# Charging Capacitors According to Maxwell's Equations: Impossible

Daniele Funaro

Dipartimento di Fisica, Informatica e Matematica
Università di Modena e Reggio Emilia
Via Campi 213/B, 41125 Modena, Italy

RÉSUMÉ. Le chargement d'un condensateur idéal conduit á la résolution de l'équation d'onde pour le champ électrique, avec des conditions initiales et aux limites. On démontre que, quelle que soit la forme du condensateur et la tension appliquée, aucune des solutions correspondant peut être compatible avec les équations de Maxwell. Le paradoxe persiste même si on affaiblit les conditions aux limites, ce qui implique l'impossibilité de d'ecrire simples phénoménes, tels que la charge d'un condensateur, par la théorie classique de l'électromagnétisme.

*ABSTRACT. The charge of an ideal parallel capacitor leads to the resolution of the wave equation for the electric field with prescribed initial conditions and boundary constraints. Independently of the capacitor's shape and the applied voltage, none of the corresponding solutions is compatible with the full set of Maxwell's equations. The paradoxical situation persists even by weakening boundary conditions, resulting in the impossibility to describe a trivial phenomenon such as the capacitor's charging process, by means of the standard Maxwellian theory.*

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## 1 Background

The subject here are the standard Maxwell's equations and their inability to handle a lot of situations that in the common practice are instead considered trivial. The main criticism is that the system is overdetermined, i.e., solutions must satisfy too many constraints without enjoying the necessary degrees of freedom. The discovery of these inconsistencies was made about ten years ago, when I started a review process of the theory of electromagnetism. The underlying motivation was based on some unsatisfactory marginal aspects. Nevertheless, in the development of the analysis, these aspects became much more relevant. The result was a renewed model (see [6], [7]) that strictly includes the solutions to Maxwell's equations, thus providing the description of a wider range of events. In particular, non-dissipating compact-support electromagnetic waves, travelling straightly at the speed of light, are very easily modelled by the new set of equations. The importance of this fact is high if we realize that one of the reasons for the split of physics into the classical and the quantum versions is actually the impossibility to represent photons via Maxwell's equations in vacuum. Indeed, an initially localized wave-packet, whose fields are successively modelled by Maxwell's equations, is soon destroyed, diffusing all around.

It is not my intention however to further self-celebrate the potentiality of my extension and the possible implications in the study of the quantum world by a classical approach. For further insight the reader is referred to [9]. Here, I will concentrate my attention on a specific application, pointing out other deficiencies of the standard Maxwell's model.

Criticizing Maxwell's equations is dangerous; one is immediately relegated as heretic. On the other hand, the power of mathematical reasoning cannot be ignored. After publishing my first report ([6]), I was contacted by Dr. W. Engelhardt (see [2], [3], [4]). He was puzzled by the excessive number of constraints that a Maxwellian solution has to satisfy. By trying to impose all of them one inevitably comes to contradictions. Usually, engineers follow a certain computational path in order to come out with solutions mimicking as much as possible reality. When a reasonable output is obtained they do not feel it is necessary to check if further restrictions apply. It is like storing all of $n+1$ objects inside $n$ boxes (one object per box). There are several paths one can follow, but none is going to be resolutive. A dirty trick is to hide the last object (whatever it is) and show to the public one of the allowed combinations.

Due to its linearity, it is easy to prove that the Maxwell's system must have, under very mild assumptions, at most one solution (uniqueness). The work of Engelhardt shows that there may be different solutions to the same given problem. How can this happen? In reality, the problem imposes $n+1$ constraints and turns out to be impossible; however, by applying $n$ steps of the solution process one can actually get something meaningful. That "something" depends on the constraint that has been discarded. Engelhardt writes the electromagnetic unknowns in terms of the potentials $\Phi$ and $\mathbf{A}$ and notes that different conclusions are reached according to the choice of the *gauge*, notwithstanding that the gauge has no influence on the expression of Maxwell's equations. The conclusion that there are different solutions contradicts uniqueness. This nonsense can only be justified by deducing that none of the solutions proposed is correct, not because of a mistake in the computation, but because they result from an incomplete procedure and that a full resolution does not exists. This observation casts dark clouds on the Maxwellian theory.

In this short note I would like to study in the easiest possible way a very simple problem: the charge of a capacitor. Solving the wave equation for the electric field I get a solution incompatible with Ampère's law. A similar question was examined in [4] via retarded potentials. There the author points out inconsistencies between the wave equation and the Faraday's law. Hence, the analysis of the distribution of an electromagnetic field inside a capacitor depends on the way the problem is approached. In the quasi-stationary regime (current flow is relatively slow) the magnetic contribution is usually neglected. If higher modes are involved, the full system of Maxwell's equations must be invoked. In this fashion, according to the nature of the phenomenon, one picks up the right tool to operate, consisting of $n$ relations chosen on purpose. Very often the outcome is convincing. If little troubles emerge they are attributed to some unavoidable inaccuracy dictated by the limits of the model. However, the question is deeper: the model itself is not mathematically correct when taken with all its constraints. One could find rigorous mathematical outcomes by getting rid of one constraint at the time, but at this point there is no a unique theory of electromagnetism.

In [4], the limit of the Maxwellian theory is attributed to the presence of nonhomogeneous terms. In my opinion, as rigorously analyzed in [8], even homogeneous Maxwell's equations in vacuum are not trouble free, displaying an extremely reduced space of solutions. The equations in this case are affected by an almost total lack of initial displacements satisfying both the conditions $\text{div}\mathbf{E} = 0$ and $\text{div}\mathbf{B} = 0$ (see my viewpoint in [9], chapter 1).

It must be honestly pointed out however that the solution's space of Maxwell's equations is far from being empty. There are in fact remarkable situations where the model has mathematical meaning and matches reality. Solutions seem however to belong to a kind of *meagre set*. This set looks closed, connected and with zero measure (with respect to standard topologies in functional spaces), though I have no rigorous proof of these statements. Common applications, as the one studied here, belong to the *complementary set* and cannot be approximated by Maxwellian solutions. My conclusion is that, despite of the extraordinary achievements of the technological world, Maxwell's[1] equations are inefficacious in most practical cases, unless they are accompanied by rough and non well justified mathematical adaptations.

## 2 Technical preliminaries

In order to make the problem as simplest as possible let us suppose we are in vacuum, though this restriction is not necessary. Maxwell's equations include the Ampère's law:

$$\frac{\partial \mathbf{E}}{\partial t} = c^2 \mathrm{curl}\mathbf{B} \tag{1}$$

where we set the current source term equal to zero. Moreover, we have the Faraday's law of induction:

$$\frac{\partial \mathbf{B}}{\partial t} = -\,\mathrm{curl}\mathbf{E} \tag{2}$$

and the two following conditions on the divergence of the fields:

$$\mathrm{div}\mathbf{E} = 0 \tag{3}$$

$$\mathrm{div}\mathbf{B} = 0 \tag{4}$$

Putting all together, there are six unknowns and eight equations.

Relations (3) and (4) are often taken as an optional. For example, if one assumes that $\mathrm{div}\mathbf{E} = 0$ holds at initial time $t = 0$, it follows from (1) that the divergence of the electric field must be zero at all times. It is enough to compute the divergence of both terms in equality (1) to obtain:

$$\frac{\partial(\mathrm{div}\mathbf{E})}{\partial t} = c^2\ \mathrm{div}(\mathrm{curl}\mathbf{B}) = 0 \qquad \forall\ t \tag{5}$$

The above passage is mathematically correct. However, it is source of big mistakes. From expression (5) we presume that it is not necessary to check whether equation $\mathrm{div}\mathbf{E} = 0$ is maintained during time evolution.

[1] Allow me to give my respects to J.C. Maxwell, who never wrote the equations in the form we are used to and never imagined that his name would have been involved in such diatribes.

Nevertheless, it would be wise not to be much confident on this fact. We shall demonstrate through a simple example that div$\mathbf{E}$ can instead spontaneously assume values different from zero, mining the foundations of the Ampère's law in vacuum.

We assume that our functions are regular, so that they can be differentiated as many times as needed. It is easy to recover the following equation regarding the time variation of the electromagnetic energy:

$$\frac{1}{2}\frac{\partial}{\partial t}(|\mathbf{E}|^2 + c^2|\mathbf{B}|^2) \ = \ -c^2 \ \mathrm{div}(\mathbf{E}\times\mathbf{B}) \tag{6}$$

Here $\mathbf{E}\times\mathbf{B}$ is the Poynting vector. To get (6) one scalarly multiplies (1) by $\mathbf{E}$, (2) by $\mathbf{B}$, and uses notions of standard calculus.

The electric field satisfies the wave equation:

$$\frac{\partial^2\mathbf{E}}{\partial t^2} \ = \ c^2\Delta\mathbf{E} \tag{7}$$

which is obtained by observing that:

$$\Delta\mathbf{E} \ = \ -\,\mathrm{curl}(\mathrm{curl}\mathbf{E}) \ + \ \nabla(\mathrm{div}\mathbf{E}) \tag{8}$$

Therefore, relation (3) is necessary in order to get (7). The wave equation also holds for the magnetic field $\mathbf{B}$.

In addition, there are initial conditions and boundary constraints. The discussion of these is a crucial issue. There are several ways to impose boundary conditions. Commonly, a list of possible choices is presented (see, e.g., [12], section I.5, or [10], section 11.6.1). One can then pick up the ones that better fit the phenomenon to be studied.

As in the case of the wave equation (7), there is the tendency to transform the first-order system (1), (2) into a second-order one. Playing with second-order derivatives in the space variables is much easier and the *well-posedness* of the problem generally follows from properties of the Laplace operator. In this circumstance, boundary conditions are naturally derived from a solid theory. It is to be noticed however that the choice of the constraints for an elliptic operator is not equivalent to that of a system of hyperbolic equations, where a preliminary study of the *characteristic lines* should be done in order to detect which part of the boundary is actually involved. This analysis looks quite difficult in the context of Maxwell's equations, where the notions of characteristic curves and wave-fronts are, in most cases, not very clear.

For practical applications, the nature of the boundary constraints comes from physical considerations, sometimes without worrying about the mathematics. It is not rare to see cases where boundary conditions are under-determined, and others in which boundary conditions are over-determined. Nevertheless, this does not seem to cause any sort of ethical problem. As far as the results are in agreement with reality there is no reason to suspect the possibility of spurious solutions or that the entire formulation is inconsistent.

By examining a specific case, let us review what possibilities are offered regarding equation (7). For a given smooth bi-dimensional domain $\Omega$, we consider a capacitor where the two plates, shaped as $\Omega$, are parallel and placed at a distance $d$. The vertical direction is the $z$-axis and the plates are situated at the positions $z = 0$ and $z = d$. We assume to work with an *ideal capacitor*. This means that the electric field stays perpendicular to each plate surface and that the charge can be uniformly modified on the plates. As initial condition we impose (capacitor completely uncharged):

$$\mathbf{E} = 0 \qquad \mathbf{B} = 0 \qquad \text{at time } \ t = 0 \tag{9}$$

Since $\mathbf{B} = 0$, one has curl$\mathbf{B} = 0$ for $t = 0$. Thanks to (1), one must have:

$$\mathbf{E} = 0 \qquad \frac{\partial \mathbf{E}}{\partial t} = 0 \qquad \text{at time } t = 0 \tag{10}$$

The behavior of the electric field on the two plates will be specified later. We assume that laterally the capacitor is totally insulated, that amounts to say that the electric field is orthogonal to the normal $\mathbf{n}$ to the boundary $\partial\Omega \times [0, d]$. By setting $\mathbf{E} = (E_x, E_y, E_z)$ and $\mathbf{n} = (n_x, n_y, n_z)$, we must have $n_z = 0$, $n_x^2 + n_y^2 = 1$ and:

$$\mathbf{E} \cdot \mathbf{n} = n_x E_x + n_y E_y = 0 \qquad \text{on } \partial\Omega \times [0, d] \qquad \forall\, t \tag{11}$$

We would like to discuss the uniqueness of the solution of the vector wave equation. To this end, relation (11), being just a scalar equality, is not sufficient. In fact, we are not assigning Dirichlet boundary conditions on both components $E_x$ and $E_y$, but only a constraint on a linear combination of them. Something more is required.

A good companion for (11) is the boundary relation $\mathbf{B} \times \mathbf{n} = 0$, $\forall\, t$. This can be differentiated in time, obtaining $(\partial \mathbf{B}/\partial t) \times \mathbf{n} = 0$, $\forall\, t$. Recalling (2), we can translate the last relation in terms of the curl of $\mathbf{E}$.

Therefore, a suitable set of boundary constraints for the lateral surface of the capacitor is:

$$\mathbf{E}\cdot\mathbf{n} \ = \ 0 \qquad \text{curl}\mathbf{E}\times\mathbf{n} \ = \ 0 \qquad \text{on } \partial\Omega\times[0,d] \qquad \forall\, t \tag{12}$$

Traducing in terms of components, one has:

$$E_x n_x + E_y n_y = 0 \qquad \frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x} \qquad \frac{\partial E_z}{\partial x} n_x + \frac{\partial E_z}{\partial y} n_y = 0 \tag{13}$$

i.e., the right number of constraints. Since $\mathbf{B}\times\mathbf{n} = 0$ implies that the magnetic field is orthogonal to the boundary, it turns out that the Poynting vector $\mathbf{E}\times\mathbf{B}$ is tangential to the surface, that is in agreement with the fact that the energy cannot escape from the walls (see later on).

We are going to show that the wave equation for the electric field has unique solution (which is not however a proof for existence). If there were two distinct solutions, their difference would satisfy the wave equation with homogeneous data. To recap, we impose the initial conditions in (10), Dirichlet homogeneous conditions on the upper and the lower plates, and conditions in (12) on the lateral surface. It is possible to show (see below) that, with these constrictions, the only admissible case is $\mathbf{E}$ identically zero everywhere at every time. Therefore, the difference of two solutions has to vanish identically, and this is against the hypothesis that they are distinct.

The uniqueness of the vector wave equation follows from the conservation of a suitable energy. By extending the proof given in [5], p. 83, to the vector case, one scalarly multiplies both members of (7) by $\partial\mathbf{E}/\partial t$ and integrates on the whole domain:

$$\int_{\Omega\times[0,d]} \frac{\partial\mathbf{E}}{\partial t}\cdot\frac{\partial^2\mathbf{E}}{\partial t^2} = c^2\int_{\Omega\times[0,d]}\left[-\frac{\partial\mathbf{E}}{\partial t}\cdot\text{curlcurl}\mathbf{E} + \frac{\partial\mathbf{E}}{\partial t}\cdot\nabla\text{div}\mathbf{E}\right] \tag{14}$$

where we recalled (8). Note that $\text{curl}\mathbf{E} = 0$ on the boundaries $\Omega\times\{0\}$ and $\Omega\times\{d\}$ (lower and upper plates). By assuming that $\mathbf{n}$ is the outer normal, Green's formulas yield:

$$\frac{1}{2}\frac{d}{dt}\left[\int_{\Omega\times[0,d]}\left(\frac{\partial\mathbf{E}}{\partial t}\right)^2 \ + \ c^2\int_{\Omega\times[0,d]}(\text{curl}\mathbf{E})^2 \ + \ c^2\int_{\Omega\times[0,d]}(\text{div}\mathbf{E})^2\right]$$

$$= \ c^2\int_{\Omega\times\{0\}}\frac{\partial\mathbf{E}}{\partial t}\cdot\mathbf{n}\ \text{div}\mathbf{E} \ + \ c^2\int_{\Omega\times\{d\}}\frac{\partial\mathbf{E}}{\partial t}\cdot\mathbf{n}\ \text{div}\mathbf{E}$$

$$+\ c^2 \int_{\partial\Omega\times[0,d]} \frac{\partial \mathbf{E}}{\partial t} \cdot (\mathrm{curl}\mathbf{E} \times \mathbf{n})\ +\ c^2 \int_{\partial\Omega\times[0,d]} \frac{\partial}{\partial t}\,(\mathbf{E}\cdot\mathbf{n})\,\mathrm{div}\mathbf{E} \quad (15)$$

All the terms on the right-hand side are zero by virtue of the homogeneous boundary conditions (note in particular that $\partial\mathbf{E}/\partial t = 0$ on the lower and upper sides). On the left-hand side we have the time derivative of a non negative quantity. Due to the initial conditions, such a quantity is zero at the beginning. Hence, it will remain zero forever. One easily finds out that the only compatible solution is $\mathbf{E} = 0$ identically. For this last check, boundary conditions must be used one more time. Note that the role of the one-dimensional boundaries $\partial\Omega \times \{0\}$ and $\partial\Omega \times \{d\}$ is negligible (in presence of regular solutions, at least).

The above reasoning is quite standard, especially in the framework of finite element approximations, where the theory is constructed on a weak formulation having the associated energy similar to the one considered here.

The problem of setting up the correct boundary conditions for the Maxwell's first-order system remains open and needs further discussion. We can guess that $\mathbf{E}\cdot\mathbf{n} = 0$ and $\mathbf{B}\times\mathbf{n} = 0$ are good candidates, because they lead to reasonable assumptions in the framework of the second-order wave equation. Nevertheless, we will weaken the second condition a little bit. Concerning with the special case we are discussing, the electromagnetic energy "flows" along characteristic curves contained in the domain $\Omega \times [0, d]$. We do not want *inflow boundaries* regarding the Poynting field $\mathbf{P} = \mathbf{E}\times\mathbf{B}$, thus we can enforce $\mathbf{P}$ to be orthogonal to the boundary. We can then replace (12) by:

$$\mathbf{E}\cdot\mathbf{n}\ =\ 0 \qquad (\mathbf{E}\times\mathbf{B})\cdot\mathbf{n}\ =\ 0 \qquad \text{on } \partial\Omega\times[0,d] \qquad \forall\ t \qquad (16)$$

Note that it is not necessary to have $\mathbf{B}\times\mathbf{n} = 0$ in order to enforce the milder condition $\mathbf{P}\cdot\mathbf{n} = 0$. We have plenty of orthogonality relations: $\mathbf{E}\perp\mathbf{n}$, $\mathbf{E}\perp\mathbf{P}$, $\mathbf{B}\perp\mathbf{P}$, $\mathbf{P}\perp\mathbf{n}$, but $\mathbf{B}$ remains undertermined.

Let us suppose for example that $\mathbf{E}$ is forced to be zero on the lower and upper plates (homogeneous Dirichlet conditions). One gets, after integrating (6) in the entire domain:

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega\times[0,d]} (|\mathbf{E}|^2 + c^2|\mathbf{B}|^2)\ =\ -\,c^2 \int_{\Omega\times[0,d]} \mathrm{div}(\mathbf{E}\times\mathbf{B})$$

$$= -c^2 \int_{\partial\Omega\times[0,d]} (\mathbf{E}\times\mathbf{B})\cdot\mathbf{n}\ =\ 0 \qquad (17)$$

where we used the divergence theorem with $\mathbf{n}$ directed outward. The last term is zero because of (16). According to (17), the electromagnetic energy, initially zero, will stay zero during time evolution. This says that the Maxwell's system admits unique solution.

In alternative to the insulation of the lateral wall, one could take into account Neumann conditions. Instead of (12), a viable option is then:

$$\frac{\partial E_x}{\partial \mathbf{n}} = 0 \qquad \frac{\partial E_y}{\partial \mathbf{n}} = 0 \qquad \frac{\partial E_z}{\partial \mathbf{n}} = 0 \qquad \text{on } \partial\Omega \times [0,d] \quad \forall\, t \tag{18}$$

By assuming div$\mathbf{E} = 0$, we multiply equation (7) by $\partial \mathbf{E}/\partial t$ and integrate, obtaining through the usual passages:

$$\frac{1}{2}\frac{d}{dt}\left[\int_{\Omega\times[0,d]} \left(\frac{\partial \mathbf{E}}{\partial t}\right)^2 + c^2 \int_{\Omega\times[0,d]} (|\nabla E_x|^2 + |\nabla E_y|^2 + |\nabla E_z|^2)\right]$$

$$= c^2 \int_{\partial\Omega\times[0,d]} \left(\frac{\partial E_x}{\partial t}\frac{\partial E_x}{\partial \mathbf{n}} + \frac{\partial E_y}{\partial t}\frac{\partial E_y}{\partial \mathbf{n}} + \frac{\partial E_z}{\partial t}\frac{\partial E_z}{\partial \mathbf{n}}\right) = 0 \tag{19}$$

where we imposed (18) and homogeneous Dirichlet conditions on the plates. Again, we deduce an uniqueness result.

## 3 Charging the capacitor

We apply to a concrete case the situation examined in the previous section. The two plates, initially short-circuited, are successively subjected to a difference of potential. As far as initial and boundary conditions are concerned, we assume (10) and (12). Moreover, for any $t > 0$:

$$\mathbf{E} = (0, 0, \alpha(t)) \qquad \text{on both plates} \tag{20}$$

where $\alpha$ is a given function with $\alpha(0) = 0$ and $\alpha'(0) = 0$. Thus, the charge starts flowing smoothly on the plates, based on a difference of potential equal to $V = \alpha d$. For symmetry reasons, we can set the ground at level $z = d/2$. With respect to the lines of force of the electric field, for $\alpha$ positive the lower surface is an *inflow boundary*, while the upper surface is an *outflow boundary*. The total flux through the boundaries is zero, so that the integral of div$\mathbf{E}$ on the whole domain is also zero. We omit to specify the boundary conditions for $\mathbf{B}$, since there is no need for them, as it will emerge from the analysis.

In the context of smooth functions, the solutions to the Maxwell's system belong to a subset of those satisfying the wave equation (7). We can construct an explicit solution by setting $\mathbf{E} = (0, 0, E_z)$, where $E_z$ does not depend on $x$ and $y$. In this fashion, we have:

$$\frac{\partial^2 E_z}{\partial t^2} = c^2 \frac{\partial^2 E_z}{\partial z^2} = c^2 \Delta E_z \tag{21}$$

with (see the last relation in (13)):

$$E_z = \alpha(t) \quad \text{on the plates} \tag{22}$$

$$\frac{\partial E_z}{\partial \mathbf{n}} = 0 \qquad \text{on the lateral surface} \tag{23}$$

$$E_z = 0 \qquad \frac{\partial E_z}{\partial t} = 0 \qquad \text{at time } t = 0 \tag{24}$$

The explicit expression of (21) is not elementary but it is recoverable through Fourier series expansion. A theory in the framework of Sobolev spaces can be found for instance in [5], section 7.2, or in [1], p. 345. Such a solutions turns out to be smooth in the domain including the boundary. Due to the uniqueness theorem cited in the previous section, there are no other possible choices for $\mathbf{E}$. Hence, we also found the solution to the Maxwell's problem.

At this point we notice that the function $E_z$ is not certainly constant with respect to $z$. In fact, the setting $E_z = \alpha(t)$, $\forall z \in [0, d]$, is in general incompatible with equation (21), because $\partial^2 E_z / \partial t^2 = \alpha''(t) \neq 0 = \partial^2 E_z / \partial z^2$. Therefore, the partial derivative $\partial E_z / \partial z$ is different from zero almost everywhere. In conclusion, we have:

$$\text{div}\mathbf{E} = \frac{\partial E_z}{\partial z} \neq 0 \tag{25}$$

We just discovered that the solution to our vector wave equation cannot be divergenceless. As a consequence, independently on how we define the magnetic field, there are no chances to solve the entire set of Maxwell's equations. This is true for any set $\Omega$, for any $d$ and for any function $\alpha$ (with zero derivative at the origin); too many degrees of freedom to argue that this is just incidental.

The problem we proposed has very smooth solutions, hence the idea that things may improve by converting it into variational form is hopeless. In truth, using a general *test function* in (14), a variational formulation is soon obtained, which can be automatically extended to functions belonging to suitable Sobolev spaces.

By the above procedure, the magnetic field turns out to be totally unspecified. As already noticed, the electric field remains parallel to the $z$-axis and does not depend on $x$ and $y$. The only reasonable choice compatible with such a behavior of the electric field is $\mathbf{B} = 0$, everywhere at all times. This is again in contradiction with (1) since we know that $\partial \mathbf{E}/\partial t \neq 0$.

The case $\alpha'(0) \neq 0$ is more difficult to handle, but this preliminary analysis induces us to guess that conclusions cannot be too much different. Time periodic conditions applied at the plates are not compatible with (10). However, even starting from $\alpha'(0) \neq 0$, after a transient, the function $\alpha$ may assume a given oscillating behavior, resulting asymptotically in a periodic evolution of the internal fields (see figure 1).

At this point, one may argue that the wave equation approach requires too strong boundary conditions. Perhaps, by weakening the insulation condition at the lateral sides one can enlarge the solution space and obtain situations that are compatible with $\mathrm{div}\mathbf{E} = 0$. So, we just enforce $\mathbf{E} \cdot \mathbf{n} = 0$, forgetting the other constraints and losing the uniqueness result for the wave equation. Unfortunately, this weaker hypothesis is again inconsistent with Maxwell's equations. To prove such a negative claim we follow very classical arguments.

We work in the neighborhood of the lower plate $S = \Omega \times \{0\}$. Due to (20), we have:

$$\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x} = \frac{\partial E_y}{\partial y} = \frac{\partial E_z}{\partial x} = \frac{\partial E_z}{\partial y} = 0 \qquad \text{in } S \quad \forall\, t \quad (26)$$

$$\frac{\partial^2 E_z}{\partial x^2} = \frac{\partial^2 E_z}{\partial y^2} = 0 \qquad \text{in } S \quad \forall\, t \quad (27)$$

Thanks to (27), the third component of the wave equation satisfies (21) for all $t$, when restricted to $S$. Moreover, the $\mathrm{div}\mathbf{E} = 0$ condition implies:

$$\frac{\partial E_z}{\partial z} = -\frac{\partial E_x}{\partial x} - \frac{\partial E_y}{\partial y} = 0 \qquad \text{in } S \quad \forall\, t \quad (28)$$

In conclusion, one obtains:

$$E_z = \alpha \qquad \frac{\partial E_z}{\partial z} = 0 \qquad \frac{\partial^2 E_z}{\partial z^2} = \frac{\alpha''}{c^2} \qquad \text{in } S \quad \forall\, t \quad (29)$$

Therefore, near the surface, one gets the Taylor expansion:

$$E_z = \alpha + \frac{\alpha'' z^2}{2c^2} + o(z^2) \qquad \forall\, t \quad (30)$$

For instance, let us assume that $\alpha$ has a quadratic growth (though similar considerations will hold for a more general choice): $\alpha(t) = at^2$, $a > 0$. The expression of $E_z$ for $z$ sufficiently small becomes $a(t^2 + z^2/c^2)$. We now take $\delta \in ]0, d]$ and consider the domain $\Omega \times [0, \delta]$. The divergence of $\mathbf{E}$ is zero inside there. The lateral boundary is insulated, i.e. $\mathbf{E} \cdot \mathbf{n} = 0$, hence the incoming flux in $\Omega \times \{0\}$ must equate the outgoing flux in $\Omega \times \{\delta\}$. If $\delta$ is suitably small this is impossible since $\mathbf{E} \cdot \mathbf{n} = at^2$ uniformly in $\Omega$ for $z = 0$, which is less than $\mathbf{E} \cdot \mathbf{n} = a(t^2 + \delta^2/c^2)$ for $z = \delta$. Again we arrive at a contradiction.

It is worthwhile to notice that, in the proof given above, the zero divergence condition has been recalled both locally (via (28)) and globally (Gauss's law applied to the box $\Omega \times [0, \delta]$).

This paradox tells us that there are troubles at the constitutive level. Relation (5) states that condition $\text{div}\mathbf{E} = 0$ must be preserved at all times, while we discovered that this cannot be true. Where is the mistake? The wrong assumption is in the writing of the Ampère's law in vacuum, where the generic vector $\partial \mathbf{E}/\partial t$ is supposed to be the *curl* of another vector. This is not necessarily true. By dropping this hypothesis one discovers that, even in absence of currents due to independent charges, there might be a sort of flow-field with the property $\text{div}\mathbf{E} \neq 0$.

One may try a correction by rewriting equation (1) as:

$$\frac{\partial \mathbf{E}}{\partial t} \;=\; c^2 \text{curl}\mathbf{B} \;+\; (0, 0, \alpha') \tag{31}$$

where the added forcing term substitutes the boundary conditions, that now become of homogeneous type. This consideration does not help, since the new term is also the *curl* of a vector (take for instance $(-\frac{1}{2}y\alpha', \frac{1}{2}x\alpha', 0)$).

To justify that relation $\text{div}\mathbf{E} \neq 0$ is physically admissible one can rely on the finiteness of the speed of light $c$, which rules the transfer velocity of the information between the two plates. As one modifies the difference of potential, the electric field inside the capacitor has to be redistributed (recall that, in the path we followed, the magnetic field remains equal to zero). This happens at speed $c$, in contrast with Coulomb's law (represented by the Gauss's law) that requires the information to travel at infinite speed. The only way for a field of the form $(0, 0, E_z)$ to establish a communication between the plates is to create compression and rarefaction waves by varying its divergence. Note however that the integral of $\text{div}\mathbf{E}$ on the entire domain remains always zero, so that sources and wells are in perfect equilibrium. These observations, based on elementary arguments, tell us that a rethinking of the equations ruling electrodynamics is unavoidable.

At this point, some readers may argue that the insulating condition $\mathbf{E} \cdot \mathbf{n} = 0$ is too "artificial" and made on purpose to guide the internal field along vertical lines of force. In order to answer this possible question, first of all, we remark that, in the framework of 3D geometries, imposing $\mathbf{E} \cdot \mathbf{n} = 0$ does not necessarily mean that $\mathbf{E}$ must be vertical (this is a consequence of the resolution of the wave equation). Secondly, we propose to remove the boundary conditions in (12) and replace them by (18). A uniqueness theorem for the wave equation is still guaranteed (see the end of section 2). In the new setting, $\mathbf{E}$ could in principle assume configurations that are more similar to those of a charged capacitor in a stationary regime, i.e., with curved lines of force that are more pronounced at the rim of the plates. By the way, such an improvement is only illusory. Indeed, let us consider again $\mathbf{E} = (0, 0, E_z)$ (with $E_z$ not depending on $x$ and $y$) satisfying the one-dimensional equation (21). Such a solution also satisfies (18), therefore it is the unique solution to the vector wave equation (7) equipped with the new set of boundary constraints. We know that $\mathbf{E}$ is not of Maxwellian type since its divergence is not zero. We also know that $\mathbf{B}$ remains equal to zero when time passes. Thus, starting from $\mathbf{E} = 0$ and $\partial \mathbf{E}/\partial t = 0$ at time $t = 0$, there is no development of horizontal component of $\mathbf{E}$ during the charging process. Such a conclusion seems counterintuitive. Every physicist would bet that there is a mistake in the reasoning. On the other hand, this is just a correct mathematical results which is a consequence of having approached the problem from a nonstandard perspective. Here the study of a dynamical situation provides results apparently not in agreement with the stationary case. As claimed in the introduction, there are many different paths one may follow and they depend on what information one would like to extrapolate. Although some partial results could be reasonably in agreement with the physical phenomenon under study, others may not. In any case the final answer is often contradictory.

One may finally wonder about the possibility to replace (18) by weaker hypotheses, as done for the case of the insulated wall. The aim is to allow the creation of a nontrivial field $\mathbf{B}$ with the consequent generation of curved lines of force for the electric field. At the same time, the hope is to restore the zero divergence condition. The risk is to invalid the uniqueness theorem for Maxwellian solutions. Certainly, one may think that there are infinite ways the magnetic field can grew up in the capacitor's charging procedure, as there are infinite ways the electric field may be deformed. Distinguishing among these solutions may require unusual physical considerations that are not comprised in the standard theory of electrodynamics. Nevertheless, such a risk does not occur, since, even with a total lack of information on the lateral boundary, we are able to end up with a nonexistence result.

The proof is simple. Let us recall relation (30) and for simplicity take $\alpha(t) = at^2$. Near the surface $S$, the electric field is of the form: $\Big(E_x, E_y, a(t^2 + z^2/c^2)\Big)$, neglecting second-order infinitesimals. This is true for any $t$. At level $z = \delta$ (for $\delta > 0$ fixed and sufficiently small), by computing the divergence one gets:

$$0 \ = \ \mathrm{div}\mathbf{E} \ \approx \ \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{2a\delta}{c^2} \qquad \forall\ t \tag{32}$$

where the approximation holds up to first-order infinitesimals. The crucial step is now to observe that, since the capacitor is initially uncharged, for $t$ small enough the sum of $\partial E_x/\partial x$ and $\partial E_y/\partial y$ cannot be equal to the fixed number $-2a\delta/c^2$. Thus, we arrived at another absurdity.

This check shows that there is conflict between boundary constraints, the wave equation and the divergence-free condition, pointing out once again the inconsistency of the model. Note that hypotheses here are really very mild. In particular, it is sufficient to have the boundary condition $\mathbf{E} = (0, 0, \alpha(t))$ only in a 2D region (of any size) included in $S$. We do not deny that there are interesting solutions of the full set of equations, but they look dotted islands in the ocean of electromagnetic phenomena (see the introduction).

One can try to "hide" the above evidences and argue as follows (the "$n$-steps-instead-of-$n$+1" technique mentioned in the introduction). Faraday's and Ampère's laws can be advanced in time. They have good physical justification, so the solution will naturally find its path, reproducing within a certain accuracy the phenomenon. Thanks to (5), there is no necessity to check what happens to the divergence of the electric field. One hopes that nobody will discover that, in some small remote region of the plates, $\mathbf{E}$ will grow up in such a way that $\partial\mathbf{E}/\partial t$ is not of curl-type, disregarding Ampère's law. A nonzero divergence starts developing and at this point the link with Maxwell's equations is definitely lost. If somebody asks to explain the reason of a non vanishing divergence, an evasive answer is that such a divergence is negligible in practical cases. That is why is hard to convince the public of the unreliability of the Maxwell's model.

## 4 Other paradoxical results

We propose the following experiment. The difference of potential between the plates is increased quadratically for a given interval of time, after which is kept constant. The information propagates from the boundary to the interior and the third component $E_z$ follows the wave equation (21). As we stop the increasing of potential, the field continues to develop and assumes an oscillating behavior.

The plots of figure 1, obtained from a simple numerical test, show that after a transient where the distribution monotonically grows (solid lines), a periodic regime follows (dashed lines). We do not specify all the parameters of the test, since the purpose here is to comment the qualitative behavior. In particular, the intensity of the field inside the capacitor assumes values that are greater than the ones attained at the boundary.

By denoting with $A$ the area of $\Omega$, the capacitance is given by $C = \epsilon_0 A/d$, while the stored energy is given by:

$$\frac{1}{2}\, CV^2 \;=\; \frac{\epsilon_0}{2}\, Ad\; \alpha^2 = \; \frac{\epsilon_0}{2}\, \mathcal{V}\; \alpha^2 \tag{33}$$

where $V = \alpha d$ is the difference of potential and $\mathcal{V}$ is the measure of the volume of the capacitor. In (33) it is assumed that, at stationary regime, the electric field inside the capacitor is uniformly equal to $V/d$. We also know that the quantity $\frac{1}{2}\epsilon_0(|\mathbf{E}|^2 + c^2|\mathbf{B}|^2)$ denotes the energy density of the electromagnetic field. In the case we are examining, this energy integrated over the volume of the capacitor is: $\frac{1}{2}\epsilon_0 \int_{\mathcal{V}} E_z^2$. As we notice before, this last quantity can be bigger than the one predicted by (33). Of course, perfect capacitors as the one we are studying here do not exist in reality and this strangeness is not noticed in practice. Nevertheless, such a suspicious theoretical result provides us with another indication that the ruling equation have flaws or, at least, that the definition of the energy stored by a capacitor is lacunary.

In the *two-capacitors paradox* (see, e.g., [11], vol. 2, p. 684), the charge present on the plates of a capacitor of capacitance $C$ is redistributed by adding another capacitor in parallel of capacitance $C$ (the new total capacity is then $2C$). This implies that the initial voltage $V$ is halved. The initial stored energy is $\frac{1}{2}CV^2$ while the final one is $\frac{1}{2}(2C)(V/2)^2 = \frac{1}{4}CV^2$. Thus, half of the energy mysteriously disappears.

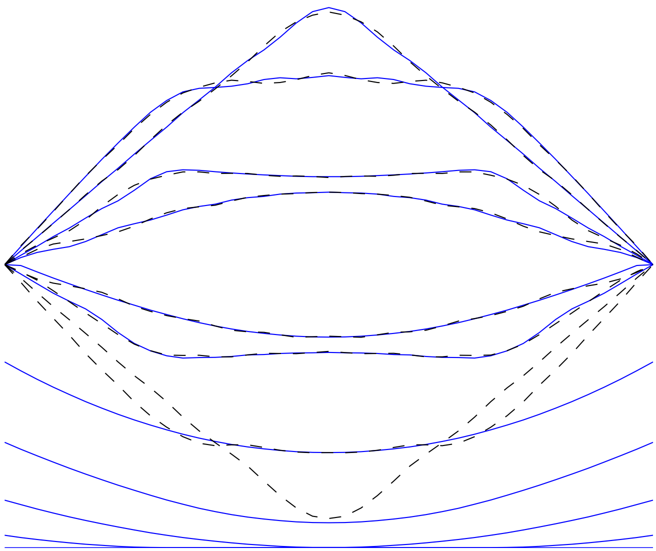

Figure 1: Behavior with respect to time of the function $E_z$. Boundary conditions are increased quadratically and suddenly stopped. As the wave equation predicts, non dissipating oscillations are developed.

Explanations of this fact have been proposed copiously. The principal track for the investigation is the analysis of losses in the charge transfer procedure. This includes: heating of the connecting wire, self inductance of the circuit (see [13]), electromagnetic emission, etc. What emerges from this paper is rather the urgent need of restating basic formulas. The stored energy in (33) is only part of the total energy that also contains the kinetic of the fields inside the capacitors. Such a dynamics in inevitably produced by the redistribution of the charges. By assuming energy conservation, internal oscillations do not dump and must contribute to the total energy. A correction of (33) should include this option.

A similarity can be made with a perfect elastic ball, that initially is at rest at distance $h$ from the ground (pure gravitational potential energy). The ball successively drops and bounces back. A barrier is posed at level $h/2$ before the ball can reach again level $h$. Oscillations develop between the ground and level $h/2$, where the potential energy is half of the initial one. However, there is no energy loss, since one has to take into account the nonzero kinetic energy when the ball hits the upper obstacle.

Formula (33) is only valid in the pure stationary case, so its application in the dynamical description of the charge transfer between two capacitors is incorrect. In practical applications, the oscillations decay in a finite time due to some internal energy dissipation. This loss of energy must be added to the one arising from the external circuitry. After an appropriate interval of time, the total amount of lost energy is, as correctly predicted, equal to half of the initial energy. We have the right to suppose that the dissipation due to the outer circuit is negligible, but, on the other hand, we must handle the dissipation of the internal oscillations in some appropriate way.

## 5 Comments

I am sure that following the above discussion many experts will start providing their explanations. An ideal capacitor does not exist. Charges are fluctuating on the plates making it impossible a uniform distribution and resulting in the creation of “magnetic currents”. The lateral boundaries of the capacitors cannot be perfectly insulated. The wave equation theoretically implies that the divergence of the electric field can be different from zero, but in a quasi-stationary regime the amount is negligible. For fast-varying fields the approach to the problem should be different. At high frequencies, the coupling of electric and magnetic fields produces electromagnetic emissions. In other words, nature is very complicated and a simple linear model, such as the Maxwellian, cannot take into account all the possible manifestations, unless one accepts to introduce some approximation.

By the way, there are no excuses! What has been studied here is a mathematical setting trying to explain a very simple phenomenon. Moreover, it is not a simple phenomenon with very rare specific parameters, since the shape of the capacitor and the applied potential are arbitrary. The results are wrong because the underlying physics is wrong: too many constraints compared to the degrees of freedom. The evolution equations rely on the finiteness of the speed of light, while the Gauss's law (ruling the stationary cases) finds its roots on the "action at the distance". Depending on circumstances, one has to choose what equations are "more meaningful".

What is the borderline between the physics of slow-varying or fast-varying potentials? Nobody can predict it; because in reality such a difference does not exist. Why do we think there should be a difference? Because the Maxwell's model is controversial. It has to be specialized based on the target, and misses the analysis of the intermediate situations. In fact, very little is known for instance about the so called *near-field* of an antenna, where, under suitable resonance conditions, oscillating fields transform into radiation waves (see my paper [8] to this regard). It is not admissible that the structure of a model changes depending on the problem to be solved.

I have a solution to propose (see the references to my papers). In my alternative model equations the divergence of the electric field can be different from zero even in vacuum (let me skip any explanation). What happens inside a capacitor? Everything one may suspect: magnetic fields are generated, information propagates at the speed of light, some electromagnetic pressure (included in the model equations) acts on the plates, and several other known (or less known) effects. It depends on the assumptions on the device and the external solicitation. The general solution can be a nightmare. The difficulty of the math reflects however the complexity of the phenomena observed in the real world, without any barrier among the different regimes. Starting from a unique formulation, negligible quantities can be simplified later, if the nature of the phenomenon allows for it. In the modified model, the wave equation does not hold and it is substituted by a suitable nonlinear hyperbolic system of equations where div$\mathbf{E}$ can be actually different from zero. A meaningful solution (from the mathematical viewpoint) can be finally recovered. This is the way a correct model should work. A more precise explanation will be given in a future paper.

I still do not know if my proposal is the optimal tool to study electromagnetism. It is certain that Maxwell's model is mathematically incorrect and, consequently, requires revision. It fails on simple questions and not just in the simulation of exotic problems.

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