en 2D et 3D ; et à pouvoir identifier certaines différences ultimes, dignes d'ultérieures études, avec le modèle orthodoxe.

1 Introduction

In previously published papers [1, 2] we provided a peculiar model for one-dimensional oscillators, embodying energy and mass fluctuations in both the quantum wave equation and the classical energy theorem. It seems to us able to reconcile (at least some) basic quantum-like features with classical-like motion. Extending calculations to the three-dimensional case (here represented by the Hydrogen atom) is in principle possible by the use of linear-motions superposition techniques ; but it requires a number of important premises, to be expounded in this paper.

Describing 2D or 3D motions by an appropriate composition of one-dimensional time-laws is a basic conceptual and mathematical tool for the classical physicist, stemming out directly from the vector character of the particle momentum $\vec{p} = m \vec{v}$. Motion composition properties also appear in quantum mechanics, although in the different form dictated by the state superposition principle. Other traces can be found in the variables separation techniques as we employ, f.i., in the template case of the Hydrogen atom model. Yet expounding this topic in quantum mechanics looks a much more complex matter compared with the classical, not only because of the many circumstances where non-linear coupling between variables must be taken into account - but essentially because of the intrinsic lack of the concept itself of particles trajectory in the former. However, from the old Bohr and Sommerfeld models via stochastic theories and Bohm mechanics, up to the fluctuation model we have recently proposed, classical-like approaches to quantum mechanical effects look to become more and more seasoned nowadays. So a rather

solid background is available to us and analysis in this field is still required. Amongst its main purposes, there is developing detailed calculations of the rotational motions which one can build up by superposing basic time-laws as the ones proposed in [1, 2].

Before attempting any practical calculation, however, we have to investigate in this paper about a major difficulty which we will certainly meet with in these future trials. It is represented by the existence of S-states with angular momentum quantum number $l = 0$. By the expression of the squared angular momentum values $l(l + 1)$ holding in quantum mechanics, this value in a S-state is obviously also zero. Now no fluctuation law of a quantity whatsoever can result in a time average and mean-square value both to be zero, except in the inconsistent case where the quantity is constantly null. So the states with zero angular momentum get out of any attempted resolution in two or three component motions. They have a spherical symmetry where any polar axis is actually suppressed. This fact seems to result merely from the statistics attached by the quantum formalism to the radial motion, summing over all directions of the radial axis in space. Yet no physical rotation of this axis may be thought without a (squared) average angular momentum different from zero. Moreover, we have to note that the S-states 3D wavefunctions are not zero on the nucleus ($r = 0$), what is connected indeed to the fact that there is no centrifugal force to keep the electron far out from it. Again, it would be hardly attempted to explain by classical thinking how the electron moves in the very proximity of the nucleus.

All of these intriguing circumstances seem to prevent "a priori" that an S-orbital is interpreted by a rotating particle model in a classical sense. But on a general point of view, even in recent literature, many efforts are still found, aimed at investigating different original paths towards the goal of identifying traces of electron trajectories in the atom. Some of the most interesting (theoretical) ones account for spin rotation in the calculation, as is basically done in Bohm mechanics [3]; or spin-orbit effects as we find f.i. in the known models by Gryzinski [4 ÷ 6]. Moreover several papers, reporting both theory (the "closed orbits" one) [7 ÷ 11] and experiments [12 ÷ 23], have also been issued lately which demonstrate classical-like trajectories of wave-packets in Rydberg atoms; with some insight even in the innermost atom core [24 ÷ 27]. However, the difficulty with S states is not removed even in the ordinary relativistic treatment, where the 3D S wavefunctions behaviour shows a (weak) singularity of the order $\approx r^{-\alpha^2/2}$ (α = fine structure constant) [28, 29].

Yet remarkably, in the (2+1)D Coulomb interaction case [28], a Chern-Simons term accounted for into the Hamiltonian is known to bring a correction to the angular momentum quantum number. This might reveal, to further analysis, beneficial to smooth out the singularity, at least in the 2D case. Besides that, arguments can be found to introduce supplemental angular momentum in S states whenever one accounts for spin or other perturbations which make l not so good a quantum number any more. In our concerns here however, we focus our interest on a non-relativistic, spinless frame. In any case indeed, we do not believe in principle that solving the S-states puzzle should help with having recourse to extrinsic variables as spin rotation, nonlocalization and magnetic effects; nor to relativity in general. The real spirit of a basic description of particles interaction should be independent of such ingredients being activated or not in the model. State of the art considered, the mentioned S-states properties look to us to substantiate peculiarly some very fundamental discrepancies between the quantum and classical-like models. For this reason the case deserves major investigation.

So we leave out in this paper any spin or magnetic effect, and resting on a basic ground we trace back the anomaly of S-states to a consequence of the standard quantization procedure historically risen to dominance. We turn out at examining the possibility that a different quantization procedure from the standard one is employed, in such a way that new interesting elements can be identified in the basic Hydrogen model. The sound variation here is introducing a constant additional value into the squared angular momentum values, and we discuss the alternative Hydrogen model set up in this way. First to say, we will see that many of the spectroscopic terms pertinent to the new model come "naturally" identical with the standard ones. Residual discrepancies are only found for small l -values, and some of the known orthodox expressions might perhaps also be recovered by appropriate redefinition of the perturbational Hamiltonians in the new context. However, even small term shifts implied by any model - if tracing in the reality - are in principle susceptible of experimental checks; but on a more concrete ground, here we have only a very basic model of the Coulomb interaction, so that the differences we can find in the term structure shall be better taken as hints for comparison purposes with analogous models. The real utility of the present framework is indeed as a new reference for discussion. By its definition, it will lend itself to be finely interpreted basing on 3D composition of fluctuating classical-like microscopic motions. Therefore

investigating the S-orbitals problem will also be found to offer further insight into the ultimate difference between a standard quantum and classical-like models of motions.

2 Background

In order to eliminate the ambiguities impending to the uncertain indications of the correspondence principle, different quantization procedures candidates for application in the multidimensional cases have been carefully examined since the beginning of quantum mechanics [30]. Particular concern has been given to their symmetry properties and ordering of non-commutative operators [31]; the regular laplacian form we use for the 3D kinetic Hamiltonian is in fact known to insure the invariance of the Schrödinger equation by rotations of axes, and general covariance properties. We are brought to think that this standard description represents the ultimate statistical appearance of the microscopical motions, once averaged all over the space directions. So we make clear that the aim of the present study is *not* at claiming that a better expression may be found. It is rather to affirm that a precursor form may exist, saving memory of some important kinetic properties - these last destined to be obliterated by the final step bringing to the standard result. In other words, we search here for a decomposition of the quantization procedure into different steps, each of them being connected (amongst other) to some peculiar motions appearances - in the quoted spirit of a motion composition methodology.

In this connection, it may be interesting to stress that in the (standard) isolated atom, separation of variables worked out, we find symmetrical states with complete oblivion of the classical ellipses. A number of authors [32 ÷ 46] developed Hydrogen models in two dimensions and some [47, 48] gave a very peculiar insight into this question. Rather rigorous trials have been made afterwards to built up ellipses again, by coherent composition of nearby quantum states; this has been done not only by theoretical means [49 ÷ 61], but also experimentally with a certain degree of success [62 ÷ 69]. However, all of these concern Rydberg atoms in quasi-classical circumstances. In a complementary way, we may think that the standard atom is straight the result we would also get starting with a mere "classical ellipse quantization" first, and averaging it afterwards over all the possible parameters and directions in space allowed by quantum theory. But exception made for the very first calculations by Sommerfeld himself, we have not yet available such a kind of detailed

model. In any case, the real difficulty it would certainly face if developed would be, first of all, how to insure the right quantum numbers spectrum at the final step - given all the previous ones. Indeed even the very solid 2D theories available for the electron staying in a fixed plane around the nucleus result into a spectrum different from the real 3D case. Since many years already, conjectures have been advanced as how to frame these differences into sound dynamical aspects of the system [38, 45, 48]. The case is emblematic of the so-called space quantization problem, soon going beyond it to point straight forward at the doublets appearance and to the Stern-Gerlach experiment interpretation.

X. Oudet [70 ÷ 73] has investigated space quantization in the light of relativistic principles with a model based on exchanges of small mass grains to describe the proton-electron interaction and the electron motion. He proposed to consider that quantification is not due properly to angular momentum but results from a so called intrinsic quantum of action h . This last is equally partitioned between the angular and translational (parallel to z) motion degrees of freedom so that motion is also quantified in the perpendicular direction to the classical plane of the Bohr Sommerfeld approach. For the S states, this means that half the action $h/2$ contributes to the orthogonal motion and the other half to an angular momentum. First prospections of the influence of these assumptions on the atomic model are given by this author, with particular insight into relativistic and quantum magnetic properties.

The method we describe in this paper is also based on a quantization step first operated in a plane, but it insures at last that the entire spectrum of quantum numbers m, l, n is identical with the known standard ; the only difference in this respect consists into the addition of a constant $\hbar^2/4$ value to the squared kinetic momentum eigenvalue. Indeed in order to overcome the mentioned difficulties with the standard model, we have to insert some additional rotation into the atom Hamiltonian. Yet we do not get this by adding any explicit energy term more to the standard Coulomb interaction (as is f.i. the case with the Chern-Simons term [28] ⁽¹⁾) : we only assume a different quantization procedure at the start. Compared to Oudet's model, however, we concluded that in the present non-relativistic, spinless treatment, only half a quantum of angular momentum $\hbar/2$ (to be interpreted as a fluctuation, with a zero average

¹The Chern-Simons correction may imply a modified (fractional) statistics [74] ; this is not at all the case in our model, where no spin is considered, l stays unperturbed and only l^2 -eigenvalues modify into $(l + 1/2)^2$.

value) in the zenithal degree of freedom is adequate to the fluctuation hypothesis. This is also because inserting the other half quantum into the procedure (at least as it is here proposed) caused essential singularities to appear in the calculations. Then with the previous assumptions, we came to the results expounded in the following section.

3 Theory

In this section, we are going to introduce our "alternative" expression for the 3D kinetic Hamiltonian of the Hydrogen atom. We ask for the following requirements to be satisfied :

a) the new form still must be a laplacian-like one (at second derivatives).

b) it must result into the same known set of the standard Hydrogen quantum numbers m, l, n, n_r with the same reciprocal relations. The energy terms must be the same that in the standard case, i.e.

$$E_n = -\frac{\hbar R}{n^2} = -\frac{me^4}{2\hbar^2 n^2} \quad (1)$$

Here R is the Rydberg constant, e the electron charge, and m the electron mass. Accidental degeneracy is therefore preserved here.

c) the squared momentum values, instead, have to take the values

$$\hbar^2 [l(l+1) + 1/4] = \hbar^2 (l + 1/2)^2 \quad (2)$$

As anticipated, this is tantamount to introduce a "zero point" fractional quantum number $1/2$ which corresponds to an additional kinetic momentum lying in the $z = 0$ plane, with alternate values $\pm \hbar/2$ in the time domain (take a square wave or a flip-flop for easy reference). This additional momentum has to be interpreted indeed as a fluctuation amplitude, so that its time-average is actually zero ⁽²⁾, and we have essentially only to deal with its squared average. Note that $(l + 1/2)^2$ are the same values we find in the quasi-classical case of the standard quantum theory ; but here we are searching the conditions for they are expressed exactly

²Fast quivering of the electron in our context may resemble a "zitterbewegung" (ZB) motion, the relativistic effect linked to interference between Dirac states. Deeper analysis, not affordable here, would be required to examine the possible connections of the present model with ZB ; interestingly, however, "simulation experiments" by Gerritsma [75] seem to show that ZB is also linked to a mass effect, which is one of the main ingredients for the fluctuation model promoted in [1,2].

by our laplacian form, without any approximation. So the resultant model may perhaps be considered as a rather exotic one, but is always a quantum-like one. More details on all of these points will be found later on.

d) The radial energy operator has not to have the regular 3D form $1/r^2 \partial/\partial r \ r^2 \partial/\partial r$ but the one pertinent to a 2D quantization in the plane, i.e. $1/r \partial/\partial r \ r \ \partial/\partial r$. In this way, we are setting up a mathematical ground to resolve our model in component motions in the future.

e) All the wavefunctions have to be zero on the nucleus ($r = 0$) ⁽³⁾. Their radial number of nodes must be the same standard number for $l \neq 0$; for $l = 0$ instead, it obviously comes out increased by 1.

Looking particularly at point b) we conclude that the azimuthal part ($\phi =$ azimuthal angle ⁽⁴⁾) of the laplacian must stay unchanged in comparison to the standard, while point d) determines the new radial part. Therefore we introduce the laplacian form :

$$\Delta' = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{j(\theta)}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \frac{\sin(\theta)}{j(\theta)} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin(\theta)^2} \frac{\partial^2}{\partial \varphi^2} \quad (3)$$

The factor $j(\theta)$ we find inserted into it actually modifies the zenithal part of the standard laplacian. As we will see in a next section, it is connected to a new metrics being defined in the 3D space, which we will have to consider by the sake of interpretation. However, before expounding this point, let us write down the Hydrogen wave equation as determined by the new context. We have

$$\begin{aligned} \frac{\hbar^2}{2m} \left[\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{j(\theta)}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \frac{\sin(\theta)}{j(\theta)} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin(\theta)^2} \frac{\partial^2}{\partial \varphi^2} \right] \Psi_{nlm}(r, \theta, \varphi) = \\ = - \left[E_n + \frac{e^2}{r} \right] \Psi_{nlm}(r, \theta, \varphi) \end{aligned} \quad (4)$$

³This condition should induce meditation on such topics as electron capture or Darwin effect, popularly calculated as functions of a certain permanence of the S electron on the nucleus (the more if some physical reality of the present model may ever appear, at some degree). But for now, we have different purposes in this work.

⁴We refer to the regular set of polar coordinates r, θ, φ with polar axis along z .

In order to discuss this equation, we note first that it can be interpreted as the sequence of two quantization steps. The first step is a quantization in the plane at constant azimuth :

$$\begin{aligned} -\frac{\hbar^2}{2m} \left[\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{j(\theta)}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \frac{\sin(\theta)}{j(\theta)} \frac{\partial}{\partial \theta} \right] \Psi_{nlm}(r, \theta, \varphi) = \\ = \left[E_n + \frac{e^2}{r} + \frac{\hbar^2}{2m} f_m(r, \theta) \right] \Psi_{nlm}(r, \theta, \varphi) \end{aligned} \quad (5)$$

The second step is quantizing the azimuthal rotation :

$$\begin{aligned} \frac{1}{r^2 \sin(\theta)^2} \frac{\partial^2}{\partial \varphi^2} \Psi_{nlm}(r, \theta, \varphi) = \\ = f_m(r, \theta) \Psi_{nlm}(r, \theta, \varphi) = -\frac{m^2}{r^2 \sin(\theta)^2} \Psi_{nlm}(r, \theta, \varphi) \end{aligned} \quad (6)$$

Indeed the intermediary function $f_m(r, \theta)$ is independent of φ at last, because by separation of variables we are able to extract the factor $Exp[i m \varphi]$ from $\Psi_{nlm}(r, \theta, \varphi)$. Then the overall procedure is a quantization in the meridian plane + rotation around the polar axis along z . Although in this paper we cannot afford accomplishing our main task (i.e. using Newton-like motions for 3D composition in Hydrogen, what is demanded to future papers) it must be stressed here that we consider the previous circumstance a clue to that purpose.

It is easy to understand now that point b) can be satisfied if wave-functions may be set up using the standard expressions multiplied by a perturbational factor equal for all of them. So we have to take

$$\Psi_{nlm}(r, \theta, \varphi) = A(r) B(\theta) \chi_{nl}(r) Y_m^l(\theta, \varphi) \quad (7)$$

$$Y_m^l(\theta, \varphi) = P_m^l(\cos(\theta)) Exp[i m \varphi] \quad (8)$$

where $Y_m^l(\theta, \varphi)$ are the regular spherical harmonics, $P_m^l(\cos(\theta))$ the associated Legendre polynomials and the $\chi_{nl}(r)$ are the radial parts of the standard Hydrogen :

$$\chi_{nl}(r) = k(n, l) Exp\left(-\frac{r}{a n}\right) \left(\frac{r}{n}\right)^l Hypergeometric1F1[1+l-n, 2+2l, \frac{2r}{a n}] \quad (9)$$

Here $k(n, l)$ is the normalization constant, a is the Bohr radius. As stated already, $n l m$ are the regular quantum numbers of the standard atom.

Concerning the $A(r)$ and $B(\theta)$ functions, we can easily find their forms by comparing a few calculation elements of our model with the standard case. In this last, we have indeed

$$-\frac{1}{\sin(\theta)^2} \left[\sin(\theta) \frac{\partial}{\partial \theta} \sin(\theta) \frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial \varphi^2} \right] Y_m^l(\theta, \varphi) = l(l+1) Y_m^l(\theta, \varphi) \quad (10)$$

From eq. (5) and point c) we have to put instead

$$\begin{aligned} -\frac{1}{\sin(\theta)^2} \left[\sin(\theta) j(\theta) \frac{\partial}{\partial \theta} \frac{\sin(\theta)}{j(\theta)} \frac{\partial}{\partial \theta} + \frac{\partial^2}{\partial \varphi^2} \right] B(\theta) Y_m^l(\theta, \varphi) = \\ = (l+1/2)^2 B(\theta) Y_m^l(\theta, \varphi) \end{aligned} \quad (11)$$

By simple developments, using both the equations we find that for all l, m

$$B(\theta) = \sqrt{j(\theta)} \quad (12)$$

and

$$Cot(\theta) \frac{j'(\theta)}{2j(\theta)} - \frac{3}{4} \frac{j'(\theta)^2}{j(\theta)^2} + \frac{j''(\theta)}{2j(\theta)} = -\frac{1}{4} \quad (13)$$

The solution for $j(\theta)$ ⁽⁵⁾ is given in *fig.1*.

⁵Equation (13) has a rather involved solution, we could deal with by use of Wolfram Mathematica 8. An expression can be found in the brief Appendix to this work.

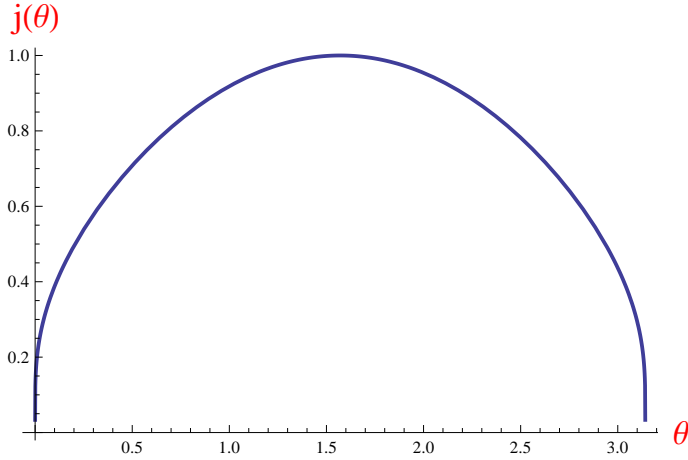


fig. 1 – Function $j(\theta)$ with $0 \leq \theta \leq \pi$.

Now we can write down equation (4) again in the form

$$\begin{aligned} -\frac{\hbar^2}{2m} \left[\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{(l+1/2)^2}{r^2} \right] A(r) \chi_{nl}(r) = \\ = \left[-\frac{me^4}{2\hbar^2 n^2} + \frac{e^2}{r} \right] A(r) \chi_{nl}(r) \end{aligned} \quad (14)$$

Comparing it with the standard equation satisfied by $\chi_{nl}(r)$ we have

$$\begin{aligned} -\frac{\hbar^2}{2m} \left[\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2} \right] \chi_{nl}(r) = \\ = \left[-\frac{me^4}{2\hbar^2 n^2} + \frac{e^2}{r} \right] \chi_{nl}(r) \end{aligned} \quad (15)$$

and we find for all n :

$$A(r) = \sqrt{r} \quad (16)$$

So our new wavefunctions take the expression ⁽⁶⁾ :

$$\Psi_{nlm}(r, \theta, \varphi) = \beta(n, l) \sqrt{r} \sqrt{j(\theta)} \chi_{nl}(r) Y_m^l(\theta, \varphi) \quad (17)$$

⁶As is obvious, the normalization factors we attached to $\chi_{nl}(r)$ in eq. (9) should take now values different from the ones pertinent to the standard states. So we leave the $\chi_{nl}(r)$ expression unchanged, but add in the expression (17) a corrective normalization factor $\beta(n, l)$.

Equation (17) shows us that the form of the orbitals in the new model changes in comparison to the standard by the factor $\sqrt{r}\sqrt{j(\theta)}$. From *fig.1* we see that $j(\theta) \rightarrow 0$ when $\theta \rightarrow 0$ or $\theta \rightarrow \pi$. So the main differences come out for $m = 0$ at small values of r and j . Indeed the $m = 0$ wavefunctions change from the finite values they take in the standard model around $(r = 0, \theta = 0, \theta = \pi)$, down to 0. For all quantum numbers besides, when r moves from 0 to greater values and θ starts with detaching from the extreme values 0 or π , the deformation may soon be considered small ⁽⁷⁾. Concerning the radial part, we see in particular that requirement e) is satisfied, i.e. even for $l = 0$ the electron has now a null probability to be found in the volume element $4\pi r^2 dr$ near $r = 0$. Concerning the angular part, the probability to find the electron at the zenithal points $(\theta = 0, \theta = \pi)$ is now equally zero for all quantum numbers. Therefore all the orbitals (take moduli in (17)) have now a torus-like symmetry. By an example, we give in *figs.2* and *3* 3D-plots of two sampled S- and F-like orbitals and their radial behaviour plotted in the additional 2D graph of *fig. 4* (all plots with Mathematica 8). Other orbitals change quite similarly, but clearly orbitals with $m \neq 0$ are not submitted to any variation in the topological sense.

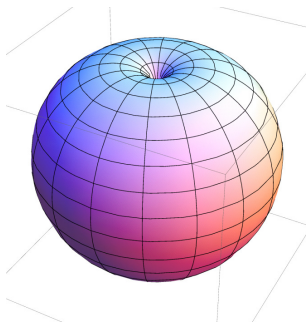


fig. 2 – Toroidal S-state

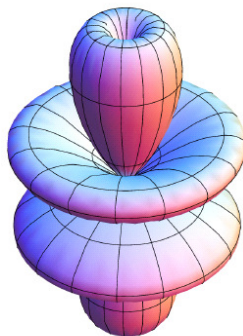


fig. 3 – Toroidal F – State ($l=3, m=0$)

⁷This explains by the overcoming effect of the l -powers and evanescence terms enclosed in the functions $\chi_{nl}(r)$, and by the fast trend to unity of the factor $j(\theta)$ when $\theta \rightarrow \pi/2$.

Amplitude and distance from the nucleus in arbitrary scales

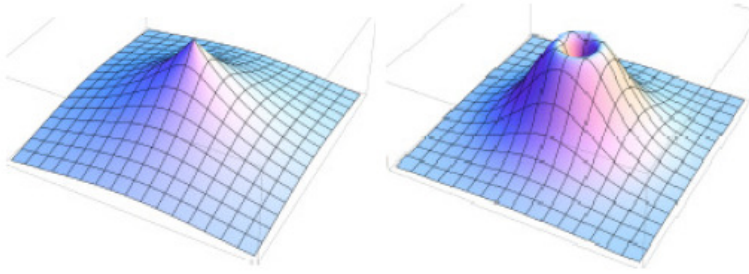


fig. 4 – S-state radial wavefunctions (standard vs non standard)

Note that not all of the functions in the set (17) turn out to be orthogonal to each other when we integrate (only) over the standard volume element $\sin(\theta)d\theta$. It can be checked by numerical calculations of the volume integrals that the θ -functions $\sqrt{j(\theta)}P_m^l(\cos\theta)$ and $\sqrt{j(\theta)}P_{m'}^{l'}(\cos\theta)$ slightly superpose to each other when $|l' - l|$ and $|m' - m|$ both are odd numbers, or both are even ones. So we find (p, q integers) :

$$\int_0^\pi j(\theta)P_m^l(\cos(\theta))P_{m\mp 2q}^{l\mp 2p}(\cos(\theta))\sin(\theta)d\theta \neq 0 \quad (18)$$

$$\int_0^\pi j(\theta)P_m^l(\cos(\theta))P_{m\mp (2q+1)}^{l\mp (2p+1)}(\cos(\theta))\sin(\theta)d\theta \neq 0 \quad (19)$$

The integrals are instead 0 when $|l' - l|$ and $|m' - m|$ have different parity.

A similar situation however, also occurs to the standard associated Legendre functions $P_m^l(\cos(\theta))$, which are also found to interfere in the θ domain when $|l' - l|$ and $|m' - m|$ both are odd numbers :

$$\int_0^\pi P_m^l(\cos\theta)P_{m\mp (2q+1)}^{l\mp (2p+1)}(\cos(\theta))\sin(\theta)d\theta \neq 0 \quad (20)$$

Yet here, the set of functions with integral overlap is not represented only by (19) but also by (18) so that it includes a greater number of them. Indeed the presence of $j(\theta)$ in the integral brings it to a non-zero value even for the cases where the standard integrand $P_m^l(\cos(\theta))P_{m\mp 2q}^{l\mp 2p}(\cos(\theta))$

amounts to a zero integral superposition (although it is an even function of the latitude around $\theta = \pi/2$). So amongst other, we see that if we integrate the non-standard wavefunctions (17) over the full space volume element $r^2 \sin(\theta) dr d\theta d\varphi$ with $m \neq m'$, we still find the result 0 - so that the orbitals can be said orthogonal to each other; but in case $m = m'$ and $|l' - l| = 2p \neq 0$, these orbitals will show (small) overlaps. As it is well known, eigenfunctions orthonormality is a property strictly linked to the mathematical asset of the linear state superposition principle in orthodox quantum mechanics, allowing easy treatment of mixed states with their interference effects. But in our view, it is not an essential property to keep the model working in all circumstances. The true and concrete matter is indeed, after all, having available a wave equation able to give solutions with any initial conditions set given by physical constraints. That one has available two solutions with certain initial conditions, and he can find a third case by linear mixing (or the inverse, i.e. easily achieve state reduction), this is obviously a very comfortable circumstance. But whenever it is not so, the only important matter to be insured is that we can find the solutions with any chosen constraint; we will always be able to investigate their relations afterwards. This situation also occurs in the framework of the fluctuation model expounded in [1,2], because there an imaginary potential term is added to the standard Hamiltonian so that the resulting wave equation is a non-linear one.

However, the cross integral values (18) and (19) fast approach the standard values (i.e. the ones with $j(\theta) = 1$) when l' and l are large enough compared to $1/2$. Also very interesting, if we integrate over the "deformed volume" with element $r^2 \sin(\theta) dr d\theta / (r j(\theta))$, the factor $r j(\theta)$ eliminates in the integrand so that the set (17) turns out to recover the standard properties in that volume; we identify this last now as a sort of non-euclidean space domain.

Looking more specifically at the meridian plane, we see from equation (5) that $1/j(\theta)$ plays the form factor role attached to $\sin(\theta)$ into it. Since $r \sin(\theta) d\theta$ is the part of metric element in spherical coordinates pertinent to a zenithal altitude variation, we give the interpretation that it is deformed into $r \sin(\theta) d\theta / j(\theta)$, thus becoming a bit longer ($j(\theta) \leq 1$). This is tantamount to say that the space element attached to the angle θ is greater than the standard cartesian value; in other words, we face here the appearance of a non-euclidean geometry tract to account for. This can be found better explained physically in the next section. We call $d\theta_{def}$ the deformed angular space element $d\theta / j(\theta)$. We plot the function

$\theta_{def}(\theta)$ in fig. 5.

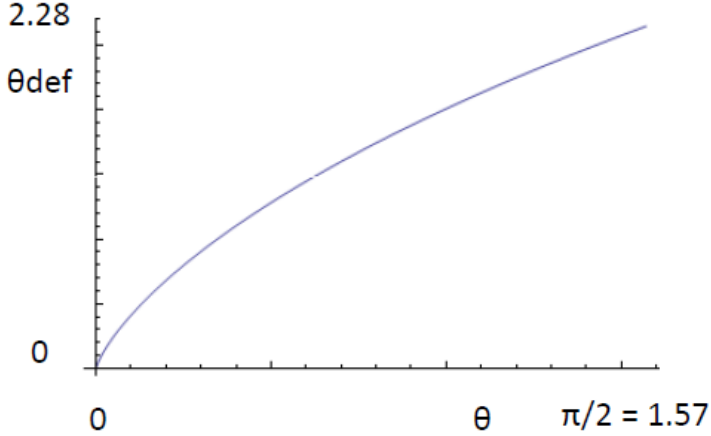


fig. 5— Non— euclidean coordinate θ_{def} as a function of θ

Note that a treatment of non-euclidean space extension in the variable θ implies the following transformations in the main operators definitions (\hat{l} = standard angular momentum operator ; \hat{l}_{ns} = a non-standard operator in the present model, fitted to absolute value calculations) :

$$\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{j(\theta)dt} \quad (\text{classical view}) \quad (21)$$

$$\frac{\partial}{\partial\theta} \Rightarrow \frac{1}{j(\theta)} \frac{\partial}{\partial\theta} \quad (\text{quantum view}) \quad (22)$$

So we define

$$\hat{l}_{ns} \equiv \hat{l}_{ns}^- = \frac{1}{j(\theta)} \hat{l} \quad (23)$$

where \hat{l}_{ns} takes the role of a kinetic momentum operator accounting for the space deformation. Here we have given a $(-)$ suffix to \hat{l}_{ns} , because it also turns out useful to define a dual operator

$$\hat{l}_{ns}^+ = j(\theta) \hat{l} \quad (24)$$

So we can write

$$\hat{l}^2 \implies \hat{l}_{ns}^2 \equiv \hat{l}_{ns}^+ \cdot \hat{l}_{ns}^- = j(\theta) \hat{l} \cdot \frac{1}{j(\theta)} \hat{l} \quad (25)$$

These equations may also be read in the sense that space deformation introduces a major distinction between the measures (quantum means) of the angular momentum itself and its absolute value. This brings us to introduce the couple of non-standard operators \hat{l}_{ns}^- and \hat{l}_{ns}^+ but only their ordered product univocally defines the squared operator ⁽⁸⁾. Eq. (25) means that we have to measure first the quantity $\hat{l}_{ns}^- = 1/j(\theta) \hat{l}$ to get the absolute mean value of the angular momentum from the deformed wavefunction; and apply $j(\theta) \hat{l}$ afterwards, in order to report the average squared momentum to the euclidean observer. Compared to the standard quantum formalism, a remarkable conceptual difference is that in order to calculate the absolute squared value we cannot simply apply twice the same standard operator \hat{l} , because the non-euclidean space includes motion modes we should not average out (see *figs. 6, 7* as an example). The standard operator \hat{l} instead, still measures the mean angular momentum and expresses as $\hat{l}_x + \hat{j} \hat{l}_y + \hat{k} \hat{l}_z$ in cartesian coordinates. Taking the scalar product in (25) and the known expressions in polar coordinates

$$\hat{l}_x = i \left[\text{Cos}(\varphi) \text{Cot}(\theta) \frac{\partial}{\partial \varphi} + \text{Sin}(\varphi) \frac{\partial}{\partial \theta} \right] \quad (26)$$

$$\hat{l}_y = i \left[\text{Sin}(\varphi) \text{Cot}(\theta) \frac{\partial}{\partial \varphi} - \text{Cos}(\varphi) \frac{\partial}{\partial \theta} \right] \quad (27)$$

$$\hat{l}_z = -i \frac{\partial}{\partial \varphi} \quad (28)$$

it can be now checked that the quantity $\hat{l}_{ns}^+ \cdot \hat{l}_{ns}^-$ comes indeed equal to

$$\hat{l}_{ns}^+ \cdot \hat{l}_{ns}^- = -\frac{1}{\text{sin}(\theta)^2} \left[\text{sin}(\theta) j(\theta) \frac{\partial}{\partial \theta} \left(\frac{\text{sin}(\theta)}{j(\theta)} \frac{\partial}{\partial \theta} \right) + \frac{\partial^2}{\partial \varphi^2} \right] \quad (29)$$

as required by (11). In this expression, we have to note that the azimuthal-dependent part of the operator is unaffected by the factor $j(\theta)$ at last.

⁸Changing their order in the definition leads to the same result when we set $j(\theta) \rightarrow 1/j(\theta)$ in the formalism, i.e. assuming a mere change of definition for $j(\theta)$.

This obviously means that the magnetic quantum numbers and corresponding φ -dependent parts of the wavefunctions also are unaffected by the definition (25). Amongst other indeed, we also find insured that (bra and kets here stay for the deformed waves (17)) :

$$\langle ml | \hat{l}_z^2 | ml \rangle = - \int_V \Psi_{nlm}^*(r, \theta, \varphi) \frac{\partial^2}{\partial \varphi^2} \Psi_{nlm}(r, \theta, \varphi) \sin(\theta) r^2 dr d\theta d\varphi = m^2 \quad (30)$$

$$\langle ml | \hat{l}_z | ml \rangle = -i \int_V \Psi_{nlm}^*(r, \theta, \varphi) \frac{\partial}{\partial \varphi} \Psi_{nlm}(r, \theta, \varphi) \sin(\theta) r^2 dr d\theta d\varphi = m \quad (31)$$

Moreover we find

$$\langle ml | \hat{l}_x | ml \rangle = \langle ml | \hat{l}_y | ml \rangle = 0 \quad (32)$$

So these diagonal terms also come out equal to the standard values i.e. are null. By numerical checks, we have found that non-diagonal terms with indices $(m, m \pm 1)$ are instead different from zero (as expected) and show very small differences ($< 1\%$) in comparison to the standard values. The same cannot be said of the operators $1/j(\theta)\hat{l}_x$ and $1/j(\theta)\hat{l}_y$, resulting in matrix terms different up to about 31 % from the standard : f.i. we have

$$\langle 01 | \frac{1}{j(\theta)} \hat{l}_x | 11 \rangle = -i \langle 01 | \frac{1}{j(\theta)} \hat{l}_y | 11 \rangle = 0.929 \quad (33)$$

$$\langle 12 | \frac{1}{j(\theta)} \hat{l}_x | 22 \rangle = -i \langle 12 | \frac{1}{j(\theta)} \hat{l}_y | 22 \rangle = 1.168 \quad (34)$$

while the standard values are $1/\sqrt{2} = 0.707$ and 1 for the two cases respectively. However, all of these differences soon decrease when the l, l' values increase. Due to the involved expression of $j(\theta)$ and consequent long pc calculation time, it is not affordable here computing the new matrix terms or integral overlaps all over the quantum numbers spectrum with deformed wavefunctions, so we leave this to a future work whenever useful. But we have attempted a physical interpretation to the main quoted findings. As anticipated, this is expounded in a next section.

Here we only add a mathematical point more, concerning the difference between the new laplacian Δ' we have previously shown and the known standard form. If we name Δ the standard 3D laplacian operator in polar coordinates, it is easy to check that this difference is

$$\hat{l}_0 = \Delta' - \Delta = -\frac{1}{r} \frac{\partial}{\partial r} - \frac{j'(\theta)}{r^2 j(\theta)} \frac{\partial}{\partial \theta} \quad (35)$$

By the present context, it is clear that the action of this operator is correlated to the appearance of the additional (fluctuation) quantum number $1/2$ in our theory. So we will name it "the zero point (rotational) operator". This operator is a linear composition of derivatives in r and θ . As such, we want to stress that it can be quite naturally framed into the possible set of extensions of quantization procedures we have formerly investigated [76, 77], and traced back to the use of non-standard, generalized Hamilton-like quadratic forms - these last include a term linear in the classical particle momentum \vec{p} and are correlated to a mass effect. By equation (35), \hat{l}_0 can be regarded as the perturbational Hamiltonian which has to be added to Δ to obtain the deformed orbitals - and viceversa, $-\hat{l}_0$ brings back from the latter to the standard model ⁽⁹⁾.

4 Physical interpretation

Orbitals deformation with toroidal symmetry, two-steps interpretation of the quantization procedure and appearance of the additional term $\hbar^2/4$ in the expression of the squared angular momentum are the essential features introduced by the new laplacian form Δ' . These properties altogether seem to provide us with a special insight into the atom model and new instruments to our description ability.

As announced before, the space extension in the meridian circle $r = \text{const}$, $\varphi = \text{const}$ is perturbed by the metric factor $1/j(\theta)$ so that the measure of the full zenithal angle coverage (π) becomes instead

$$\int_0^\pi d\theta_{def} = \int_0^\pi \frac{d\theta}{j(\theta)} = 4.558 \simeq 1.45 \pi \quad (36)$$

We interpret this non-euclidean measure assuming that the electron is submitted to fast oscillations in the meridian plane, perturbing the standard dynamics we generally have when we set $j(\theta) = 1$. In this paper, calculations fitted to the energy-mass fluctuation model as given

⁹So $-\hat{l}_0$ suppresses the $\hbar/2$ fluctuation and obviously brings us back to the orthodox treatments and results, also for the entire perturbational environment of the atom.

in [1,2] are not affordable ; but a simulated, simple insight can be given by the following equations, treading in a basic fluctuation frame. Would it be $\theta_c(t)$ an electron time-law in the meridian plane, it might have a form (f.i. the simplest) including a slow (s) varying term, and a fast (f) one (take time derivatives ; in this example, $\dot{\theta}_s(t)$ and $\dot{\theta}_{def}(t) \leq 0$) :

$$\dot{\theta}_c(t) = \dot{\theta}_s(t) (1 + a(t) \sin(\omega_f t)) \quad (37)$$

with

$$\omega_f \gg |\dot{\theta}_s(t)| \quad (38)$$

$$\langle \dot{\theta}_s(t) a(t) \sin(\omega_f t) \rangle_{2\pi p/\omega_f} \simeq 0 \quad (39)$$

$$\langle \dot{\theta}_c(t) \rangle_{2\pi p/\omega_f} \simeq \dot{\theta}_s(t) \quad (40)$$

The two last equations here indicate how to average out fast fluctuations ; indeed the time averages $\langle \rangle$ are intended to be taken over a small interval $2\pi p/\omega_f$ (f.i. $p \approx 2$ or 3) around t . We have also, by the same rule ⁽¹⁰⁾ :

$$\theta_{def}(t) = \theta_{def}(0) - \langle \int_0^t |\dot{\theta}_c(t')| dt' \rangle_{2\pi p/\omega_f} \quad (41)$$

Obviously the time laws must be such that (call τ the time needed to go from $\pi/2$ to 0) :

$$\int_{-\tau}^{\tau} \dot{\theta}_c(t) dt = \int_{\pi}^0 d\theta_c = -\pi \quad (42)$$

$$\int_{-\tau}^{\tau} \dot{\theta}_{def}(t) dt = -\varepsilon \pi = -1.45 \pi \quad (43)$$

All of this can be resumed saying that when a fast fluctuation with zero time average is superimposed to some predefined $\dot{\theta}_s(t)$ time law, the total angular path effectively covered by the electron increases by a factor ε due to forward/backward vibration steps around any position θ_s . This motion is much analogous to the fluctuating motion of the pointer in a mechanical watch affected by vibrations.

¹⁰The sign minus in eq. (41) is due to the fact that our time laws bring the zenithal altitude from ($\theta = \pi/2$ in $t = 0$) to ($\theta = 0$ in $t = \tau$), i.e. time derivatives are negative quantities.

To enlighten our example, we have plotted in *figs.* 6 and 7 the functions $\theta_c(t)$, $\theta_{def}(t)$ consequent to a simple time law (44) in time units of τ , with $\pi/2 \geq \theta_c \geq 0$ and $0 \leq t/\tau \leq 1$. We purposely have taken a rather small frequency ω_f , to allow the reader appreciate fluctuations in the graphics. For an effective application to the electron orbital high frequencies some coefficients in the equations should be rescaled (so it was not worthy to us achieving high numerical precision in the plots 6, 7 and in eq. (46), all of this is for demonstrative purposes). The case can be therefore simply illustrated as follows : taking f.i. (pulsations in *rad/s*)

$$\dot{\theta}_s(t) = -\frac{\pi}{2\tau} \quad (44)$$

$$\omega_f = \frac{61.1}{\tau} \quad (45)$$

and

$$a(t) \simeq -(0.074 + 0.019 \frac{t}{\tau}) \left[3.71 - \ln \left(1 - \frac{t}{\tau} \right) \right]^2 \quad (46)$$

we get the following graphs for $\theta_c(t)$, $\theta_{def}(t)$:

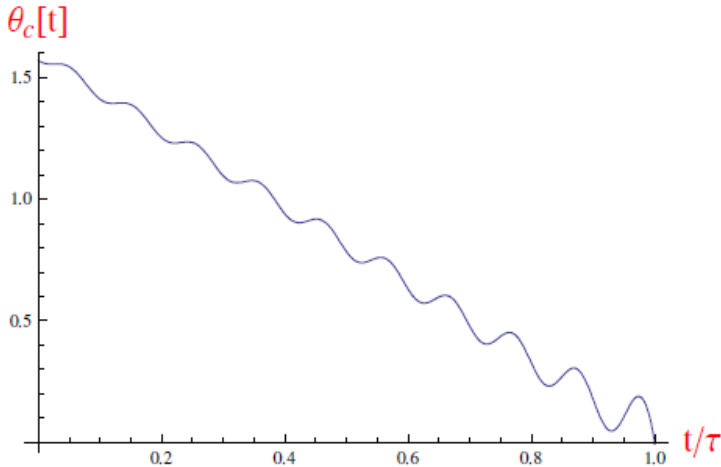


fig. 6— Example of classical time—law with fluctuations

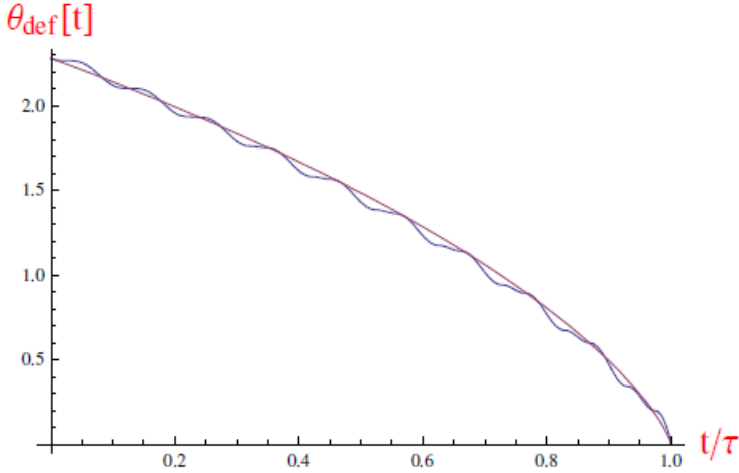


fig. 7 – Non-euclidean coordinate θ_{def} associated to $\theta_c(t)$

So the coordinate $\theta_{def}(t)$ appears to us as the measure of the average (red line in fig. 7) absolute angular path covered when following the law $\theta_c(t)$ (this behaviour confirms the one shown in fig. 5, since the time t is proportional to θ_s ⁽¹¹⁾ in the example. This is thanks to the peculiar expression given in eq. (46)).

In a very rough model with fluctuating mass $m(t)$, just to fix ideas for the S orbital case in the present context, we might think to a family of basic electron trajectories in the meridian plane of the form ($c \ll 1$)

$$m(t)r(t)^2\dot{\theta}_c(t) = \pm \frac{\hbar}{2} \sqrt{1 + c \sin(\omega t + \phi)} \quad (47)$$

where we assume that the number of trajectories with a (+) sign is balanced with the (−) ones; then the time-averaged, squared angular momentum in the (x, y) plane should be

$$\frac{1}{T} \int_0^T m(t)^2 r(t)^4 \dot{\theta}_c^2(t) dt = \frac{\hbar^2}{4} \quad (48)$$

We can also cast (in a future refined model) by simple logics these properties into a final expectation for the quantum squared momentum, at

¹¹ $\theta_s \equiv \theta$ in fig. 5.

least qualitatively : a classical-like motion interpretation should exist, where by upgraded equations similar to (47) (48) the quantum number l turns out unchanged, but the quantum squared value $l(l+1)$ must be increased by a factor $1/4$ - the value we have indicated in this paper indeed.

When such kind of refined model will be expounded to take full account of energy, momentum and mass fluctuations, we believe it can provide a classical-like motion interpretation even for the case of the S-orbitals. Yet for now, it removes the apparent incongruence of these orbitals with respect to zenithal elevation changes; but it still remains to be understood how azimuthal displacement is possible with the specific constraint $l_z = 0$. An explanation can be found, still staying in a fluctuation model frame. Although investigation is still required in this direction, we want to propose it here for the sake of an interesting perspective being brought to the model.

Looking at the toroidal orbitals, we see that the probability to find the electron at the extreme zenithal angles $0, \pi$ is now null. But in a mass fluctuation model, it will be different. This model will bring its own peculiarities, leading to some differences with the quantum models. It has been shown [2] that due to the mass effect the particles density ρ will not take zero values at nodes as in the standard quantum model : zeros actually transform into minima, so that the singularities of the inverse of density become rather high values, but not infinite ⁽¹²⁾. Since the mean velocity v_D of the equivalent particle (or the packet if preferred) is given by the equation

$$v_D = 2 \frac{\nu}{\rho} \quad (49)$$

(ν is the inverse of the rotation period here) the particles group will be found to cross over the zenithal points at a rather high velocity, but not infinite in that model. Now suppose that during the passages over $\theta = 0$ or π , at each time, a transverse ⁽¹³⁾ fluctuation impulse of force, available in the external ("vacuum") background, deflects the electron at a different random azimuthal angle or meridian plane : no violation of the condition $l_z = 0$ will occur, since $\sin(\theta) = 0$ in that very moment. After many passages at last, the trajectories clew will fill the orbital in full.

¹²This fact brings back the orbitals *genus* to zero again.

¹³i.e. parallel to the (x, y) plane and orthogonal to the particle velocity at zenith.

This mechanism would also help us with insuring that orbital angular momentum in the (x, y) plane is able to flip between $\hbar/2$ and $-\hbar/2$ with zero average as previously assumed.

We have to stress that this sudden (and random) meridian plane rotation consequent to the passages at zenith is only a proposed explanatory mechanism herein, whose real occurrence we cannot prove by our present knowledge. Further research is obviously required, in order to investigate about any possible appearance of our precursor stage in the physical reality. So by now, the proposed escapement towards a classical interpretation rests on the mathematical anomaly of the l_z definition in the very points where $\sin(\theta) = 0$: this looks to us as the very subtle boundary edge between the quantum description and the classical view of motion. On another hand, the impulse discontinuity at zenith is strictly linked to the pointlike character of the electron in this model, at least when the magnetic quantum number m is zero; yet we might be able to remove it introducing electron dimensions or some non-local effect (both in classical or quantum views) in further investigations. When m is different from zero instead, in order to eliminate the singularities dues to the $l_z = \text{const}$ constraint imposed by quantum mechanics, we can also invoke the favourable circumstance that in a fluctuating model interpretation the operator eigenvalues must be generally thought as time-averages of oscillating quantities.

Squared angular momentum values equal to $(l + 1/2)^2$ are also found in the standard quantum model when a quasi-classical limit is taken. Yet there, the result is obtained heuristically, by imposing phase-match of the radial wavefunction with the Kramers conditions; no special interpretation beyond quasi-classical applications seems ever to have followed. However, quasi-classical conditions are only insured for r greater than the Bohr radius, great l values, small latitudes. The present model seems instead to us able to shed new light with some rigour and completeness on the overall context.

5 Spectroscopic terms (comments)

Since the expression of the energy eigenvalues (1) is identical to the known standard, all of the basic Lyman, Balmer, Paschen, Brackett and Pfund transition frequencies are obviously also preserved in our model. As a general rule, all of the spectroscopic terms corrections depending linearly on the operators \hat{l}_z and \hat{l}_{ns}^2 come identical to the standard ones,

because the differences in m values and in $[l(l+1) + 1/4]$ values stay unchanged. Concerning fine structure instead, a rather subtle point must be made. Would we assume brutally that the additional fluctuating momentum makes no interference with the spin vector, the spin-orbit correction would not change compared to the standard model. Indeed it essentially depends on the value of the scalar product $\hat{l} \cdot \hat{s}$ which would remain the same : we should then simply add $1/4$ to both $j(j+1)$ and $l(l+1)$ in the known RS expression

$$< \hat{l} \cdot \hat{s} > = \frac{1}{2} [j(j+1) - l(l+1) - s(s+1)] \quad (50)$$

leaving it therefore unperturbed (¹⁴). But it may not be so, if the scalar product has to account for an interference term arising by the product of the equatorial parts of spin with the flipping $\pm \hbar/2$. So deeper investigation, including a detailed spin model in the fluctuation frame, is needed to enlighten this point. A specific remark can also be reserved here to the Darwin term, since generally it is expressed as

$$\frac{\pi e^2 \hbar^2}{2mc^2} \Psi_{n00}(0, \theta, \varphi)^2 = \frac{2n}{mc^2} E_n^2 \quad (51)$$

where the wavefunctions $\Psi_{n00}(0, \theta, \varphi)$ are different from zero in the standard model. The wavefunctions we have produced in this paper are instead null on the nucleus, so that eq. (51) fails in our case. Then the standard treatment of the Darwin term should be revised to cope with the present model, probably having recourse to non-locality electron properties. Note however, that the Darwin term is known to be coincident with the limit $l \rightarrow 0$ of the general RS-coupling expression of the spin-orbit energies [82]. Thus we can link this question to the previous one ; yet the point remains a subtle one, as in all cases where relativistic corrections are treated with non-rigorous, non-relativistic ingredients. But as is obvious, further discussion extending to the Dirac model or examining QED-based corrections cannot be afforded here.

At least for weak magnetic fields, Zeeman terms are instead unchanged compared to the known expressions, because they only depend on the z component of the angular momentum (and spin). We have seen indeed that the angular momentum z component matrix elements are

¹⁴Even so, we may not find the same situation for multielectron atoms where spin-spin interaction must be considered, since then we have to account for a quadratic term in $\hat{L} \cdot \hat{S}$, which might be affected by the corrective term $1/4$.

identical to the standard ones. Correspondingly, the fluctuating electrical current in the meridian plane due to the added $\pm\hbar/2$ makes a zero time-averaged mechanical momentum with the magnetic field, so that the added energy correction is effectively zero. On another hand, the popular expression for the gyromagnetic ratio given by the relativistic theory of the electron [78, 79]

$$g = \frac{k}{k + 1/2} \quad (52)$$

where k is the so-called screening quantum number [80] ⁽¹⁵⁾, also remains unchanged by the present context. Clearly however, and pointing to possible generalizations of our model to multielectron atoms, the present comments are only based on the assumption that a RS coupling with a quantum number l good enough is insured, so that any deeper refinement of theoretical calculations besides this assumption still waits for future research.

6 Conclusion

Searching for traces of classical motion in quantum mechanics, we have introduced an "alternative" model for the (non-relativistic) Hydrogen atom, as it stems from a non-standard quantization procedure. We identified the so-called "zero point operator" \hat{l}_0 in eq. (35), useful to get across two quantization sets as a sort of mathematical bridge : if we start with our toroidal Hamiltonian model and subtract \hat{l}_0 to it, we recover the standard quantized model ; if we start with this last, adding \hat{l}_0 , we turn back to the former. Giving sense to the appearance of the non-euclidean metrics, we might also say [83] that we have transformed the quantum radial motion mode expressed by $\frac{1}{r} \frac{\partial}{\partial r}$ in the laplacian Δ into another one, $-\frac{j'(\theta)}{r^2 j(\theta)} \frac{\partial}{\partial \theta}$ (i.e. an orthogonal mode playing in the meridian plane) to obtain Δ' . We have also given the interpretation that the standard quantum atom model can be obtained by the suite of two precursor quantization steps (5) and (6) (quantization in the meridian plane + azimuthal rotation) and a third step removing the fluctuation quantum number $1/2$ by the action of $-\hat{l}_0$. The first two steps offer a quantized model seeming to us well fitted for a classical-like interpretation, by summing ergodic microscopic motions with variable mass (to

¹⁵In the atomic shells nomenclature pertinent to such models, k is either equal to l or to $-(l + 1)$, see also f.i. [81].

be attempted in next papers). If these future calculations will be successful, residual discrepancies between their results and the standard quantum model may then be ascribed to the action of $-\hat{l}_0$ so that a further, peculiarly focused insight may hopefully be given into the elusive incongruence between classical and quantum theories.

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Appendix

The function $j(\theta)$ solving eq. (13) with the conditions

$$j\left(\frac{\pi}{2}\right) = 1 \quad (53)$$

$$j(\theta) = j(\pi - \theta) \quad (54)$$

takes the expression

$$j(\theta) = Exp\left[\int_{\frac{\pi}{2}}^{\theta} f(\theta')d\theta'\right] \quad (55)$$

where

$$f(\theta') = \frac{\pi Cot(\theta')EllipticK\left[\sin^2\left(\frac{\theta'}{2}\right)\right] - 0.5\pi^2 \csc(\theta')LegendreP\left[\frac{1}{2}, \cos(\theta')\right]}{\pi EllipticK\left[\sin^2\left(\frac{\theta'}{2}\right)\right] + \pi LegendreQ\left[-\frac{1}{2}, \cos(\theta')\right]} + \quad (56)$$

$$+ \frac{\pi Cot(\theta')LegendreQ\left[-\frac{1}{2}, \cos(\theta')\right] - \pi \csc(\theta')LegendreQ\left[\frac{1}{2}, \cos(\theta')\right]}{\pi EllipticK\left[\sin^2\left(\frac{\theta'}{2}\right)\right] + \pi LegendreQ\left[-\frac{1}{2}, \cos(\theta')\right]} \quad (57)$$

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