# From Classical to Wave-Mechanical Dynamics

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ABSTRACT. The time-independent Schroedinger and Klein-Gordon equations - as well as any other Helmholtz-like equation - turn out to be associated with exact sets of Hamiltonian ray-trajectories (coupled by a "Wave Potential" function, encoded in their structure itself) describing any kind of wave-like features, such as diffraction and interference. This property suggests to view Wave Mechanics as a direct, causal and realistic, extension of Classical Mechanics, based on *exact* trajectories and motion laws of point-like particles "piloted" by de Broglie's mono-energetic matter waves and avoiding the probabilistic content and the wave-packets both of the standard Copenhagen interpretation and of Bohm's theory.

*RÉSUMÉ - Les équations indépendantes du temps de Schroedinger et de Klein-Gordon, ainsi que toutes les autres équations d'Helmholtz, sont associées à des systèmes de trajectoires hamiltoniennes couplées par une function (le "potentiel d'onde") codée dans leur structure même, qui permettent de décrire tous le phénomènes ondulatoires, comme le diffraction et l'interférence. Cette propriété suggère d'envisager la Mécanique Ondulatoire comme une extension directe, causale et réaliste de la Mécanique Classique (basée sur les trajectoires exactes de particules ponctiformes pilotées par des ondes materielles monochromatiques) évitant à la fois le probabilisme et les paquets d'ondes de l'interpretation de Copenhagen et de la théorie de Bohm.*

KEYWORDS : *Helmholtz equation - Electromagnetic waves - Eikonal approximation - Ray trajectories - Classical dynamics - Relativistic dynamics - Hamilton equations - Hamilton-Jacobi equations - Wave Mechanics - de Broglie's matter waves - Pilot waves - Schrödinger equation -*

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*Klein-Gordon equation - Quantum potential - Bohm's theory - Quantum trajectories - Wave potential.*

# 1 Introduction

> "[La Mécanique Quantique], que je connais bien, puisque je l'ai longtemps enseignée, est très puissante et conduit à un très grand nombre de prévisions exactes, mais elle ne donne pas, à mon avis, une vue exacte et satisfaisante des phénomènes qu'elle étudie. Cela est un peu comparable au rôle joué naguère par la thermodinamique abstraite des principes qui permettait de prévoir exactement un gran nombre de phénomènes et était par suite d'une grande utilité, mais qui ne donnait pas une idée exacte de la réalité moleculaire dont le lois de la thermodynamique des principes ne donnaient que les conséquences statistiques".
> (*Louis de Broglie, 1972* [**1**])

Any kind of monochromatic wave phenomena may be dealt with, as we shall see, in terms of an exact, *ray-based* kinematics, encoded in the structure itself of Helmholtz-like equations. The ray trajectories and motion laws turn out to be coupled by a dispersive "*Wave Potential*" function, which is responsible for *any typically wave-like features such as diffraction and interference*, while its absence or omission confines the description to the geometrical optics approximation.

We extend this wave property, in the present paper, to the case of Wave Mechanics, thanks to the fact that both the *time-independent* Schrödinger and Klein-Gordon equations (associating monochromatic *de Broglie's matter waves* [**2, 3**] to particles of assigned total energy) are themselves Helmholtz-like equations, allowing to formulate the Hamiltonian dynamics of point-like particles in terms of *exact trajectories and motion laws* under the coupling action of a suitable *Wave Potential*, in whose absence they reduce to the usual laws of classical dynamics. We make use of *relativistic* equations, in agreement both with de Broglie's firm belief that Wave Mechanics is an essentially relativistic theory [**4-7**] and with the first (unpublished) approach considered by Schrödinger, before his non-relativistic choice [**8, 9**].

As long as the association of exact ray-trajectories with any kind of monochromatic waves was not yet known, the intrinsically probabilistic interpretation of Wave Mechanics turned out to be the most plausible one : if indeed, according to de Broglie, the current interpretation "*ne donne pas une vue exacte et satisfaisante des phénomènes qu'elle étudie",* which is the hidden reality ? The newfound general possibility of ray-trajectories [**10-12**] appears now to suggest that an "*exacte et satisfaisante*" description may be obtained from *the same Wave-Mechanical equations* viewed so far as unavoidably probabilistic. We show in Sect.2 how to obtain, both in Classical and Wave-Mechanical cases, consistent sets of exact kinematic and/or dynamic ray-based Hamiltonian equations, and discuss in Sects.3 and 4 the proposal of viewing Wave Mechanics as an *exact, non-probabilistic* physical theory *running as close as possible to Classical Mechanics* and based on a Hamiltonian dynamical system whose treatment doesn't require any simultaneous solution of Schrödinger or Klein-Gordon equations.

We do not aim to develop here a systematic theoretical structure akin to the Copenhagen and/or Bohmian Mechanics, but to face - and, possibly, to avoid - the radical change of the classical vision involved by those interpretations.

*We are not proposing,* with respect to the Copenhagen and/or Bohmian interpretations, an equivalent route, nor a different level of approximation, nor a particular case : *we are proposing a different conception of physical reality.*

## 2 Dispelling commonplaces on wave trajectories

**2.1 -** We shall assume, in the following, both *wave monochromaticity* (strictly required by any such typically wave-like features as diffraction and interference) and *stationary media* (usually imposed by the experimental set-up). Although our considerations may be easily extended to most kinds of waves, we shall refer in this sub-Section, in order to fix ideas, to *classical electromagnetic waves* traveling according to a scalar wave equation of the simple form

$$\nabla^2\psi - \frac{n^2}{c^2}\ \frac{\partial^2\psi}{\partial t^2} = 0 \quad , \tag{1}$$

where $\psi(x,y,z,t)$ represents any component of the electric and/or magnetic field and $n(x,y,z)$ is the (time independent) refractive index of the medium.

By assuming

$$\psi = u\,(\vec{r},\omega)\,e^{-i\;\omega\;t}, \tag{2}$$

with $\vec{r} \equiv (x, y, z)$, we get from eq.(1) the well-known [**13**] Helmholtz equation

$$\nabla^2 u + (n\, k_0)^2 u = 0 \tag{3}$$

(where $k_0 = \frac{2\pi}{\lambda_0} \equiv \frac{\omega}{c}$), and look for solutions of the (quite general) form

$$u(\vec{r}, \omega) = R(\vec{r}, \omega)\;\; e^{\,i\,\phi\,(\vec{r},\omega)}, \tag{4}$$

with real $R(\vec{r}, \omega)$ and $\phi\,(\vec{r}, \omega)$, which represent respectively, *without any probabilistic meaning*, the amplitude and phase of the monochromatic waves.

*Contrary to the commonplace* that a treatment in terms of *ray-trajectories* is only possible for a limited number of physical cases (such as reflection and refraction) ascribed to the so-called *geometrical optics approximation,* eq.(3) was shown in Refs.[**10-12**] to determine the *stationary* frame on which an *exact*, *ray-based* description is possible. By defining, in fact, the *wave-vector*

$$\vec{k} = \vec{\nabla}\,\phi\,(\vec{r}, \omega) \quad , \tag{5}$$

a set of *rays*, orthogonal to the phase surfaces $\phi\,(\vec{r}, \omega) = const$, turns out to travel, in stationary media, along *stationary* trajectories given by a simple Hamiltonian system of *kinematical* equations (both ray geometry and motion laws : see *Appendix*), under the basic action of a “Wave Potential” function

$$W(\vec{r}, \omega) = \; - \frac{c}{2\,k_0} \frac{\nabla^2 R(\vec{r}, \omega)}{R(\vec{r}, \omega)}, \tag{6}$$

inducing a mutual *perpendicular coupling* between the relevant *monochromatic* ray-trajectories, *which is the one and only cause of wave-like features such as diffraction and interference.* An important consequence of this *perpendicularity* is the fact of leaving the intensity of the ray velocity unchanged, *confining the coupling action to a mere deflection*, while any possible variation of the speed amplitude is due to the refractive index of the medium. The limit of *geometrical optics* is reached when the space variation length $L$ of the wave amplitude $R(\vec{r}, \omega)$ turns out to satisfy the condition $k_0\, L >> 1$. In this case the role of the Wave Potential is negligible, and the rays travel independently from one another under the only action of the refractive index, according to the "*eikonal equation*" [**13**]

$$k^2 \equiv (\vec{\nabla}\;\phi)^2 \cong (n\;k_0)^2 \tag{7}$$

obtained in the *Appendix* as eq.(A7).

**2.2 -** Let us pass now to the *dynamics of single (spinless) particles* with rest mass $m_0$ and assigned energy $E$, launched into a force field deriving from a stationary potential energy $V(\vec{r})$. Their *relativistic* behavior is described by the *time-independent* Hamilton-Jacobi equation **[14-17]**

$$[\vec{\nabla}S(\vec{r},E)]^2 = [\frac{E-V(\vec{r})}{c}]^2 - (m_0\, c)^2 \tag{8}$$

where the basic property of the function $S(\vec{r},E)$ is that the particle momentum is given by the relation

$$\vec{p} = \vec{\nabla}S\,(\vec{r},E). \tag{9}$$

In other words, the Hamilton-Jacobi surfaces $S(\vec{r},E) = const$, perpendicular to the momentum of the moving particles, *"pilot" them, in Classical Mechanics, along a set of fixed trajectories,* determining also their motion laws.

One of the main forward steps in modern physics, giving rise to Wave Mechanics, was performed by de Broglie's association of *mono-energetic* material particles **[2, 3]** with suitable *monochromatic "matter waves",* according to the correspondence

$$\vec{p}/\hbar \equiv \vec{\nabla}S(\vec{r},E)/\,\hbar \rightarrow \vec{k} \equiv \vec{\nabla}\phi. \tag{10}$$

The Hamilton-Jacobi surfaces $S(\vec{r},E) = const$ were assumed therefore as the phase-fronts of these *matter waves*, while maintaining their original role of "piloting" the particles - *just as in Classical Mechanics* - according to eq.(9).

The successive step was the assumption **[8, 9]** that these monochromatic matter waves satisfy a Helmholtz-like equation of the form (3), and that *the laws of Classical Mechanics* - represented here by eq.(8) - provide *the eikonal approximation* of this equation. By recalling, therefore, eqs. (7)-(10), one may perform the replacement

$$(n\,k_0)^2 \cong k^2 \rightarrow p^2/\hbar^2 \equiv (\vec{\nabla}\frac{S}{\hbar})^2 \equiv [\frac{E-V(\vec{r})}{\hbar\,c}]^2 - (\frac{m_0\,c}{\hbar})^2 \tag{11}$$

into eq.(3), reducing it to the *time-independent* Klein-Gordon equation

$$\nabla^2 u + [(\frac{E-V}{\hbar\,c})^2 - (\frac{m_0\,c}{\hbar})^2]\,u = 0, \tag{12}$$

holding for the de Broglie waves associated with particles of total energy $E$ moving in a stationary external potential $V(\vec{r})$.

As is well known, the existence of the waves predicted by de Broglie was very soon confirmed by the experiments performed by Davisson and Germer on electron diffraction by a crystalline nickel target [**18**] : de Broglie's (mono-energetic) pilot waves were therefore shown to be *objective physical quantities* (in configuration space), as testified by their measurable properties.

*Contrary to the commonplace* that no *exact* particle trajectory may be defined in a Wave-Mechanical description, the same treatment providing the stationary ray-trajectories of the Helmholtz eq.(3) may now be applied to the particle trajectories associated with the *Helmholtz-like* eq.(12). Recalling eqs.(4) and (10), we assume therefore, in eq.(12),

$$u(\vec{r}, E) = R(\vec{r}, E)\, e^{\,i\, S(\vec{r},E)/\hbar} \quad , \tag{13}$$

where the real functions $R(\vec{r}, E)$ and $S\ (\vec{r}, E)$ represent, respectively, *without any probabilistic meaning*, the amplitude and phase of de Broglie's mono-energetic matter waves, whose objective reality is established beyond any doubt by their observed properties of diffraction and interference. After the separation of real and imaginary parts, eq.(12) splits then into the system

$$\begin{cases} \vec{\nabla} \cdot (R^2\, \vec{\nabla} S) = 2\, R\, \vec{\nabla}\, R\ \cdot\ \vec{\nabla} S\ +\ R^2\, \vec{\nabla}\ \cdot\ \vec{\nabla} S\ = 0 & (14) \\ (\vec{\nabla} S)^2 - [\dfrac{E-\ V}{c}]^2 + (m_0\, c)^2\ = \hbar^2 \dfrac{\nabla^2 R(\vec{r}, E)}{R(\vec{r}, E)} & (15) \end{cases}$$

and the differentiation $\frac{\partial H}{\partial \vec{r}} \cdot d\vec{r} + \frac{\partial H}{\partial \vec{p}} \cdot d\vec{p} = 0$ of the relation

$$H(\vec{r}, \vec{p}, E) \equiv V(\vec{r})\ \ +\ \ \sqrt{(pc)^2 + (m_0 c^2)^2 - \hbar^2 c^2 \frac{\nabla^2 R(\vec{r}, E)}{R(\vec{r}, E)}} = E \tag{16}$$

obtained from eq.(15) is seen to be satisfied by the exact and *self-*

*contained* Hamiltonian set of wave-dynamical equations

$$\begin{cases} \frac{d\vec{r}}{dt} = \frac{\partial H}{\partial \vec{p}} \equiv \frac{c^2\,\vec{p}}{E - V(\vec{r})} & (17) \\ \frac{d\vec{p}}{dt} = -\frac{\partial H}{\partial \vec{r}} \equiv -\vec{\nabla} V(\vec{r}) - \frac{E}{E - V(\vec{r})}\,\vec{\nabla} Q(\vec{r}, E) & (18) \\ \vec{\nabla} \cdot (R^2 \vec{p}) = 0 & (19) \end{cases}$$

with

$$Q(\vec{r}, E) = -\frac{\hbar^2\,c^2}{2\,E}\,\frac{\nabla^2 R(\vec{r}, E)}{R(\vec{r}, E)} \quad , \qquad (20)$$

describing the particle motion along a set of *stationary trajectories.* The exact dynamical laws $\vec{r}(E,t)$ *and* $\vec{p}(E,t)$ are completely determined by the initial conditions $\vec{r}(E,t=0)$ *and* $\vec{p}\,(E,t=0)$ and by the wave amplitude distibution $R(\vec{r}(E,\ t=0))$over the wave launching surface.

It is interesting to observe that eq.(17) coincides with the "*guidance velocity*" proposed by de Broglie in his relativistic "*double solution theory*" [**4-7**], and that $\vec{v} \equiv \frac{d\vec{r}}{dt} \neq \vec{p}/m$, although maintaining itself parallel to the momentum $\vec{p}$.

The function $Q(\vec{r}, E)$, which we call once more, for simplicity sake, *"Wave Potential",* has the same basic structure and coupling role of the function $W(\vec{r}, \omega) = -\frac{c}{2\,k_0}\frac{\nabla^2 R(\vec{r},\omega)}{R(\vec{r},\omega)}$ of eq.(6) : *it has therefore not so much a "quantum", as a "wave" origin,* entailed into quantum theory by de Broglie's matter waves. Just like the external potential $V(\vec{r})$, the Wave Potential $Q(\vec{r}, E)$ is "encountered" by the particles along their motion, and plays, once more, the basic role of *mutually coupling the trajectories* relevant to each mono-energetic matter wave. Once more, the *presence* of the Wave Potential is *the one and only cause of diffraction and/or interference of the waves*, and *its absence reduces the system* (17)-(19) *to the classical set of dynamical equations, which constitute therefore*, as expected, its *geometrical optics* approximation. Another interesting observation is that such phenomena as diffraction and interference *do not directly concern particles, but their (stationary) trajectories.* The overall number of traveling particles is quite indifferent, and may even be vanishingly small, so that, for instance, speaking of *self-diffraction* of a single particle is quite inappropriate. Each particle simply follows, according to the dynamical motion laws (17)-(19), the stationary trajectory, *pre-fixed* *from the very outset*, along which it's launched : although, in the XVIII century, Maupertuis attributed this kind of behaviour to the "*wise action of the Supreme Being*", we limit ourselves to state that it is encoded in the structure itself of Helmholtz-like equations.

Eq.(14) plays the double role of *"closing"* the Hamiltonian system (17)-(19) by providing step by step, after the assignment of the wave amplitude distribution $R(\vec{r}, E)$ over a launching surface, the necessary and sufficient condition for the determination of $R(\vec{r}, E)$ over the next wave-front (thus allowing a consistent "closure" of the Hamiltonian system), and of *allowing* to show (as in the previous case : see *Appendix*) that the coupling "force" $\vec{\nabla} Q(\vec{r}, E)$, of wave-like origin, is *perpendicular* to the particle momentum $\vec{p}$, so that that *no energy exchange is involved by its merely deflecting action* : any possible energy change may only be due to the external field $V(\vec{r})$. The assignment of the distribution $R(\vec{r}, E)$ on the launching surface has the role, in its turn, of describing the essentials of the experimental set up. Let us finally notice that, in the particular case of *massless* particles (i.e. for $m_0 = 0$), the Klein-Gordon equation (12), by assuming the Planck relation

$$E = \hbar\omega, \tag{21}$$

reduces to the form

$$\nabla^2 u \; + \; (n\,\omega\,/c)^2 \;\; u \; = \; 0, \tag{22}$$

with

$$n(\vec{r}, E) = 1 - V(\vec{r})/E. \tag{23}$$

Eq.(22) coincides with eq.(3), which may be therefore viewed as the time-independent Klein-Gordon equation holding for massless point-like particles.

**2.3** - The same procedure applied in Sect.2.2 to obtain the stationary *relativistic* Klein-Gordon equation (12) was applied by Schrödinger [**8, 9**] to obtain his *non-relativistic* time-independent equation

$$\nabla^2 u(\vec{r}, E) + \frac{2m}{\hbar^2}[E - V(\vec{r})]\; u(\vec{r}, E) = 0 \tag{24}$$

from the *non-relativistic* time-independent Hamilton-Jacobi equation

$$(\vec{\nabla} S)^2 = 2\,m\,[E - V(\vec{r})] \quad , \tag{25}$$

The association of eq.(24) with the *self-contained* non-relativistic Hamiltonian system

$$\begin{cases} \dfrac{d\,\vec{r}}{d\,t} = \dfrac{\partial H}{\partial \vec{p}} \equiv \dfrac{\vec{p}}{m} & (26) \\ \dfrac{d\,\vec{p}}{d\,t} = -\dfrac{\partial H}{\partial\,\vec{r}} \equiv -\,\vec{\nabla}[V(\vec{r}) + Q(\vec{r}, E)] & (27) \\ \vec{\nabla} \cdot (R^2\,\vec{p}) \equiv\, 0 & (28) \end{cases}$$

may be obtained along the same lines of the previous case, in terms of the trajectory-coupling "Wave Potential"

$$Q(\vec{r}, E) = \;-\,\frac{\hbar^2}{2m}\frac{\nabla^2 R(\vec{r}, E)}{R(\vec{r}, E)} \tag{29}$$

and of the Hamiltonian function

$$H(\vec{r}, \vec{p}, E) = \frac{p^2}{2m} + V(\vec{r}) + Q(\vec{r}, E). \tag{30}$$

The time-independent Schrödinger equation (24) directly provides, in conclusion, the *exact, non-probabilistic point-particle* Hamiltonian system (26)-(28), reducing to the usual (non-relativistic) point-particle dynamical description when the Wave Potential $Q(\vec{r}, E)$ is neglected, i.e., once more, in the limit of geometrical optics.

**2.4** - Many examples of numerical solution of the Hamiltonian particle dynamical system (26)-(28) in cases of diffraction and/or interference were given in Refs.[**10-12**], by assuming, for simplicity sake, the absence of external fields and a geometry allowing to limit the computation to the $(x, z)$-plane, for waves launched along the $z-$axis. The particle trajectories and the corresponding evolution both of de Broglie's wave intensity and of the Wave Potential were computed, with initial momentum components $p_x(t = 0) = 0$; $p_z(t = 0) = p_0 = 2\pi\hbar/\lambda_0$, by means of a *symplectic* numerical integration method. We limit ourselves to present in Fig.1, on the $(x, z)-$plane, the case of the diffraction of a *Gaussian* particle beam traveling along $z$ and starting, from a vertical slit centered at $x = z = 0$, in the form $R(x; z = 0) \div\; exp(-\,x^2/w_0^2)$, where the length $w_0$ is the so-called *waist radius* of the beam. We plot on the *left-hand side* of Fig.1 the particle trajectory pattern, and on the *right-hand* side the initial and final transverse intensity distributions of the wave.

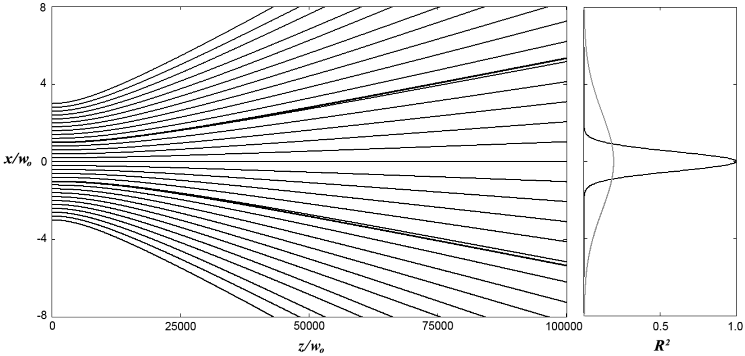


**Fig. 1 -** Particle trajectories and transverse initial and final wave intensity distribution on the $(x, z)$-plane for a Gaussian beam with $\lambda_0/w_0 = 2 \times 10^{-4}$.

The two heavy lines represent the *analytical approximation*

$$x(z) = \pm \sqrt{w_0^2 + \left(\frac{\lambda_0 \, z}{\pi \, w_0}\right)^2} \quad , \tag{31}$$

given by the so-called *paraxial* theory [**19**], of the trajectories starting (at $z = 0$) from the *waist positions* $x = \pm w_0$ . The agreement between the *analytical* expression (31) and the *numerical* results provides, of course, an excellent test of our approach and interpretation. It was shown in Ref.[**12**] that the uncertainty relation $\Delta x \; \Delta p_x > h$ *turns out to be violated close to the slit, but is asymptotically verified far enough from the slit,* thanks to the trajectory-coupling role of the Wave Potential $Q(\vec{r}, E)$.

## 3 The Copenhagen and Bohmian approaches

**3.1** - Let us now recall that, starting from eqs.(2), (21) and (24), one obtains [**14**] the *ordinary-looking* wave equation

$$\nabla^2 \psi = \frac{2m}{E^2} [E - V(\vec{r})] \; \frac{\partial^2 \psi}{\partial t^2}, \tag{32}$$

describing the propagation and dispersive character of *mono-energetic* de Broglie matter waves. By means, however, of the same eqs. (2), (21) and (24) one may also get the equation [**8, 9, 14, 15**]

$$\nabla^2\psi - \frac{2m}{\hbar^2}\, V(\vec{r})\,\psi = -\frac{2m}{\hbar^2} E\,\psi \equiv -\frac{2m\,i}{\hbar}\,\frac{E}{\hbar\,\omega}\,\frac{\partial\psi}{\partial t} = -\,\frac{2m\,i}{\hbar}\,\frac{\partial\psi}{\partial t}, \quad (33)$$

which is the usual form of the *time-dependent Schrödinger equation* for particles moving in a stationary potential field $V(\vec{r})$. Since eq.(33) is *not* a wave equation, any wave-like implication is due, in its case, to its connection with the *time-independent* Schrödinger equation (24), *from which it is obtained*. Eq.(24) admits indeed, as is well known, a (discrete or continuous, according to the boundary conditions) set of energy eigen-values and ortho-normal eigen-modes, which (referring for simplicity to the discrete case) we indicate respectively by $E_n$ and $u_n(\vec{r})$ ; and it's a standard procedure, making use of eqs.(2) and (21) and defining both the *eigen-frequencies* $\omega_n \equiv E_n/\hbar$ andthe *eigen-functions*

$$\psi_n(\vec{r},t) = u_n(\vec{r})\, e^{-i\,\omega_n\,t} \equiv u_n(\vec{r})\, e^{-i\,E_n\,t/\hbar} \quad , \quad (34)$$

to verify that any linear superposition (with constant coefficients $c_n$) of the form

$$\psi(\vec{r},t) = \sum_n c_n\; \psi_n(\vec{r},t), \quad (35)$$

is a general solution of eq.(33).

Since eqs.(32) and (33) hold *at same level of mathematical truism,* one may wonder what different roles they play in the treatment of de Broglie's waves.

At first glance, the time-dependent Schrödinger equation (33) appears to describe the deterministic evolution of an *arbitrary* superposition of monochromatic waves $\psi_n(\vec{r},t)$, each one of which travels according to a wave equation of the form (32) (with $E = E_n$) along the Helmholtz trajectories determined by the relevant *time-independent* Schrödinger equation (24).

Mainly because, however, of the energy-independence of eq.(33) and of the property of the $u_n(\vec{r})$ of constituting a complete ortho-normal basis, Born [**20**] proposed for the function (35) a role going much beyond that of a simple superposition : although eq.(33) is not - by itself - a wave equation, its solution (35) was assumed, under the name of "Wave-Function", to represent *the most complete description of the physical state of a particle whose energy is not determined* : a vision giving to eq.(33) a dominant role both with respect to eq.(32) and to eq.(24).

Even though “no generally accepted derivation has been given to date" [**21**], *this "Born Rule" aroused, together with Heisenberg's uncertainty relations, an intrinsically probabilistic conception of physical reality*, associating moreover to the *continuous and deterministic* evolution given by eq.(33) the *further Postulate* of a *discontinuous and probability-dominated* evolution process, after interaction with a measuring apparatus, in the form of a *collapse* (according to the probabilities $|c_n|^2$, in duly normalized form) into a single eigen-state. Because of the Born Rule, a sharp distinction is made, therefore, between a superposition of *independent* mono-energetic waves $\psi_n(\vec{r}, t)$ and their *inextricable coupling* (due to the assumption of $|\psi(\vec{r}, t)|^2$ as a probability density) in a single "Wave-Function" $\psi(\vec{r}, t)$ evolving as a whole : a statistical mixture which becomes even more inextricable (and non-local) for a system of N particles, still assumed to be described by a single time-dependent Schrödinger equation and Wave-Function.

The Born Rule provided a plausible interpretation of Schrödinger's equations (24) and (33) *as long as the possibility of associating exact ray-trajectories with any kind of Helmholtz-like equation was not yet known*. Such a newfound possibility suggests, however, the possibility of a less striking interpretation of physical reality, limiting itself to view the function (35) (in duly normalized form) as a simple average taken over a superposition of *independent mono-energetic* waves $\psi_n(\vec{r}, t)$, each one piloting point-like particles along exact Helmholtz trajectories pertaining to *their own* energy $E_n$, under the action of *their own* Wave Potential $Q(\vec{r}, E_n)$. Being a merely mathematical assembling of either observational or hypothetical information (the set of coefficients $c_n$), this average is not bound, of course, to respect the locality properties of classical observables. Notice that, in any case, the property of any $\psi_n(\vec{r}, t)$ of undergoing its own Wave Potential $Q(\vec{r}, E_n)$leads in general to the progressive spreading of the "wave-packet" $\psi(\vec{r}, t)$.

Reminding that, according to E.T. Jaynes [**22**], "*our present quantum mechanical formalism is (...) an omelette that nobody has seen how to unscramble, and whose unscrambling is a prerequisite for any future advance in basic physical theory*", the mono-energetic de Broglie waves - when considered as *independent* from one another (i.e. not *scrambled* together) - could allow a non-probabilistic description in terms of exact point-particle trajectories, providing a straightforward Wave-Mechanical extension of Classical Dynamics.

**3.2 -** It's worthwhile reminding that, although the *time-dependent* Schrödinger equation (33) is a simple consequence of the *time-independent* equation (24), its “stronger" version

$$\nabla^2\psi - \frac{2m}{\hbar^2}\, V(\vec{r},\ t)\,\psi = -\,\frac{2m\,i}{\hbar}\,\frac{\partial\psi}{\partial t} \quad , \tag{36}$$

containing a time-dependent external potential $V(\vec{r},t)$, may only be considered as a *Postulate*, and is often assumed, indeed, as a First Principle at the very beginning of standard textbooks.

Even though it cannot be obtained from de Broglie’s basic assumption (10), it was accepted by de Broglie himself [**5**] with these words : “*The form of* [eq.(33)] *allows us to go beyond single monochromatic waves and to consider superpositions of such waves. In addition, it suggests the way to extend the new Mechanics to the case of fields varying with time. Indeed, since it permits us to go beyond monochromatic waves, time no longer plays a special part, and* ***it is then natural to admit*** *that the form of the equation must be preserved when V depends on time as the general form of the equation of propagation of $\psi$ waves in the non-relativistic Wave Mechanics of a single particle*”. We limit ourselves to observe that eq.(36) cannot lose, in its induction from eq.(33), its statistical character.

**3.3 -** Let us finally come to the case of Bohm’s theory **[23-34],** and to its connections with the present analysis. Bohm’s first step was to express Born’s Wave-Function (35) (preserving its statistical interpretation) in the form

$$\psi(\vec{r},t) = R(\vec{r},t)\; e^{\;i\,G(\vec{r},t)/\hbar} \tag{37}$$

which leads, when introduced into eq.(36) - after separation of real and imaginary parts - to a couple of well-known fluid-like equations (which we shall omit here for brevity sake) where the main role is played by a so-called “Quantum Potential” term of the form

$$Q_B(\vec{r},t) = -\frac{\hbar^2}{2m}\,\frac{\nabla^2 R(\vec{r},t)}{R(\vec{r},t)} \tag{38}$$

*formally* coinciding with the mono-energetic $Q(\vec{r},E) = -\,\frac{\hbar^2}{2m}\frac{\nabla^2 R(\vec{r},E)}{R(\vec{r},E)}$ of eq.(29), of which it represents in fact, for stationary external potentials $V(\vec{r})$, a time-evolving average.

Bohm’s replacement (37) - shaped on eq.(13), i.e on de Broglie’s mono-energetic pilot-waves, *whose objective reality was established once* *and for all by the Davisson and Germer experiments* - depicts the function $\psi(\vec{r},t)$ of eq.(35) as a physical wave *hopefully sharing the same objective reality of de Broglie's (mono-energetic) waves.* Bohm's approach is indeed a strong attempt to dress with plausibility the Born Rule by presenting the Wave-Function $\psi(\vec{r},t)$ as the most general "pilot-wave".

The use of the Quantum Potential term is avoided (but implicit) in modern Bohmian Mechanics (which remains fully equivalent to Bohm's original formulation) by means of the time integration, starting from an initial position $\vec{r}(t=0)$, of the apodictic "guidance equation"

$$\frac{d\,\vec{r}(t)}{d\,t} = \vec{\nabla}\,G\,(\vec{r},t)/m \equiv \frac{\hbar}{m\,i}\,Im\,(\frac{\vec{\nabla}\,\psi}{\psi}) \equiv \frac{\hbar}{2\,m\,i}\,\frac{\psi^*\,\vec{\nabla}\,\psi - \psi\,\vec{\nabla}\,\psi^*}{\psi\,\psi^*}, \tag{39}$$

where the term $\vec{\nabla}\,G\,(\vec{r},t)$ is immediately obtained from eq.(37). Eq.(39) requires of course the simultaneous solution, step by step, of a time-dependent Schrödinger equation.

Since, as is shown in any standard textbook of Quantum Mechanics [**14**, **15**], the Born Wave-Function $\psi$ is associated with *a probability current density* of the form $\vec{J} \equiv \frac{\hbar}{2\,m\,i}\,(\psi^*\,\vec{\nabla}\,\psi - \psi\,\vec{\nabla}\,\psi^*)$ (a quantity whose statistical nature extends to *stationary states,* for which $\vec{J}$ may be shown to be time-independent), the vector field $\vec{\nabla}G/m$ turns out to coincide with $\vec{J}/(R^2)$, representing therefore, in the general case, the average velocity at which a *fluid-like probability density* is transported along the flux lines.

While, indeed, de Broglie's basic Ansatz $p \equiv \hbar\,\vec{k} = \vec{\nabla}S$, concerning *monochromatic matter waves*, draws its objective reality from the Davisson-Germer experiments, no experimental evidence allows to pass off the (most general) probabilistic expression (35) of Born's Wave-Function as a "pilot wave" with an exact particle momentum of the form $p = \hbar\,\vec{k} \equiv \vec{\nabla}G$.

We outline our comments on the Bohmian approach as follows :

1) the quantity $|\psi|^2$, in Bohm's own words, is "*a probability density belonging to a statistical ensemble*" ;

2) The Bohmian interpretation doesn't differ too much, therefore, from the "Copenhagen" one, which it basically endows with a visual representation of the lines of probability flow ;

3) A statistical wave appears to be logically inadequate to act as a force (deriving from Bohm's Quantum Potential) on the moving particle ;

4) In case the Bohmians remain apodictically convinced of having obtained a set of exact and objective point particle trajectories, holding in the most general case (35), they don't appear to have ever stressed the contradiction of this fact with the Uncertainty Principle ;

5) no distinction is made, in Bohmian works, *between the (exact) stationary case and the (statistic) general case (*35), and no perception is shown that, at least in the stationary case, the presence of exact trajectories would disprove the Uncertainty Principle ;

6) no Bohmian paper appears to have ever pointed out *the most important property of de Broglie's (monochromatic) pilot waves* : the presence of a wave-like Potential acting *perpendicularly* to the relevant trajectories, limiting itself to deflect the particle motion *without any energy exchange* ;

7) last not least, the integration of eq.(39) requires the simultaneous solution, step by step, of a Schrödinger time-dependent equation - a practical obstacle which is fully bypassed by our dynamical system (26)-(28), and may explain the exiguous number of trajectory patterns computed, for instance, in Ref.[**31**].

## 4 Discussion and conclusions

Our discussion is summarized in Tables I and II :

| **TAB.I**<br>EXACT (POINT-PARTICLE) DESCRIPTION | **TAB.II**<br>PROBABILISTIC (WAVE-PACKET) DESCRIPTION |
|---|---|
| $\frac{d\,\vec r}{d\,t} = \frac{\vec p}{m}$<br>$\frac{d\,\vec p}{d\,t} = -\,\vec\nabla[\,V(\vec r) - \frac{\hbar^2}{2\,m}\,\frac{\nabla^2 R(\vec r, E)}{R(\vec r, E)}]$<br>$\vec\nabla \cdot (R^2\,\vec p) = 0$ | $i\hbar\,\frac{\partial\,\psi}{\partial\,t} = \frac{\hbar^2}{2\,m}\;\nabla^2\,\psi \,+\, V(\vec r)\;\psi$<br>$\frac{d\,\vec r}{d\,t} = \frac{\hbar}{m\,i}\;\;Im\;\frac{\vec\nabla\,\psi(\vec r, t)}{\psi(\vec r, t)}$ |

TAB.I refers to *our own approach*, whose basic equations are *encoded in the structure itself of Schrödinger's time-independent equation*, and provide the *exact trajectories of point-particles* of assigned energy $E$, piloted by a de Broglie matter wave of amplitude $R$ (a wave whose objective reality is shown by its diffractive properties), without requiring the simultaneous solution of any Schrödinger equation;

TAB.II, referring to the *Bohmian approach*, provides, on the other hand, a set of *probability flux-lines* built up step-by-step by *the simultaneous solution, starting from an assigned wave packet, of Schrödinger's time-dependent equation.*

We are *not proposing,* in conclusion - with respect to the Copenhagen and/or Bohmian interpretations - *an equivalent route* (in the sense, for instance, of the Newtonian/Hamiltonian equivalence in Classical Mechanics), nor a different level of approximation, nor a particular case : *we are proposing a different conception of physical reality.*

The Bohmian Mechanics –whatever its role may be - is far from the spirit of the present paper, whose aim is to suggest *an exact, non-probabilistic, uncertainty-free Wave Mechanics, running as close as possible to Classical Mechanics.*

## APPENDIX

After the use of eq.(4) and the separation of real and imaginary parts, the Helmholtz eq.(3) splits into the system

$$\begin{cases} \vec{\nabla} \cdot (R^2\, \vec{\nabla}\, \phi) \equiv 2\, R\, \vec{\nabla}\, R\, \cdot \vec{\nabla}\phi \;+\; R^2\, \vec{\nabla} \cdot \vec{\nabla}\phi \;= 0 & (A1) \\ D(\vec{r},\ \vec{k},\ \omega) \equiv \dfrac{c}{2\,k_0}[k^2 - (n\,k_0)^2] + W(\vec{r},\omega) = 0 & (A2) \end{cases}$$

where

$$W(\vec{r},\omega) = \; - \frac{c}{2\,k_0}\frac{\nabla^2 R(\vec{r},\omega)}{R(\vec{r},\omega)} \qquad , \qquad (A3)$$

and the differentiation $\frac{\partial D}{\partial \vec{r}} \cdot d\vec{r} + \frac{\partial D}{\partial \vec{k}} \cdot d\vec{k} = 0$ of eq.(A2) is satisfied by the kinematic *self-contained* Hamiltonian system

$$\begin{cases} \dfrac{d\vec{r}}{dt} = \dfrac{\partial D}{\partial \vec{k}} \equiv \dfrac{c\vec{k}}{k_0} & (A4) \\ \dfrac{d\vec{k}}{dt} = -\dfrac{\partial D}{\partial \vec{r}} \equiv \vec{\nabla}[\dfrac{ck_0}{2} n^2(\vec{r},\omega) - W(\vec{r},\omega)] & (A5) \\ \vec{\nabla} \cdot (R^2\, \vec{k}) = 0 & (A6) \end{cases}$$

associating with the Helmholtz equation an *exact stationary set of trajectories* along which the *monochromatic* rays (each one characterized by its launching position and wave vector) are driven. Since no new trajectory may arise in the space region spanned by the wave trajectories (so that $\vec{\nabla} \cdot \vec{\nabla}\phi = 0$), eq.(A1) tells us that $\vec{\nabla} R \cdot \vec{\nabla}\phi = 0$ : the amplitude $R(\vec{r},\omega)$ of the monochromatic wave - together with its derivatives and functions, including $W(\vec{r},\omega)$ - is therefore distributed over the relevant wave-front, normal to $\vec{k} \equiv \vec{\nabla}\phi(\vec{r},\omega)$, and the coupling term $\vec{\nabla} W(\vec{r},\omega)$ acts *perpendicularly* to the relevant ray trajectories. Eq.(A1) provides moreover step by step, after the assignment of the wave amplitude distribution $R(\vec{r},\omega)$ over the launching surface, the necessary and sufficient condition for the determination of $R(\vec{r},\omega)$ over the next wave-front, thus allowing a consistent *"closure"* of the Hamiltonian system.

When, in particular, the space variation length $L$ of the wave amplitude $R(\vec{r},\omega)$ turns out to satisfy the condition $k_0 L >> 1$, eq.(A2) reduces to the well-known **[13]** *eikonal equation*

$k^2 \equiv (\vec{\nabla}\phi)^2 \cong (n\, k_0)^2$. (A7)

In this *geometrical optics approximation* the coupling role of the Wave Potential is neglected, and the rays travel independently from one another under the only action of the refractive index.

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