Revisiting the Schwarzschild and the Hilbert-Droste Solutions of Einstein Equation and the Maximal Extension of the Latter: Part I

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ABSTRACT. In this first part of a series of pedagogical notes, the differences between the Schwarzschild and the Hilbert-Droste solutions of Einstein equation are scrutinized through a rigorous mathematical approach, based on the idea of warped product of manifolds. It will be shown that those solutions are indeed *different* because the topologies of the manifolds corresponding to them are different. After establishing this fact beyond any doubt, the maximal extension of the Hilbert-Droste solution (the Kruskal-Szekeres spacetime) is derived with details and its topology compared with the ones of the Schwazschild and the Hilbert-Droste solutions.

In the second part of our notes, we will study the problem of the imbedding of the Hilbert-Droste solution in a vector manifold, hopefully clarifying the work of Kasner and Fronsdal on the subject. In an Appendix, we present a rigorous discussion of the Einstein-Rosen Bridge.

A comprehensive bibliography of the historical papers involved in our work is given at the end.

Contents

1 Introduction

The journal *General Relativity and Gravitation* reprinted in 2003 the famous paper in which Schwarzschild consecrated himself as the first person to find an exact solution of the Einstein field equation [1]. Following the same volume of that journal, S. Antoci and D.-E. Liebscher published an editorial note claiming that the solution presented by Schwarzschild in 1916 (which describes the gravitational field generated by a point of mass) is not equivalent to the one currently taught in textbooks on General Relativity. The latter being a solution which was, however, found by J. Droste and D. Hilbert just a year after Schwarzschild's publication. This event culminated in a series of papers concerned with the equivalence or the nature of these two solutions.

Three years after the editorial note of Antoci and Liebscher, a rectification was published in the above journal (cf. ref. [2]) claiming that the solutions of Schwarzschild and of Hilbert-Droste are indeed equivalent, based on the existence of a coordinate transformation for which the metric found originally by Schwarzschild can be written in the same coordinate form as the one found by Droste and Hilbert. This opinion is shared by the authors of ref. [3], published in 2007, and of ref. [4], published in 2013.

However, the latter authors ignored that a spacetime is not only defined by a metric, but also by the topology of the corresponding manifold. And in fact, as we shall explain in details later, while the Schwarzschild manifold is homeomorphic to $\mathbb{R}\times]0,\infty[\times S^2$, leaving no room for a black hole and dispensing a procedure of maximal extension, the topology of the Hilbert-Droste manifold is homeomorphic to $\mathbb{R}\times(]0,\infty[-\{\mu\})\times S^2$ (for some real $\mu > 0$), being consequently a *different* solution of the Einstein equation. We remark that the latter solution having a disconnected manifold require a maximal extension in order to become a satisfactory spacetime (cf. Definition 9).

This was recognized by N. Stavroulakis in his writings entitled *"Mathématiques et trous noirs"* (cf. ref. [5]), which appeared in the *Gazette des mathématiciens*, and "*Vérité scientifique et trous noirs" (cf. refs.* [6]–[9]), published just four years before the Antoci & Liebscher editorial note. (We shall comment briefly on Stavroulakis's articles in the second part of our notes). Another author, who seems to be one of the first to advocate that the solutions of Schwarzschild and of Hilbert-Droste are really different, was L. Abrams, publishing about the subject already in 1979 (cf. ref. [20]).

It is important to remark that because the Hilbert-Droste solution has a disconnected topology (which as we will show below, is not the case of the manifold in Schwarzschild's solution), the Relativity community was lead to the "Maximal Extension" research programme, which grown from a J. Synge's letter to the editor in a Nature's volume which dates from 1949, and culminated in the Kruskal-Szekeres spacetime and in the Fronsdal imbedding of the Hilbert-Droste manifold – a procedure which was based in a work of E. Kasner from 1921 (almost four decades before Fronsdal's paper was published). This, of course, inaugurated the physics of black holes.

Our paper revisit this issues from a mathematically rigorous standpoint and is organized as follows. In Section 2, we present the mathematical formalism which will be adopted in rest of our work. In particular, we discuss the warped product of manifolds, which is a powerful tool in constructing spacetimes in General Relativity, some issues concerning the extension of manifolds (which is complemented by the Appendix A) and the properties of null (or lightlike) geodesics which are useful in verifying that a given manifold is maximal. In Section 3, we set a framework in which both the Schwarzschild and the Hilbert-Droste solutions can be constructed, in such a way that a parallel between their derivations and the origin of their topological differences will be shown.

In the second part of our series of notes, motivated by the disconnectedness of the Hilbert-Droste solution, we begin the search for its maximal extension, covering details normally omitted by the present literature leading to the Kruskal-Szekeres spacetime. A *brief* summary of the relevant historical developments will be presented. Lastly, we will discuss the works of Kasner and Fronsdal that culminated in the embedding of the Hilbert-Droste spacetime in a 6-dimensional vectorial manifold, thus ending this chapter in the history of General Relativity.

2 Mathematical Formalism

In order to fix our notation and refresh the memory, we review in Subsections 2.1 and 2.2 some elementary facts concerning pseudo-Riemannian geometry, Minkowski vector spaces and spacetimes.

Then, the following two subsections are dedicated to a discussion of the *warped product*, a powerful tool that can be employed in the construction of some spacetimes in General Relativity. As we shall see, its use has at least two advantages: it can elegantly simplify calculations related to geometric quantities, as the Ricci curvature tensor, and even more important, when a spacetime is given in the form of a warped product, its manifold topology is stated without ambiguities since the beginning.

Finally, in the Subsection 2.5, we discuss some properties of null geodesics which shall be useful for the second part of our work in devising construction of the maximal extension of the Hilbert-Droste solution (the Kruskal-Szekeres spacetime), a subject which is normally treated very informally in the current literature.

In Appendix A, our discussion of the extension of manifolds is continued from a topological point of view. There, we discuss some topological issues which may arises when two topological spaces are glued together through a continuous identification of its topological subspaces. That Appendix is however unnecessary for our main developments, but will be used in the rigorous construction of the Einstein-Rosen bridge presented in Appendix of the second part.

2.1 Manifolds and Exponential Mapping

First, recall that

Definition 1 *A pseudo-Riemannian manifold is an ordered pair* (M, g)*, where* M *is a smooth manifold and* $g \in \sec T_2^0 M$ *is a metric tensor, i.e., a symmetric and non degenerate* 2*-covariant tensor field in* M *with the same index in all tangent spaces of* M*. We may say that* M *have a pseudo-Riemannian structure.*

Remember that the *index* of a symmetric bilinear form g is the greatest integer υ such that there is a subspace W with the properties: $\dim W = \upsilon$ and $g(x, x) < 0$ for all $x \in W$.

When there is no fear of confusion, we may refer to a pseudo-Riemannian manifold (M, g) just by M.

Definition 2 *Let M be a pseudo-Riemannian manifold and let γ be a curve from $I \subset \mathbb{R}$ into M. Let $\hat{D}_\gamma$ be the induced Levi-Civita connection (of g) on γ. So we will call γ a geodesic if $\hat{D}_\gamma \gamma'(t) = 0$ for all $t \in I$.*

In what follows, unless we use the adjective *segmented*, all geodesics are defined on a interval which contains $0 \in \mathbb{R}$.

Recall that a geodesic γ defined on $I \subset \mathbb{R}$ is called inextendible if and only if, for all geodesics σ defined on $J \subset \mathbb{R}$ such that $\sigma'(0) = \gamma'(0)$, we have that $J \subset I$. To each $x \in T_pM$, we will denote by γ_x the unique inextendible geodesic such that $\gamma_x^{'}(0) = x$.

The idea of approximate the neighborhood of a point in a manifold through the tangent space in that point can be made precise by using the exponential mapping:

Definition 3 *Let M be a pseudo-Riemannian manifold and let $p \in M$. Let D_p be the subset of T_pM such that, for all $x \in D_p$, the domain of γ_x contains $[0, 1] \subset \mathbb{R}$. The exponential mapping $\exp_p$ at p is the mapping from D_p into M such that $x \to \exp_p(x) = \gamma_x(1)$.*

Remark 1 *Let γ be a geodesic with induced Levi-Civita connection $\hat{D}_\gamma$. As, in coordinates, $\hat{D}_\gamma \gamma'(t) = 0$ corresponds to a system of ordinary differential equations of second order, the solution depends smoothly on the initial values. Then the exponential mapping is a well-defined smooth mapping.*

In this paragraph, to each $\theta \in T_p^*M$, we will denote by $\mathrm{d}\boldsymbol{\theta}$ the differential mapping of θ as being a function from T_pM into $\mathbb{R}$, and *not* the exterior derivative of θ as being a *covector field*. In the proof of the following Lemma, given $x \in M$, the natural homomorphism ϕ between $T_x(T_pM)$ and T_pM is the mapping such that, for all covector $\theta \in T_p^*M$, $\theta\left[\phi(v_x)\right] = \mathrm{d}\boldsymbol{\theta}(v_x)$, for all $x \in T_pM$.

Lemma 1 *Let M be a pseudo-Riemannian manifold. For each $p \in M$, there is a neighborhood $V \subset T_pM$ of $0 \in T_pM$ such that $\exp_p |V$ is a diffeomorphism.*

Proof. Let ϕ be the natural homomorphism between $T_x(T_pM)$ and T_pM. Let $v_0 \in T_0(T_pM)$, let $v = \phi(v_0)$ and let $\lambda(t) = vt$ be a mapping from $\mathbb{R}$ into T_pM. So, as $\lambda'(0) = v_0$,

$$\exp_{p*}(v_0) = \exp_{p*}\left[\lambda'(0)\right] = \left(\exp_{p*} \circ \lambda\right)'(0) = v$$

Hence $\exp_{p*}$ is the natural homomorphism ϕ. By Remark 1 and the inverse mapping theorem, the result follows. ■

Definition 4 *Let M be a pseudo-Riemannian manifold and let $p \in M$. A neighborhood U of p will be called normal if there is a neighborhood $V \subset T_pM$ of $0 \in T_pM$ such that $\exp_p |V$ is a diffeomorphism between V and U and, for all $x \in V$, $\{tx : t \in [0,1] \subset \mathbb{R}\} \subset V$.*

So the last Lemma ensures that we can always find a normal neighborhood for a given point.

Lemma 2 *Let M be a pseudo-Riemannian manifold, let $p \in M$ and let U be a normal neighborhood of p. So, for all $q \in U$, there is a unique geodesic γ_{pq} from $[0,1] \subset \mathbb{R}$ into U such that $\gamma_{pq}(0) = p$, $\gamma_{pq}(1) = q$ and $\gamma'_{pq}(0) = \exp_p^{-1}(q)$.*

Proof. Let $v = \exp_p^{-1}(q)$ and let $\lambda(t) = vt$ be a mapping from $\mathbb{R}$ into T_pM. Let $\sigma(t) = \exp_p \circ \lambda(t)$ be a mapping from $[0,1] \subset \mathbb{R}$ into U. By the hypothesis on V, σ is well-defined, and by the Definition 3, σ is a geodesic. But

$$\sigma'(0) = \left(\exp_{p*} \circ\lambda\right)'(0) = \exp_{p*}\left[\lambda'(0)\right] = v$$

by the proof of the last Lemma. Hence the existence assertion. The proof of the uniqueness will be left as an easy exercise. ■

Let γ be a curve from $[a,b] \subset \mathbb{R}$ into a pseudo-Riemannian manifold M. We will say that γ is a broken-geodesic if there is a partition $(J_i)_{i\in F\subset\mathbb{N}}$ of $[a,b]$ such that each restriction $\gamma|J_i$, for $i \in F$, is a segmented geodesic. In this case, we say that $\gamma(a)$ and $\gamma(b)$ are connected by a broken-geodesic.

Corollary 3 *A pseudo-Riemannian manifold M is connected if and only if, for all points $p, q \in M$, there exists a broken-geodesic γ defined on $[a,b] \subset \mathbb{R}$ such that $\gamma(a) = p$ and $\gamma(b) = q$.*

Proof. Let S be the subset of M of all points that can be connected by a broken-geodesic and let $p \in M$. Let U be a normal neighborhood of p. So, by Lemma *2*, if $p \in S$, $U \subset S$. But if $p \notin S$, then $U \cap S = \varnothing$, and M cannot be connected. Hence the result. ■

In what follows, we will call a neighborhood U in a pseudo-Riemannian manifold *convex* if U is a normal neighborhood for all $p \in U$. To see a proof that a convex neighborhood always exists around any given point, see Chapter 5 of [12].

2.2 Spacetimes

Spacetimes are the manifolds upon which the General Relativity Theory is established. To define them, we need to recall some facts about Lorentz vector spaces:

Definition 5 *A Lorentz vector space is an ordered par* (V, g)*, where* V *is a finite-dimensional linear space with dimension* $\dim V \geq 2$ *and* g *is a symmetric and non degenerate bilinear form on* V *with index* 1*.*

A sequence $(e_i)_{i \in F \subset \mathbb{N}}$ of vectors in a given Lorentz vector space (V, g) will be called orthonormal if $|g(e_i, e_j)| = \delta_{ij}$, where δ_{ij} is the Kronecker delta (i.e., $\delta_{ij} = 0$ when $i \neq j$ and $\delta_{ii} = 1$).

Lemma 4 *Let* (V, g) *be a Lorentz vector space. So there is an orthonormal basis for* V*.*

Proof. *(i)* As g is non degenerate, there is a $x \in V$ such that $g(x, x) \neq 0$. *(ii)* If $(e_i)_{i \in [1,k]}$ is a sequence of orthonormal vectors (for some $k < \dim V$), there is a vector e_{k+1} such that $(e_i)_{i \in [1,k+1]}$ is also orthonormal, by *(i)* and by the fact that g is non degenerated in the subspace $\{x \in V : g(x, e_i) = 0, i \in [1, k] \subset \mathbb{N}\}$. The result follows then by induction. ■

Definition 6 *Let* (V, g) *be a Lorentz vector space. A vector* $x \in V$ *will be called timelike if* $g(x, x) < 0$*, spacelike if* $g(x, x) > 0$ *and null (or lightlike) if* $g(x, x) = 0$*. A vector is causal if it is timelike or null. A subspace* $W \subset V$ *is called timelike, spacelike or null if all vectors in* W *are timelike, spacelike and null, respectively.*

On what follows, given a Lorentz vector space (V, g), the orthogonal complement of $x \in V$ is the subset $x^{\perp} = \{z \in V : g(x, z) = 0\}$. The reader may prove that $x^{\perp}$ is, in fact, a subspace.

Let $(e_i)_{i \in [1,n]}$ be an orthonormal basis for a n-dimensional Lorentz vector space (V, g) and let $(\varepsilon_i)_{i \in [1,n]}$ be a sequence numbers such that

$g(e_i, e_j) = \varepsilon_i \delta_{ij}$. For the proof of the next Lemma, recall [11] that the Sylvester Theorem ensures that there is one and only one $k \in [1, n] \subset \mathbb{N}$ such that $\varepsilon_k = -1$.

Lemma 5 *Let (V, g) be a Lorentz vector space and let $x \in V$. So $x^\perp$ is timelike if x is spacelike.*

Proof. Let $n = \dim V$ and suppose that x is timelike. By the proof Lemma *4*, there is an orthonormal sequence $(e_i)_{i \in [1,n-1]}$ of vectors in V such that $(e_i)_{i \in [1,n]}$ is an orthonormal basis for V, where $e_n = x/\sqrt{g(x,x)}$. Let $y \in x^\perp$. So there is a sequence $(a_i)_{i \in [1,n]}$ of real numbers such that $y = \sum_{i \in [1,n]} a_i e_i$. By hypothesis, $a_n = 0$. Hence, by Sylvester Theorem, $g(y, y) = \sum_{i \in [1,n]} (a_i)^2 > 0$, i.e., y is spacelike. ■

¿From now on, the set of all timelike vectors in a given Lorentz vector space (V, g) will be denoted by τ, while that the set of all null vectors will be denoted by Λ. These are normally called, respectively, the *timecone* and the *lightcone* of V. The union $\tau \cup \Lambda$ will be called the *causalcone* and denoted by Υ.

Exercise 1 *Using Lemma* 5, *prove that the timecone, lightcone and the causalcone of a given Lorentz vector space have two disjoint components. Also, prove that the closure of a component of the timecone is a component of the lightcone.* (*For details, see Chapter 5 of* [12] *or Chapter 1 of* [14]).

Then we shall denote by τ^+ and τ^- the disjoint components of the timecone τ, and by Λ^+ and Λ^- their respective boundaries (which are, of course, the disjoint components of Λ). The closure of τ^+ and τ^-, which will be denoted by Υ^+ and Υ^-, respectively, are the components of the causalcone Υ.

Lemma 6 *Let (V, g) be a Lorentz vector space and let $x \in \tau^+$. So $y \in \Upsilon^+$ if and only if $g(x, y) < 0$ and $z \in \Upsilon^-$ if and only if $g(x, z) > 0$.*

Proof. Let f be the continuos mapping from Υ into $\mathbb{R} - \{0\}$ such that $v \to f(v) = g(x, v)$. As $f(x) < 0$ and Υ^+ is connected, $f(\Upsilon^+) = (-\infty, 0) \subset \mathbb{R}$. If $z \in \Upsilon^-$, thus -$z \in \Upsilon^+$, hence the result. ■

Now, we are ready to generalize this to a manifold:

Definition 7 *A Lorentzian manifold is an orientable 4-dimensional pseudo-Riemannian manifold whose index of the metric is* 1*.*

Given a Lorentzian manifold M, let π be the natural projection from TM onto M. An element $x \in TM$ will be called timelike, spacelike and null (or lightlike) if x, as an element of the Lorentz vector space $T_{\pi(x)}M$, is timelike, spacelike or null, respectively. As before, $x \in TM$ is causal if it is timelike or null.

Let γ be a curve from $I \in \mathbb{R}$ into a pseudo-Riemannian manifold M and let $\hat{D}_\gamma$ be the induced Levi-Civita connection on γ. For the proof of the following Lemma, remember that, given a vector field $X \in \sec T\gamma$ over γ, we say that X is parallel if $\hat{D}_\gamma X = 0$. Let $x \in T_{\gamma(a)}M$ for some $a \in I$. By the theory of differential equations, there is one and only one parallel vector field $X \in \sec T\gamma$ such that $X_a = x$. In this case, $y \in T_{\gamma(b)}M$ (for some $b \in I$) will be called the parallel transport (from a to b) of x along γ if $X_b = y$.

Lemma 7 *Let M be a connected Lorentzian manifold. The subset $\tau(M) \subset TM$ of all causal vectors is connected or have two components.*

Proof. Let $p \in M$ and let A be the set of all broken-geodesics in M. By the Corollary 3 and the axiom of choice, there is a mapping δ_p from M into A such that each $\delta_p(q)$ is a broken-geodesic from p into q. Let Υ_r^+ and Υ_r^- be the components of the causalcone $\Upsilon_r \subset T_rM$ for any $r \in M$. To each $q \in M$, define $\hat{\Upsilon}_q^+, \hat{\Upsilon}_q^- \subset T_qM$ to be such that $\hat{x} \in \hat{\Upsilon}_q^+$ and $\hat{y} \in \hat{\Upsilon}_q^-$ if and only if there exists $x \in \Upsilon_p^+$ and $y \in \Upsilon_p^-$ such that $\hat{x}$ and $\hat{y}$ are, respectively, the parallel transport of x and y along $\delta_p(q)$. Hence, by virtue of the Levi-Civita connection, $\hat{x} \in \Upsilon_q^+$ and $\hat{y} \in \Upsilon_q^-$, and $\tau(M) = \cup_{q \in M} \left(\hat{\Upsilon}_q^+ \cup \hat{\Upsilon}_q^-\right)$. Consequently, $\tau(M)$ have at most two components, and the result follows. ■

Exercise 2 *Let M be a connected Lorentzian manifold and let $X \in \sec TM$ be a smooth timelike vector field, i.e., $X_p \in T_pM$ is a timelike vector for all $p \in M$. So the subset $\tau(M) \subset TM$ of all causal vectors have two components.*

Solution. Let g be the metric of M and let f be the continuos mapping from $\tau(M)$ onto $\mathbb{R} - \{0\}$ such that $V_p \to f(V_p) = g(X_p, V_p)$.

As $f(X_p) < 0$ for all $p \in M$, $f^{-1}(-\infty, 0)$ and $f^{-1}(0, \infty)$ must be two disconnected components, and the result follows from Lemma 7.

In the case of the last Problem, we usually say that a vector $Y \in TM$ is future-pointing if it is in the same component of $\tau(M)$ as X.

Finally,

Definition 8 *A connected Lorentzian manifold is time-orientable if and only if the subset $\tau(M) \subset TM$ of all causal vectors has two components.*

Definition 9 *A spacetime (in General Relativity) is a connected orientable ad time-orientable Lorentzian manifold (M, g) equipped.with the Levi-Civita connection D of g.*

Remark 2 *A physical motivation for the last Definition is that, if we assume that the thermodynamics holds for any process in a given spacetime, it must be possible to select a "time arrow" for the physical phenomena from the second law, given a time orientation in that spacetime.*

2.3 Product of Manifolds

In what follows, given two manifolds M and N, the natural projections π_M and π_N of $M \times N$ are the mappings from $M \times N$ into M and N, respectively, such that $\pi_M(p, q) = p$ and $\pi_N(p, q) = q$.

Lemma 8 *Let (M, g_M) and (N, g_N) be pseudo-Rimannian manifolds and let π_M and π_N be the natural projections of $M \times N$. Then $(M \times N, g)$, where*

$$g = \pi_M^*(g_M) + \pi_N^*(g_N)$$

is itself a pseudo-Rimannian manifold, called the product manifold of (M, g_M) and (N, g_N).

The proof is a direct application of Definition 1 and will be left as an easy exercise.

In order to transport mappings, vectors and tensors from manifolds M and N to the product manifold $M \times N$, the notion of a lift will be introduced below. For the sake of brevity, consider the following notation:

$$T_{(p,q)}M = T_{(p,q)}(M \times \{q\})$$

$$T_{(p,q)}N = T_{(p,q)}(\{p\} \times N)$$

for all $(p, q) \in M \times N$.

Lemma 9 *Let M and N be smooth manifolds. So to each $(p, q) \in M \times N$, $T_{(p,q)}(M \times N)$ is the direct sum of $T_{(p,q)}M$ and $T_{(p,q)}N$.*

Proof. By definition, $\pi_M|(\{p\} \times N)$ is a constant function. So $\pi_{M*}(T_{(p,q)}N) = \{0\}$. But $\pi_{M*}|T_{(p,q)}M$ is an isomorphism onto T_pM. Hence $T_{(p,q)}M \cap T_{(p,q)}N = \{0\}$. The result follows then by $\dim T_{(p,q)}(M \times N) = \dim T_{(p,q)}M + \dim T_{(p,q)}N$. ■

Because of the identifications between $T_{(p,q)}M$ and T_pM and between $T_{(p,q)}N$ and T_qN, one normally recall the last Lemma in applications as saying that $T_{(p,q)}(M \times N) = (T_pM) \times (T_qN)$.

Hereafter, given a manifold M, the set of all smooth mappings from M into $\mathbb{R}$ will be denoted by $F(M)$.

Definition 10 *Let M and N be smooth manifolds let π_M and π_N be the natural projections of $M \times N$. We define the lifts in $M \times N$ of the mappings $f \in F(M)$ and $g \in F(N)$ to be the functions $\mathbf{f} = f \circ \pi_M$ and $\mathbf{g} = g \circ \pi_N$, respectively. We also define the lifts in $M \times N$ of the vectors $x \in T_pM$ and $y \in T_pN$ as the unique $\mathbf{x} \in T_{(p,q)}M$ and $\mathbf{y} \in T_{(p,q)}N$, respectively, such that $\pi_{M*}\mathbf{x} = x$ and $\pi_{N*}\mathbf{y} = y$.*

Remark 3 *The uniqueness assertion in the last definition is ensured by Lemma 9.*

We can extrapolate the above definition to vector fields in the following way:

Definition 11 *Let M and N be smooth manifolds. Let $X \in \sec TM$ and $Y \in \sec TN$ be a vector fields. We define the lifts in $M \times N$ of X and Y to be the unique vector fields $\mathbf{X}, \mathbf{Y} \in \sec T(M \times N)$ such that $\mathbf{X}_p$ is the lift in $M \times N$ of $X_p \in T_pM$ and $\mathbf{Y}_p$ is the lift in $M \times N$ of $Y_p \in T_pN$. We will say that $\mathbf{X}$ is a horizontal lift in $M \times N$, while that $\mathbf{Y}$ is a vertical lift.*

Remark 4 *Using coordinates, one can prove that the lift of a smooth vector field is by itself smooth.*

Example 1 *In $\mathbb{R}^2$ with natural coordinates (x, y), $\frac{\partial}{\partial x}$ is the horizontal lift of $\frac{d}{dt}$, while that $\frac{\partial}{\partial y}$is the vertical one.*

¿From now on, in the terminology of Definition 11, the set of all horizontal lifts in $M \times N$ will be denoted by $\pounds(M)$, whereas the set of all vertical lifts will be denoted by $\pounds(N)$.

Finally, we need to define the lift of a r-covariant tensor field:

Definition 12 *Let M and N be smooth manifolds and let π_M and π_N be the natural projections of $M \times N$. Let $A \in \sec T^r M$ and $B \in \sec T^r N$ be r-covariant tensor fields. We define the lifts in $M \times N$ of A and B to be the unique r-covariant tensor fields $\mathbf{A}, \mathbf{B} \in \sec T^r(M \times N)$ such that, for all $(p, q) \in M \times N$ and $(v_i)_{i \in [1,r]} \in T_{(p,q)}(M \times N)$, $\mathbf{A}(v_1, ..., v_r) = A(\pi_{M*}(v_1), ..., \pi_{M*}(v_r))$ and $\mathbf{B}(v_1, ..., v_r) = B(\pi_{N*}(v_1), ..., \pi_{N*}(v_r))$.*

Remark 5 (a) *Using Lemma* 9*, one can prove the uniqueness assertion.* (b) *This definition cannot be used to lift an arbitrary (s, r)-tensor field, since that π_M^* and π_{M*} goes in "opposite" directions. But using Definition* 11*, the reader is invited to inquire how to lift a $(s, 1)$-tensor field.*

2.4 Warped Product

In General Relativity, many spacetimes can be constructed in the following way:

Definition 13 *Let (B, g_B) and (F, g_F) be pseudo-Rimannian manifolds and π_M and π_N be the natural projections of $B \times F$. Let f be a smooth mapping from F into $\mathbb{R}^+$ (the set of positive real numbers). We define the warped product $B \times_f F$ to be the pseudo-Rimannian manifold $(B \times F, g)$ such that*

$$g = \pi_B^*(g_B) + (f \circ \pi_F)^2 \pi_F^*(g_F)$$

The function f may be called the warping mapping of $B \times_f F$.

Example 2 *Let r be the identity mapping in $\mathbb{R}^+$ and let (ϕ, φ) be polar coordinates in $S^2 = S^2(1)$. Let*

$$\eta = d\phi \otimes d\phi + \sin^2 \phi d\varphi \otimes d\varphi$$

be the Euclidean metric in S^2. Then $\mathbb{R}^+ \times_r S^2$ is isometric to the Euclidean space $\mathbb{R}^3 - \{0\}$.

Exercise 3 *Let n be a positive integer and let $v \in [0, n) \subset \mathbb{N}$. Let $(x^i)_{i\in[0,n]}$ be the natural coordinates of $\mathbb{R}^{n+1}$. So $(\mathbb{R}^{n+1}, \zeta)$ is the pseudo-Euclidean n-space of index v when*

$$\zeta = -\sum_{i\in[1,v]} dx^i \otimes dx^i + \sum_{i\in[v+1,n+1]} dx^i \otimes dx^i.$$

Then the pseudo-Euclidean n-sphere S^n_v of index v is the n-sphere $S^n \subset \mathbb{R}^{n+1}$ with the induced connection of $(\mathbb{R}^{n+1}, \zeta)$. Show how S^n_v can be written as a warped product of S^{n-v}.

Recall that, given a smooth mapping f from pseudo-Rimannian manifold M (together with a metric tensor g) into $\mathbb{R}$, grad(f) is the vector field metric equivalent to df, that is,

$$g(\mathrm{grad}(f), X) = df(X) = X(f)$$

for all vector $X \in TM$. Then the Hessian of f is defined to be the 2-covariant tensor field such that

$$(V, W) \to H^f(V, W) = VW(f) - (D_V W) = g(D_V(\mathrm{grad}(f)), W)$$

and the Laplacian of f is simply the contraction of H^f, i.e., $\Delta(f) = CH^f$.

The following Lemma will be our bridge between the geometry of B and F and its warped product $B \times_f F$:

Lemma 10 *With the notation of Definition 13, let $(M, g_M) = B \times_f F$, let Ric^M be the Ricci curvature tensor of M and let $\mathbf{Ric}^B \in \pounds(B)$ and $\mathbf{Ric}^F \in \pounds(F)$ be the lifts of the Ricci tensors of B and F, respectively. Suppose that $\dim F > 1$ and define the mapping*

$$\Im(f) = \frac{\Delta(f)}{f} + (\dim F - 1)\frac{g_M(\mathrm{grad}(f), \mathrm{grad}(f))}{f^2}$$

from M into $\mathbb{R}$. Hence, for all $\mathbf{X}, \mathbf{Y} \in \pounds(B)$ and $\mathbf{V}, \mathbf{W} \in \pounds(F)$,

$$Ric^M(\mathbf{X}, \mathbf{W}) = 0,$$
$$Ric^M(\mathbf{V}, \mathbf{W}) = \mathbf{Ric}^F(\mathbf{V}, \mathbf{W}) - \Im(f) g_M(\mathbf{V}, \mathbf{W}),$$
$$Ric^M(\mathbf{X}, \mathbf{Y}) = \mathbf{Ric}^B(\mathbf{X}, \mathbf{Y}) - \frac{\dim F}{f} H^f(\mathbf{X}, \mathbf{Y}).$$

The proof of this Proposition follows a tedious application of definitions and will then be omitted. The interested reader may consult the Chapter 7 of [12].

2.5 Null Geodesics and Maximal Extensions

Definition 14 *A pseudo-Riemannian manifold M will be called maximal when, for all pseudo-Riemannian manifolds N with the same dimension of M for which M is isometric to an open submanifold, $M = N$.*

Differently from the Riemannian case, we cannot use the Hopf-Rinow Theorem to decide when our spacetime is maximal. However, we can do it by studying the behavior of the null geodesics.

Lemma 11 *Let M be a spacetime and let U be a convex neighborhood in M. So for all points $p, q \in U$, there is one $r \in U$ such that the unique geodesics from p into r and from r into q are nulls.*

To prove this, we will use the Gauss Lemma and introduce some terminology first.

For the last of this section, let (M, g) be a pseudo-Riemannian manifold, let $p \in M$, let $x \in T_pM$ and let ϕ_x be the natural homomorphism between $T_x(T_pM)$ and T_pM (recall the comment above Lemma 1). In what follows, a vector $v \in T_x(T_pM)$ will be called radial if there is a real $k \neq 0$ such that $\phi(v) = kx$, and we will denote just by g the metric for both T_pM and $T_x(T_pM)$.

Lemma 12 (Gauss Lemma) *Let (M, g) be a pseudo-Riemannian manifold and let $p \in M$. Let $v, w \in T_x(T_pM)$ and suppose that v is radial. Then*

$$g(v, w) = g(\exp_{p*} v, \exp_{p*} w).$$

Proof. Let $\lambda(t, r) = t\,[\phi_x(v) + s\phi_x(w)]$ be a mapping from $\mathbb{R} \times \mathbb{R}$ into T_pM and let $x(t, r) = \exp_p \circ \lambda(t, r)$ be a mapping from $\mathbb{R} \times \mathbb{R}$ into U. As $(D_1\lambda)(1, 0) = v$ and $(D_2\lambda)(1, 0) = w$, we have

$$(D_1x)(1, 0) = \exp_{p*} v \qquad (D_2x)(1, 0) = \exp_{p*} w.$$

But, by the definition of the exponential mapping, $t \mapsto x(t, r)$ is a geodesic. Hence $D_1^2x = 0$ and $g(D_1x, D_1x) = g(\phi_x(v) + s\phi_x(w), \phi_x(v) + s\phi_x(w))$. Thus

$$D_1g(D_1x, D_2x) = g(D_1x, D_2D_1x) = \frac{1}{2}D_2g(D_1x, D_1x) = g(\phi_x(w), \phi_x(v){+}s\phi_x(w))$$

which implies

$$[D_1 g(D_1 x, D_2 x)]\,(t,0) = g(\phi_x(v), \phi_x(w)).$$

The result follows then from the fact that $g(D_1 x(0,0), D_2 x(0,0)) = 0$ and an elementary calculation. ■

¿From now on, the *position vector field* $\mathrm{P} \in \sec T(T_pM)$ in T_pM is defined to be the vector field such that $\mathrm{P}_x = \phi_x^{-1}(x)$, and the *quadratic form* Q_p in $T(T_pM)$ is the mapping into $\mathbb{R}$ given by $Q_p(x) = g(x,x)$. Then we may write that $Q_p = g(\mathrm{P},\mathrm{P})$.

Exercise 4 *Let $\tilde{D}$ be the Levi-Civita connection on the vector space T_pM. Prove that, if* P *is the position vector field, then $\tilde{D}_v\mathrm{P} = v$ for all $v \in T(T_pM)$. (Hint: if you feel lost, appeal to coordinates).*

Lemma 13 *Let M be a pseudo-Riemannian manifold and let $p \in M$. Let* P *and Q be the position vector field and the quadratic form in T_pM, respectively. So*

$$\operatorname{grad} Q = 2\mathrm{P}$$

Proof. Let $v \in T_x(T_pM)$ for some $x \in T_pM$. Then:

$$g(\operatorname{grad} Q, v) = dQ(v) = v\,[g(\mathrm{P},\mathrm{P})] = 2g(P, v)$$

by the last exercise, and the proof is over. ■

The exponential mapping can extend the position vector field and the quadratic form over a normal neighborhood in the following way. We define the (transported) *position vector field* $\mathbf{P} \in \sec TU$ to be the vector field over U given by $\mathbf{P} = \exp_{p*} P$, and the (transported) *quadratic form* $\mathbf{Q}$ to be the mapping on U such that $x \to \mathbf{Q}(x) = Q \circ \exp_p^{-1}(x)$.

Lemma 14 *Let U be a normal neighborhood in a given pseudo-Riemannian manifold M and let $\mathbf{P}$ and $\mathbf{Q}$ be the transported position vector field and the transported quadratic form, respectively. So*

$$\operatorname{grad} \mathbf{Q} = 2\mathbf{P}.$$

Proof. Let $y \in T_qU$ for some $q \in U$. Then

$$g(\text{grad}\,\mathbf{Q}, y) = d\left(Q \circ \exp_p^{-1}\right)(y) = g(\text{grad}\,Q, \exp_p^{-1} y) = 2g(P, \exp_p^{-1} y\ )$$

and the result follows by the Gauss Lemma. ■

Proof. [Proof of Lemma 11] Let Υ_q^+ and Υ_q^- be the disjoint components of the causalcone of T_qM. As U is a convex neighborhood, there is a unique geodesic σ from p into q. Without loss of generality, suppose that $\sigma'(0) \in \Upsilon_q^-$ (or in intuitive terms, p is in the past of q). Let γ be a null geodesic defined on $I \subset \mathbb{R}$ such that $\gamma(0) = p$. Let $\mathbf{P}$ and $\mathbf{Q}$ be the transported position vector field and the transported quadratic form in T_qM, respectively. So

$$(\mathbf{Q}\circ\gamma)'(t) = d\mathbf{Q}\left[\gamma'(t)\right] = 2g(\mathbf{P}_{\gamma(t)}, \gamma'(t)).$$

But $\mathbf{Q}\circ\gamma(0) = \mathbf{Q}(p) \geq 0$, by hypothesis. If $\mathbf{Q}\circ\gamma(0) = 0$, the result follows trivially. Then suppose that $\mathbf{Q}\circ\gamma(0) > 0$. By Lemma 6 and by the Gauss Lemma, the equation above shows that $(\mathbf{Q}\circ\gamma)'(t) < 0$. Hence, there is a $k \in I$ such that $\mathbf{Q}\circ\gamma(k) = 0$. Then let $r = \gamma(k)$ and the proof is over. ■

Lemma 15 *Let M be a spacetime and let N be an open submanifold of M with the induced connection. Assume that, if γ is a null geodesic from $I \subset \mathbb{R}$ into M such that $\gamma(I) \cap N \neq \varnothing$, then $\gamma(I) \subset N$. Hence $M = N$.*

Proof. Suppose that $M \neq N$ and let U be a convex neighborhood in M such that $U \cap \partial N \neq \varnothing$. By hypothesis, there are $p \in U - N$ and $q \in U \cap N$. By the Lemma 11, there is some $r \in U$ such that the unique geodesics γ_{pr} from p into r and γ_{rq} from r into q are nulls. By hypothesis, γ_{pr} lies on N. Hence $r \in N$. So γ_{rq} lies on N. Thus $q \in N$. Contradiction. ■

Remark 6 *Physically, the last Corollary means that a spacetime is maximal if one cannot "see" beyond it.*

3 Schwarzschild and Hilbert-Droste Solutions

It is well-known that K. Schwarzschild [16] was the first to find the exact gravitational field of a point of mass in General Relativity. However, a year later, the same problem was differently approached by D. Hilbert [17] and J. Droste [18], differences which will be discussed below.

In Section 3.1, we shall build a "spacetime" model in which the Schwarzschild and Hilbert-Droste are particular cases. Hence, we show in Section 3.2 how to generate solutions from such a model and we illustrate with a simple example.

Finally, we present in the last two sections the derivation of the Hilbert-Droste and Schwarzschild solutions and we finish by discussing if these are actually the same or not.

3.1 Building the Model

Let (t, r) be be the natural coordinates of $\mathbb{R}^2$ and let $P \subset \mathbb{R} \times \mathbb{R}^+$ be an open submanifold. In what follows, (t, h) will be called *special coordinates* of P if and only if there is a diffeomorphism ϕ from $r(P)$ into $\mathbb{R}$ such that $h = \phi \circ r$.

Definition 15 *A Schwarzschild model is an ordered list $(P, (t, h), f, g, \alpha)$, where $P \subset \mathbb{R} \times \mathbb{R}^+$ is an open submanifold, (t, h) is some special coordinates of P and f, g, α are smooth mappings from $h(P)$ into $\mathbb{R}^+$ such that*

$$\lim_{h\to\infty} f(r) = \lim_{h\to\infty} g(r) = 1$$

Definition 16 *Let $M = (P, (t, h), f, g, \alpha)$ be a Schwarzschild model. We define the corresponding Schwarzschild plane Π_M to be the pseudo-Riemannian manifold (P, ζ) such that*

$$\zeta = -(f \circ h)dt \otimes dt + (g \circ h)dh \otimes dh$$

In building a manifold through the warped product, the first step is to study the geometry of its parts. In our case, we start by

Lemma 16 *Given a Schwarzschild model $M = (P, (t, h), f, g, \alpha)$, let D be the Levi-Civita connection of its Schwarzschild plane Π_M. Thus*

$$D_{\partial_t}\partial_t = \frac{f'(h)}{2g(h)}\frac{\partial}{\partial h} \qquad D_{\partial h}\partial_h = \frac{g'(h)}{2g(h)}\frac{\partial}{\partial h}$$

$$D_{\partial_t}\partial_h = D_{\partial h}\partial_t = -\frac{f'(h)}{2f(h)}\frac{\partial}{\partial t}$$

Proof. As the dimension of P is 2, a direct computation is viable. So let $(\Gamma^k_{i,j})_{(k,i,j)\in[1,2]^3}$ be the Christoffel symbols. We use the well known equation

$$\Gamma^k{}_{ij} = \frac{1}{2}\sum_{m\in[1,2]} \eta^{km}\left(\frac{\partial\zeta_{im}}{\partial x^j} + \frac{\partial\zeta_{jm}}{\partial x^i} - \frac{\partial\zeta_{ij}}{\partial x^m}\right)$$

where $\zeta_{ij} = \zeta(\partial/\partial x^i, \partial/\partial x^j)$ and $x^1 = t, x^2 = h$. Thus, for example,

$$\Gamma^1{}_{11} = 0$$

$$\Gamma^2{}_{11} = -\frac{1}{2}\zeta^{22}\frac{\partial\zeta_{11}}{\partial h} = \frac{f'(h)}{2g(h)}$$

and the identity for $D_{\partial_t}\partial_t$ follows. The last two will be left as an easy exercise. ■

So now we define our spacetime model:

Definition 17 *Let S^2 be the Euclidean 2-sphere and let $M = (P,(t,h),f,g,\alpha)$ be a Schwarzschild model. So the (Schwarzschild-like) spacetime $\mathbf{S}_M$ associated with M is the warped product*

$$\Pi_M \times_\alpha S^2$$

Remark 7 *As a Schwarzschild-like spacetime $\mathbf{S}$ is a pseudo-Riemannian manifold, it can have distinct representations as a Schwarzschild model. Indeed, to each possible choice of special coordinates (t,h) of P, there are mappings f,g,α such that $M = (P,(t,h),f,g,\alpha)$ implies $\mathbf{S} = \mathbf{S}_M$. The submanifold P of $\mathbb{R}\times\mathbb{R}^+$ (see Definition 16) is, in the other hand, fixed: it is a part of the manifold of $\mathbf{S}$. We only introduced the notion of a "Schwarzschild model" because, in finding a solution to Einstein equation (see next section), it is important to keep a track of the coordinate system which we are using.*

Remark 8 (a) *The spacetime in Definition* 17 *can be time oriented by lifting the coordinate vector $\partial/\partial t$; for more in time orientability, see Chapter* 5 *of* [12]. (b) *The traditional physical motivations for the last Definition are that its corresponding spacetime is "static" with respect to the "time" t (see Chapter* 12 *of* [12] *for a rigorous definition), spherically symmetric and, as $h\to\infty$, Π_M approach the Minkowski "plane" (see Chapter* 1 *of* [14]).

3.2 Generating Solutions

Recall that a spacetime obeys the Einstein field equation in vacuum if and only if it is Ricci flat.

In the following Proposition we will use Lemma 10 to find the restrictions that the Einstein equation imposes upon our spacetime model:

Proposition 17 *Let* $M = (P, (t, h), f, g, \alpha)$ *be a Schwarzschild model. Its spacetime* $\mathbf{S}_M$ *satisfies the Einstein field equation in vacuum if and only if*

$$K = \frac{\alpha'(r)f'(r)}{\alpha(r)f(r)g(r)} = \frac{2}{\alpha(r)g(r)}\left[\alpha''(r) - \frac{g'(r)}{2g(r)}\alpha'(r)\right]$$

$$\Im(\alpha) = \frac{1}{[\alpha(h)]^2}$$

where K *is the sectional curvature of the Schwarzschild plane of* M*, given by*

$$K = -\frac{1}{2\sqrt{f(r)g(r)}}\left[\frac{f'(r)}{\sqrt{f(r)g(r)}}\right]'$$

and

$$\Im(\alpha) = \frac{1}{\alpha(h)}\left\{\frac{\alpha''(h)}{g(h)} + \frac{\alpha'(h)}{2g(h)}\left[\frac{f'(h)}{f(h)} - \frac{g'(h)}{g(h)}\right] + \frac{[\alpha'(h)]^2}{g(h)\alpha(h)}\right\}$$

In the proof of this Proposition, the following two Lemma will be used:

Lemma 18 *Let* (M, g) *be a pseudo-Riemannian surface (that is, a pseudo-Riemannian manifold such that* $\dim M = 2$*). Let Ric be its Ricci curvature tensor and let* K *be its sectional curvature. Then*

$$Ric = Kg$$

Proof. Let (u, v) be orthogonal coordinates in some neighborhood of M (which always exists since we can employ a frame field; see, e.g., Chapter 3 of [12]) and let R be the Riemannian curvature tensor of (M, g). Remember that, if $x, y \in T_pM$ are linearly independent vectors (for some $p \in M$),

$$K(x, y) = \frac{g(R_{xy}x, y)}{Q(x, y)}$$

where

$$Q(x,y) = g(x,x)g(y,y) - g(x,y)^2$$

Since M has dimension 2, K is a smooth mapping in $F(M)$. But by definition

$$\begin{aligned} Ric(x,x) &= \frac{g(R_{x.\partial_u}x,\partial_u)}{g(\partial_u,\partial_u)} + \frac{g(R_{x.\partial_v}x,\partial_v)}{g(\partial_v,\partial_v)} \\ &= K_p\left[\frac{Q(x,\partial_u)}{g(\partial_u,\partial_u)} + \frac{Q(x,\partial_v)}{g(\partial_v,\partial_v)}\right] \end{aligned}$$

Then the result follows by a direct substitution in the above identity, and the details are left as an easy exercise. ■

As usual, in the following Lemma the partial derivative $\partial f/\partial x$ of a mapping f will be denoted just by f_x.

Lemma 19 *Let (M,g) be a pseudo-Riemannian surface with sectional curvature K. Let (u,v) be an orthogonal coordinate system over M, let $e,g \in \sec F(M)$ be positive real-valued mappings and let $\varepsilon_1^2 = \varepsilon_2^2 = 1$ be real numbers such that $\varepsilon_1 e^2 = g(\partial_u,\partial_u)$ and $\varepsilon_2 g^2 = g(\partial_v,\partial_v)$, where ∂_u and ∂_v are the coordinate vectors of (u,v). Therefore*

$$K = -\frac{1}{eg}\left[\varepsilon_1\left(\frac{e_v}{g}\right)_v + \varepsilon_2\left(\frac{g_u}{e}\right)_u\right]$$

Proof. [Proof of Proposition 17] Let Ric^{Π_M} and Ric^{S^2} be the Ricci curvature tensors of the Schwarzschild plane Π_M and of the Euclidean 2-sphere S^2, respectively. By Lemma 10, the Einstein field equation in vacuum ($Ric = 0$) is equivalent to

$$\mathbf{Ric}^{\Pi_M}(\mathbf{X},\mathbf{Y}) = \frac{2}{\alpha(r)}H^\alpha(X,Y)$$

$$\mathbf{Ric}^{S^2}(\mathbf{V},\mathbf{W}) = \Im(\alpha)g(\mathbf{V},\mathbf{W})$$

for all $V,W \in \sec TS^2$ and $X,Y \in \sec TP$, where $\mathbf{V},\mathbf{W} \in \pounds(S^2)$ and $\mathbf{X},\mathbf{Y} \in \pounds(P)$ are their respective lifts.

By Lemma 16 it is

$$H^\alpha(\partial_t,\partial_t) = -\frac{f'(h)}{2g(h)}\alpha'(r),$$

$$H^\alpha(\partial_h,\partial_h) = \alpha''(h) - \frac{g'(h)}{2g(h)}\alpha'(h).$$

Let K be the sectional curvature of Π_M. Using Lemma 18, we find that

$$\frac{f'(h)\alpha'(h)}{f(h)g(h)\alpha(h)} = K = \frac{2}{g(h)\alpha(h)}\alpha''(h) - \frac{g'(h)}{[g(h)]^2\alpha(h)}\alpha'(h)$$

and the expression for K is a direct use of Lemma 19.

Finally, Lemma 18 gives that

$$\mathbf{Ric}^{S^2}(\mathbf{V},\mathbf{W}) = \frac{g(\mathbf{V},\mathbf{W})}{[\alpha(r)]^2}$$

Hence the second equation of the Proposition follows. The last is only a direct computation, and will be left as an exercise (for the Definition of $\Im(\alpha)$, see Lemma 10). ■

Exercise 5 *Fix the submanifold $P \subset \mathbb{R} \times \mathbb{R}^+$.* (a) *Is a Schwarzschild model $M = (P,(t,h),f,g,\alpha)$ uniquely determined by Proposition* 17*?* (b) *Is the associated spacetime $\mathbf{S}_M$ of M, which satisfies Einstein equation in vacuum, uniquely determined?*

Solution. *(a)* No. *(b)* Yes. One can add to the Einstein field equation some "coordinate condition" in order to determine f, g, α uniquely. After this, we have a pseudo-Riemannian manifold *(*in particular, a spacetime*)* whose metric and *(*if P was given*)* topology is well-defined. If we use some different "coordinate condition", we must find another set of $\mathbf{f}, \mathbf{g}, \boldsymbol{\alpha}$, but this is because we are using distinct coordinate systems *(*see Remark *7).*

Indeed, the reader must already known the "Schwarzschild" solution which is normally presented in the current literature (see the next section). We illustrate in the following example another possible choice for f, g, α which also satisfies Proposition 17 and the conditions of Definition 15:

Example 3 *Let $\alpha(h) = r + \mu$. Assume that the spacetime $\mathbf{S}_M$ of a Schwarzschild model $M = (P,(t,h),f,g,\alpha)$ satisfies Einstein equation in vacuum. So by the first equation in Proposition* 17*,*

$$[f(h)g(h)]' = 0$$

But by Definition 15 (recall the limit conditions), $f(r)g(r) = 1$). Then by the second pair of equations of the same Proposition, one finds that

$$g'(h) = \frac{g(h)\,[1 - g(h)]}{r + \mu}$$

Solving this equation, we find as a possible solution

$$g(h) = \frac{h + \mu}{h - \mu}$$

Of course that $\lim_{r\to\infty} g(r) = 1$, as we needed. Hence, in terms of coordinates, the metric $\mathbf{S}_M$ reads:

$$-\frac{h - \mu}{h + \mu} dt \otimes dt + \frac{h + \mu}{h - \mu} dr \otimes dr + (r + \mu)^2 \zeta_{S^2}$$

where ζ_{S^2} is the Euclidean metric of S^2.

3.3 Hilbert-Droste Solution

The lesson which we must take from Problem 5 and Example 3 is that, in order to find some solution of Einstein equation, one needs to impose some "coordinate condition".

The way followed by Hilbert was very simple and elegant, and can be summarized in the following definition:

Definition 18 *A Schwarzschild model $M = (P, (t, h), f, g, \alpha)$ will be called a Hilbert model if and only if the special coordinates (t, h) of P were chosen such that $\alpha \circ h = \mathrm{id}_{\mathbb{R}}$.*

In what follows, let (w_1, w_2, w_3, w_4) be the natural coordinates of $\mathbb{R}^4$. So in Hilbert words [17] (translation from [19]),

> According to Schwarzschild, if one poses
>
> $$\begin{aligned} w_1 &= r \cos\vartheta \\ w_2 &= r \sin\vartheta \cos\varphi \\ w_3 &= r \sin\vartheta \sin\varphi \\ w_4 &= l \end{aligned}$$

the most general interval corresponding to these hypotheses is represented in spatial polar coordinates by the expression

$$(42)\quad F(r)dr^2 + G(r)(d\vartheta^2 + \sin^2\vartheta d\varphi^2) + H(r)dl^2$$

where $F(r)$, $G(r)$ and $H(r)$ are still arbitrary functions of r. If we pose

$$r^* = \sqrt{G(r)}$$

we are equally authorized to interpret r^*, ϑ and φ as spatial polar coordinates. If we substitute in (42) r^* for r and drop the symbol $*$, it results the expression

$$M(r)dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) + W(r)dl^2$$

where $M(r)$ and $W(r)$ means the two essentially arbitrary functions of r.

With the last definition, we are able to derive the Hilbert-Droste *metric*:

Proposition 20 *Let* $M = (P, (t, h), f, g, \alpha)$ *be a Hilbert model. Its spacetime* $\mathbf{S}_M$ *obeys Einstein equation in vacuum if and only if*

$$f(h) = \frac{1}{g(h)} = 1 - \frac{\mu}{h}$$

for some real μ.

Proof. By the first equation of Proposition 17, we have (like in Example 3),

$$[f(h)g(h)]' = 0.$$

But by Definition 16 (recall the limit conditions), $f(h)g(h) = 1$. Then by the second pair of equations of the same Proposition, one finds that

$$g'(h) = \frac{g(h)\,[1 - g(h)]}{h}$$

Hence, there is a real number μ such that

$$g(h) = \frac{1}{1 - \mu/h}$$

and the proposition is proved. ■

We do not have, however, the complete *Hilbert-Droste solution*. We only have its metric, which is just half the story. To have in hands a *proper solution*, we must set up a topology, which in this case means to pick up some $P \subset \mathbb{R} \times \mathbb{R}^+$ (recall Definitions 16 and 17).

Note that the largest submanifold of $\mathbb{R} \times \mathbb{R}^+$ in which the mappings in the last Proposition are smooth is $\mathbb{R} \times (\mathbb{R}^+ - \{\mu\})$. Then, we are motivated to state the Hilbert-Droste solution:

Definition 19 (Hilbert-Droste solution) *Given a real number μ, the Hilbert-Droste solution $H(\mu)$ is the spacetime $\mathbf{S}$ for which there is a Hilbert model $M = (P, (t, h), f, g, \alpha)$ such that $\mathbf{S} = \mathbf{S}_M$,*

$$P = \mathbb{R} \times (\mathbb{R}^+ - \{\mu\}),$$

and

$$f(h) = \frac{1}{g(h)} = 1 - \frac{\mu}{h}.$$

So by Proposition 20, the Hilbert-Droste solution obeys the Einstein equation in vacuum.

What distinguish the coordinate expression for the metric in the above Proposition and in Example 3 is the choice of coordinates. However, are Example 3 and Definition 19 describing the same solution?

Exercise 6 *Let $\mu \in \mathbb{R}$ and let $M = (P, (t, h), f, g, \alpha)$ be as in Example 3. Choose $P \subset \mathbb{R} \times \mathbb{R}^+$ to be the largest submanifold for which f, g are smooth (and the corresponding metric non degenerated). Is the spacetime $\mathbf{S}_M$ a Hilbert-Droste solution?*

Solution. Yes. But taking into account that such a metric have a singularity in $r = \mu$ we see that the largest possible P is $\mathbb{R} \times (\mathbb{R}^+ - \{\mu\})$, hence it has the same topology as Hilbert-Droste.

Remark 9 *Playing with Proposition* 17*, one can generate an infinite set of metrics for a Schwarzschild-like spacetime which satisfies Einstein equation. In principle, one can find a coordinate transformation* *which transform these metric expressions into each other. However, if we are presented with two spacetimes whose metric expressions can be transformed into each other in some coordinate chart, it does not means that they are the same solution: it is necessary to take care about the topology, which in the approach of this paper depends on a submanifold* $P \subset \mathbb{R} \times \mathbb{R}^+$.

3.4 Schwarzschild Solution

In this paragraph, let (u^1, u^2, u^3, u^4) be a coordinate system on a given spacetime with metric g. When Schwarzschild found his solution in 1916, he used the following form of the Einstein field equations in vacuum

$$\sum_{k\in[1,4]} \frac{\partial \Gamma^k{}_{ij}}{\partial u^k} + \sum_{(k,l)\in[1,4]^2} \Gamma^k{}_{il}\Gamma^l{}_{kj} = 0,$$
$$\sqrt{-\det g} = 1,$$

for all $(i, j) \in [1, 4]^2$, where $(\Gamma^k{}_{ij})_{(k,i,j)\in[1,4]^3}$ are the Christoffel symbols and $\det g$ is the determinant of the matrix whose elements are $g_{ij} = g(\partial/\partial u^i, \partial/\partial u^j)$ (see [23]). The second equation is such that only unimodular coordinate transformations preserves the "mathematical form" of the field equations.

Schwarzschild started his work by setting the spacetime manifold to be $\mathbb{R} \times \{\mathbb{R}^3 - \{0\}\}$. As he wanted a spherically symmetric solution, it was natural for him to introduce spatial polar coordinates. But the transformation from the natural coordinates of $\mathbb{R}^3$ to polar coordinates is not, of course, unimodular. In his own words [16] (translation from [22]):

> When one goes over to polar co-ordinates according to $x = r \sin\vartheta \cos\phi$, $y = r \sin\vartheta \sin\phi$, $z = r \cos\vartheta$ *(...)* the volume element *(...)* is equal to $r^2 \sin\vartheta dr d\vartheta d\phi$, *[so]* the functional determinant $r^2 \sin\vartheta$ of the old with respect to the new coordinates is different from 1; then the field equations would not remain in unaltered form if one would calculate with these polar co-ordinates, and one would have to perform a cumbersome transformation.

Then Schwarzschild proceeded in the following way (also from [16]):

> However there is an easy trick to circumvent this difficulty. One puts:
>
> $$x_1 = \frac{r^3}{3}\ ,\ \ x_2 = -\cos\vartheta\ ,\ \ x_3 = \phi$$
>
> Then we have for the volume element: $r^2 \sin\vartheta dr d\vartheta d\phi = dx_1 dx_2 dx_3$. The new variables are then polar co-ordinates with the determinant 1. They have the evident advantages of polar co-ordinates for the treatment of the problem, and at the same time *(...)* the field equations and the determinant equation remain in unaltered form.

The reader must take in mind that, in the Schwarzschild approach, the coordinate condition need to solve the equations of Proposition 17 (recall also Problem 5) must satisfies the Einstein's determinant equation.

However, thanks to the warped product, we do not need to concern with any "polar coordinates with determinant 1" here. Indeed, using the following definition, we can do the whole derivation without any mention to the coordinates of S^2:

Definition 20 *A Schwarzschild model* $M = (P, (t, h), f, g, \alpha)$ *will be called a unimodular model if and only if the special coordinates* (t, h) *of* P *were chosen such that*

$$f(h)g(h)\left[\alpha(h)\right]^4 = 1$$

Remark 10 *Let* ζ_{S^2} *be the Euclidean metric of* S^2*, let* $M = (P, (t, h), f, g, \alpha)$ *be a unimodular model and let* g *be the metric of the spacetime* $\mathbf{S}_M$ *of* M*. In the notation of the first paragraph, if we give a coordinate expression to* ζ_{S^2} *such that* $\det\zeta_{S^2} = 1$*,*

$$-\det g = f(h)g(h)\left[\alpha(h)\right]^4 \det\zeta_{S^2} = 1$$

we have the Schwarzschild "original" coordinate condition.

To see how Definition 20 together with Proposition 17 determine the warping mapping α up to *two* constants, we state the following Lemma:

Lemma 21 *If the spacetime of a unimodular model* $(P,(t,h),f,g,\alpha)$ *satisfies the Einstein field equation in vacuum, then there are real numbers* λ,μ *such that*

$$\alpha(h)=\lambda\left(3h+\mu^3\right)^{1/3}$$

Proof. By hypothesis,

$$[f(h)g(h)]'\,[\alpha(h)]+4\,[f(h)g(h)]\,\alpha'(h)=0$$

So, using the first equation of Proposition 17 we get

$$\alpha''(h)=-2\frac{[\alpha'(h)]^2}{\alpha(h)}$$

whose solution is

$$\alpha(h)=\lambda\left(3h+\mu^3\right)^{1/3}$$

for $\lambda,\mu\in\mathbb{R}$, and the proof is done. ■

Remark 11 *In the notation of the last Lemma, Schwarzschild put the constant* $\lambda=1$ *by requiring that*

$$\lim_{h\to\infty}\frac{[\alpha(h)]^2}{(3h)^{2/3}}=1$$

since he wanted that his solution in "polar coordinates with determinant 1" approximate the Minkowski spacetime as $h\to\infty$*. In our derivation, we are free to set* $\lambda\neq 0$ *to whatever we want, since this means only a change in the scale of special coordinates* (t,h)*. We will, however, stay with the Schwarzschild choice.*

Now we can derive the Schwarzschild metric like we did for the Hilbert-Droste case or in Example 5:

Proposition 22 *Let* $M=(P,(t,h),f,g,\alpha)$ *be a unimodular model. Its spacetime* $\mathbf{S}_M$ *obeys the Einstein field equation in vacuum and the limit of Remark* 11 *if and only if there are real numbers* k,μ *such that*

$$\alpha(h)=\left(3h+k^3\right)^{1/3}$$

$$f(h)=\frac{[\alpha(h)]^4}{g(h)}=1-\frac{\mu}{\alpha(h)}$$

Proof. The first equation follows from last Lemma. Computing the derivatives of α and using the condition that

$$g(h) = \frac{[\alpha(h)]^4}{f(h)}$$

(recall Definition *20*) we get, by the second equation of Proposition *17*,

$$f'(h) = \frac{1 - f(h)}{3h + \mu}$$

and the result follows simply by solving this equation. ■

Remark 12 *In the coordinates* (t, h)*, defined by Definition* 20 *and Remark* **??***, the metric described by the last Proposition reads*

$$-\left[1 - \frac{\mu}{\alpha(h)}\right] dt \otimes dt + \frac{1}{[\alpha(h)]^4} \frac{1}{1 - \mu/\alpha(h)} dh \otimes dh + [\alpha(h)]^2 \zeta_{S^2} \quad (1)$$

where $\alpha(h) = \left(3h + k^3\right)^{1/3}$ *and* ζ_{S^2} *is the Euclidean metric of* S^2*. As* α *is a diffeomorphism from* $\mathbb{R}^+$ *onto* $\mathbb{R}^+$*, we can define* (t, R) *to be the special coordinates of* $P \subset \mathbb{R} \times \mathbb{R}^+$ *such that* $R = \alpha \circ h$ *(recall first paragraph of Section 3.1). Hence, in the* (t, R) *coordinates, the last metric reads as*

$$-\left(1 - \frac{\mu}{R}\right) dt \otimes dt + \frac{1}{1 - \mu/R} dR \otimes dR + R^2 \zeta_{S^2} \quad (2)$$

Thus we have the Schwarzschild metric and we known how to make a transformation such that its coordinate expression is like that of Hilbert-Droste. But we do not have yet the *Schwarzschild solution.* As we did in last section, we must select some submanifold P of $\mathbb{R} \times \mathbb{R}^+$ to fix the topology and the spacetime manifold itself.

Remark 13 *In his original work, Schwarzschild imposed the condition that the metric components must be smooth except in the origin of his coordinate system. However, since our spacetime manifold is* $\mathbb{R} \times \mathbb{R}^+ \times S^2$*, the only way to realize that condition is by introducing the manifold with boundary* $\mathbb{R} \times [0, \infty[\times S^2$ *and extending continuously the mappings* f, g *and* α *from* $\mathbb{R}^+$ *to* $[0, \infty[$*. Thus, as in the boundary of* $\mathbb{R} \times \{0\} \times S^2$ *the functions* f *and* g *satisfy*

$$\frac{1}{k^4} f(0) g(0) = 1 - \frac{\mu}{k} \ ,$$

the Schwarzschild condition is in fact equivalent to

$$k = \mu \ .$$

So now we are motivated to state the Schwarzschild solution:

Definition 21 (Schwarzschild solution) *Given a real number μ, the Schwarzschild solution $S(\mu)$ is the spacetime $\mathbf{S}$ for which there is an unimodular model $M = (P, (t, h), f, g, \alpha)$ such that $\mathbf{S} = \mathbf{S}_M$,*

$$P = \mathbb{R} \times \mathbb{R}^+$$

$$\alpha(h) = \left(3h + \mu^3\right)^{1/3}$$

$$f(h) = \frac{[\alpha(h)]^4}{g(h)} = 1 - \frac{\mu}{\alpha(h)}$$

Exercise 7 *Given a real number μ, are the Schwarzschild $S(\mu)$ and Hilbert-Droste $H(\mu)$ solutions equivalents?*

Solution. No, as they have a different topologies. The manifold which describe the Hilbert-Droste solution is

$$\mathbb{R} \times (\mathbb{R}^+ - \{\mu\}) \times S^2,$$

while that the Schwarzschild manifold is simply

$$\mathbb{R} \times \mathbb{R}^+ \times S^2 \ .$$

Because of its topology, the Hilbert-Droste solution can be sliced into two parts, one called the exterior, whose manifold is $\mathbb{R} \times (\mu, \infty) \times S^2$, and another called the interior (or the *black hole*), whose manifold is $\mathbb{R} \times (0, \mu) \times S^2$. As we shall see in the next section, this topological property allow the Hilbert-Droste manifold to be glued together with another manifold (recall Section A), constituting what is known by the Kruskal spacetime.

However, since the Schwarzschild manifold is homeomorphic to $\mathbb{R} \times (\mathbb{R} - \{0\}) \times S^2$ (see, for instance, Example 2), we cannot find any manifold to which the Schwarzschild manifold can be glued to, in the sense of Definition 26. Even if we found in Remark 12 a coordinate transformation such that the Schwarzschild metric acquire the same form as the Hilbert-Droste, in the former, the metric expression holds only for $R > \mu$.

Remark 14 *On the other hand, differently from what the author of* [20] *did, based only on the above discussion, we cannot jump to the conclusion that black holes do not exist as appropriated solutions of Einstein equation. Indeed, the fact that the Einstein field equation have many solutions with black holes seems well established, and in some cases, according to General Relativity, black holes are unavoidable (in gravitational collapses). What is important to keep in mind, however, is that there is no internal mechanism in the theory to decide between the topologies of the Schwarzschild solution and the Hilbert-Droste solution. And, in the last analysis, the existence of black holes or the decision between the above solutions is an experimental quest.*

A Topological Extension of Manifolds

In this Appendix, we analyze the extension of manifolds from a careful topological point of view. Specifically, we give a rigorous procedure (summarized in the following list of Definitions and Lemmas) that justify the process of gluing topological spaces, manifolds and pseudo-Riemannian structures.

We hope that the following developments might be useful for relativists working in the construction of spacetimes containing black holes, wormholes, bridges or any object with exotic topology.

Our approach is based in ref. [13].

Definition 22 *A gluing structure is an ordered list* (M, N, U, V, ξ)*, where* M *and* N *are topological spaces,* $U \subset M$ *and* $V \subset N$ *are subspaces and* ξ *is a homeomorphism between* U *and* V*.*

Recall that, given a family of sets $(A_n)_{n\in F}$, the disjoint union of this family is defined to be

$$\widetilde{\cup}_{n\in F} A_n = \cup_{i\in F}\{(x, i) : x \in A_i\}$$

Definition 23 *Let* $G = (M, N, U, V, \xi)$ *be a gluing structure. Define* $\approx$ *to be the equivalence relation on the disjoint union* $M\widetilde{\cup}N$ *such that* $p \approx q$ *if and only if* $p = q$*,* $p = \xi(q)$ *or* $q = \xi(p)$*. So the quotient space* $M\widetilde{\cup}N/\approx$ *will be called the glued space* Q_G *of* G *and* $\approx$ *the equivalence of the gluing structure* G*.*

In what follows, given a gluing structure $G = (M, N, U, V, \xi)$ and its respective glued space Q_G, the natural injections i and j from G into Q_G are the mappings from M and N, respectively, into Q_G such that

$$i(p) = p \text{ if } p \in M - U \text{ and } i(p) = \{p, \xi(p)\} \text{ if } p \in U,$$
$$j(q) = q \text{ if } q \in N - V \text{ and } j(q) = \{q, \xi(q)\} \text{ if } q \in V.$$

A subset $S \subset Q_G$ will be considered open if and only if $i^{-1}(S)$ and $j^{-1}(S)$ are open in M and in N, respectively.

Lemma 23 *Let G be a gluing structure, Q_G its glued space and i and j the natural injections from G into Q_G. Then i and j are homeomorphisms between M and $i(M)$ and between N and $j(N)$, respectively.*

Proof. By the last remark, i and j are continuous. Let $X \subset M$ be an open subset. So $i(X)$ is open in Q_G if and only if $i^{-1}(i(X))$ and $j^{-1}(i(X))$ are open in M and in N respectively. The first is open since that $i^{-1}(i(X)) = X$. But for the second:

$$j^{-1}(i(X)) = j^{-1}(i(X) \cap j(N)) = j^{-1}(i(X \cap U)) = \xi(X \cap U)$$

Hence $i(X)$ is open. The result follows for i since that it is injective, and the proof is the same for j. ■

Remark 15 *Because of the last Lemma, one may ignore the natural injections and think about $i(M)$ and $j(N)$ as being actually equal to M and N, respectively. Then $M \cap N$, U and V are all identified.*

Lemma 24 *Assuming the hypothesis of Lemma 23, let P be a topological space and let ϕ_M and ϕ_N be continuos mappings from M and N, respectively, into P. Suppose that $\phi_M|U = \phi_N \circ \xi$. Hence, there is a unique continuos mapping ϕ from Q_G into P such that $\phi \circ i = \phi_M$ and $\phi \circ j = \phi_N$.*

Proof. Define ϕ to be such that $\phi(p) = \phi_M(i^{-1}(p))$ if $p \in i(M)$ and $\phi(q) = \phi_N(j^{-1}(q))$ if $q \in j(N)$. This is well-defined since that when $p \in i(M) \cap j(N)$, $p = \{x, \xi(x)\}$ for $x = i^{-1}(x)$. Hence

$$\phi(p) = \phi_M(x) = \phi_N(\xi(x)) = \phi(p)$$

Finally, ϕ is continuos by Lemma 23. ■

Lemma 25 *Let* $G = (M, N, U, V, \xi)$ *and* $G' = (M', N', U', V', \xi')$ *be gluing structures,* Q_G *and* $Q'_G = Q_{G'}$ *their respective glued spaces and* i, j *and* i', j' *their respective natural projections. Let* ϕ_M *and* ϕ_N *be continuos mappings from* M *and* N*, respectively, into* M' *and* N'*, respectively. Assume that* $\xi' \circ \phi_M | U = \phi_N \circ \xi$*. Thus, there is a unique continuous mapping* ϕ *from* Q_G *into* Q'_G *such that* $\phi \circ i = i' \circ \phi_M$ *and* $\phi \circ j = j' \circ \phi_N$*.*

Proof. Define ϕ to be such that $\phi(p) = i'(\phi_M(i^{-1}(p)))$ if $p \in i(M)$ and $\phi(q) = j'(\phi_N(j^{-1}(q)))$ if $q \in j(N)$. This is well-defined since that, if $p \in i(M) \cap j(N)$, $p = \{x, \xi(x)\}$ for $x = i^{-1}(p)$. So

$$\phi(p) = i'(\phi_M(x)) = (j' \circ \xi')(\phi_M(x)) = j'(\phi_N(\xi(x))) = \phi(p)$$

Finally, ϕ is continuos by Lemma 23. ■

The last two Lemmas are normally called the *Mapping Lemmas.*

Exercise 8 *(a) Let* $G = (\mathbb{R}, \mathbb{R}, \mathbb{R}^+, \mathbb{R}^+, \mathrm{id}_{\mathbb{R}})$ *be a gluing structure. (For any set* A*,* id_A *means the identity in* A*). Is* Q_G*, the glued space, Hausdorff? (b) Let* H^+ *be the north hemisphere of* S^2 *without the equator, and let* $N \in S^2$ *be its north pole. Let* $G = (S^2, S^2, H^+ - \{N\}, H^+ - \{N\}, \mathrm{id}_{S^2})$ *be a gluing structure. Is* Q_G *Hausdorff?*

The above exercise is then the motivation for the following definition:

Definition 24 *A gluing structure* (M, N, U, V, ξ) *will be called Hausdorff if* M *and* N *are Hausdorff and if there is no convergent sequence* $(p_n)_{n \in \mathbb{N}}$ *of points in* M *such that* $\lim p_n \in M - U$ *and* $\lim \xi(p_n) \in N - V$*.*

Lemma 26 *Let* (M, N, U, V, ξ) *be a Hausdorff gluing structure. So the glued space* Q_G *is Hausdorff.*

Proof. Let $x, y \in Q_G$ be distinct points. The result is obvious if both x and y belongs to $i(M)$ (or to $j(N)$). So, suppose that $x \in i(M) - j(V)$ and $y \in j(N) - i(U)$. Let $(N_n)_{n \in \mathbb{N}}$ and $(N'_n)_{n \in \mathbb{N}}$ be a basis for the neighborhoods of x and y, respectively. Assume that $N_n \cap N'_n$ is not empty for all n. So by the axiom of choice, there is a sequence $(x_n)_{n \in \mathbb{N}}$ such that $x_n \in N_n \cap N'_n$ for all n. Let i and j be the natural injections of G into Q_G. So $(i^{-1}(x_n))_{n \in \mathbb{N}}$ and $(j^{-1}(x_n))_{n \in \mathbb{N}}$ do not respect Definition 24. Hence, by contradiction, there is some $n \in \mathbb{N}$ such that $N_n \cap N'_n = \varnothing$, and the proof is over. ■

Exercise 9 *Let* $U = V = \{(x, y) \in \mathbb{R}^2 : x, y < 0\}$ *and* $G = (\mathbb{R}^2, \mathbb{R}^2, U, V, \xi)$*. Is the glued space* Q_G *Hausdorff if* (a) $\xi = \mathrm{id}_{\mathbb{R}^2}$ *and* (b) $\xi(x, y) = (x, y/x)$*?*

We can now extrapolate our results for manifolds:

Definition 25 *A gluing structure* $G = (M, N, U, V, \xi)$ *will be called a "manifold gluing" when* G *is Hausdorff,* M *and* N *are manifolds with the same dimension,* U *and* V *are submanifolds and* ξ *is a diffeomorphism.*

Remark 16 *It must be clear from the above definition that, in the "manifold gluing" case, the Mapping Lemmas holds for smooth mappings rather than just for continuos ones.*

Remember that a chart in a manifold M is an ordered pair (X, ψ) such that $X \subset M$ is an open subset and ψ is a homeomorphism between X and $\mathbb{R}^{\dim M}$. Recall also that an atlas in M is a set A of charts such that $M \subset \cup_{(U,\psi)\in A} U$ (we say that A *covers* M) and, given two charts $(X, \psi), (Y, \omega) \in A$ such that $X \cap Y \neq \varnothing$, both $\psi \circ \omega^{-1}$ and $\omega \circ \psi^{-1}$ are smooth (we say that A *overlaps smoothly*).

Lemma 27 *Let* $G = (M, N, U, V, \xi)$ *be a "manifold gluing". The glued space* Q_G *is itself a manifold.*

Proof. By hypothesis, Q_G is Hausdorff. Now, let A_M and A_N be atlases for M and N respectively, and define

$$A = \{(i(X), \psi \circ i^{-1}), (j(Y), \omega \circ j^{-1}) : (X, \psi) \in A_M, (Y, \omega) \in A_N\}.$$

Of course that A *covers* Q_G. To prove that they *overlap smoothly*, let $(X, \psi) \in A_M$ and $(Y, \omega) \in A_N$ be charts such that $i(X) \cap j(Y) \neq \varnothing$. So

$$(\psi \circ i^{-1}) \circ (\omega \circ j^{-1})^{-1} = \psi \circ (i^{-1} \circ j) \circ \omega^{-1}$$

is smooth and the proof is over. ■

Finally, we finish with the pseudo-Riemannian case:

Definition 26 *A Hausdorff gluing structure $G = (M, N, U, V, \xi)$ will be called a pseudo-Riemannian gluing if M and N have pseudo-Riemaniann structures (see Definition 1) and if ξ is an isometry.*

Proposition 28 *Let $G = (M, N, U, V, \xi)$ be a pseudo-Riemannian gluing and g_M and g_N the metric tensors of M and N, respectively. So, there is an unique metric tensor g_G such that (Q_G, g_G) is a pseudo-Riemannian manifold.*

Proof. Let i and j be the natural projections of G into Q_G and let $V, W \in \sec TQ_G$. So the mappings $x \to \phi_M^{(V,W)}(x) = g_M(i_*^{-1}(V_x), i_*^{-1}(W_x))$, from M into $\mathbb{R}$, and $y \to \phi_N^{(V,W)}(y) = g_N(j_*^{-1}(V_y), j_*^{-1}(W_y))$ from N into $\mathbb{R}$, are smooth. By the *Mapping Lemmas* and Remark 16, there is a unique smooth mapping $p \to \phi^{(V,W)}(p)$ from Q_G into $\mathbb{R}$ such that $\phi^{(V,W)} \circ i = \phi_M^{(V,W)}|U$ and $\phi^{(V,W)} \circ j = \phi_N^{(V,W)}|V$. Hence, just define $g_G \in \sec T^2 Q_G$ to be such that $g_G(V_p, W_p) = \phi^{(V,W)}(p)$ and it is left to the reader to show why g_G is smooth. ■

Thus the title of this Appendix is justified by the fact that Q_G can be called, suggestively, the extension of the manifolds M and N.

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