Ohm's Law and Maxwell's Equations

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RÉSUMÉ. Le rapport entre le champ électrique et la densité du courant dans la loi d'Ohm est différent dans sa structure, si on le compare à la connection entre les mêmes champs vectoriels selon les équations maxwelliennes. Est examinée la compatibilité des rapports divers dans des situations statiques et quasi-statiques. On constate que, en général, le champ électrique dans la loi d'Ohm ne peut pas être identifié avec le champ électrique tel qu'il est défini par les équations de Maxwell. Proposition d'expériences qui clarifient le problème.

ABSTRACT. The relation between electric field and current density in Ohm's law is different in structure compared to the connection following from Maxwell's equations between the same vector fields. The compatibility of the different relationships is investigated for static and quasistatic situations. It is found that, in general, the electric field in Ohm's law cannot be identified with the electric field as defined by Maxwell's equations. Experiments to clarify the problem are proposed.

KEYWORDS: Ohm's law; hydrodynamic flow; Maxwell's equations

1 Introduction

In Maxwell's Treatise [1] Ohm's law had the status of an extra 'Maxwell equation'. In modern textbooks, however, only a reduced set of equations is reproduced [2] as the basis of classical electrodynamics:

$$\nabla \cdot \vec{D} = \rho \tag{1}$$

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$$\nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j} \tag{2}$$

$$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{3}$$

$$\nabla \cdot \vec{B} = 0 \tag{4}$$

In order to solve these equations, constitutive relations are needed which are written in their simplest form as:

$$\vec{D} = \epsilon \, \vec{E} \tag{5}$$

$$\vec{B} = \mu \, \vec{H} \tag{6}$$

taking ϵ and μ as constants. Ohm's law, also formulated in its simplest version:

$$\vec{E} = \eta \, \vec{j} \tag{7}$$

is taken as an additional condition to be satisfied in matter, similarly like (5) and (6).

There is, however, a difference as to the status of (7) in comparison to (5) and (6): Whereas the constitutive relations may be considered as the definitions of the fields $\vec{D}$ and $\vec{H}$ which are necessary to make the set of equations (1 – 4) solvable, Ohm's law relates the electric field to the current density in a way which competes with the relation between the same vector fields following from (1– 6) after elimination of $\vec{D}, \vec{H}$ and $\vec{B}$:

$$\nabla \times \nabla \times \vec{E} = -\mu \, \frac{\partial}{\partial t} \left(\vec{j} + \epsilon \, \frac{\partial \vec{E}}{\partial t} \right) \tag{8}$$

$$\epsilon \, \nabla \cdot \vec{E} = \rho \tag{9}$$

The question arises whether (8) and (9) which hold both in matter and in vacuo are compatible with (7) holding in matter only.

We investigate this question for the static case (Sec. 2) and for the case of slowly varying fields where the 'displacement current' $\partial \vec{D} / \partial t$ in (2) may be neglected (Sec. 3). We find that (7) is at variance with (8) and (9) in general. This indicates that the electric field entering Ohm's law cannot be simply identified with the field as defined by Maxwell's equations. Experiments to clarify the problem are proposed.

2 A static electric field and Ohm's law

In a flashlight battery electric charges are driven to the poles by electrochemical processes resulting in a constant voltage across the poles. The electrostatic field outside the battery may be calculated from (9). It has the characteristics of a dipole field as sketched in Fig. 1a. At large

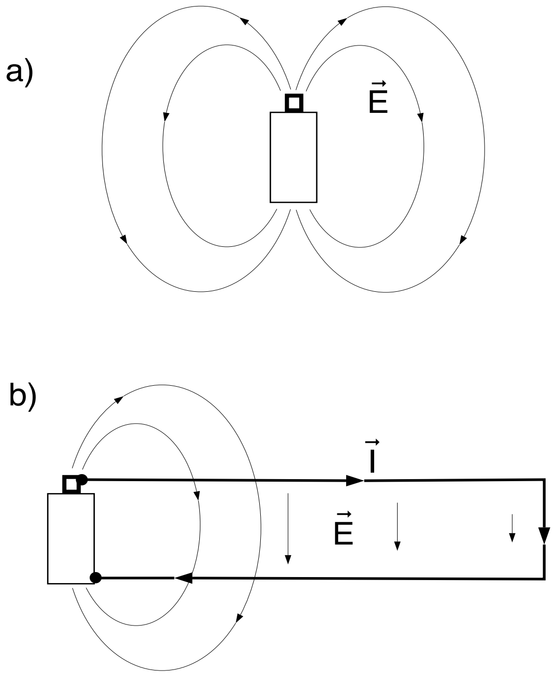

Figure 1 a) Electric field of a battery b) Electric field after connection to a wire loop distance the field strength is proportional to r^{-3}. When we connect the battery to a resistive wire loop, electrons flow from the negative to the positive pole through the wire. The structure of the electric field is drastically changed: There is now an electric field between the wire pieces opposite to each other with a field strength varying linearly with the distance along the wire (Fig. 1b). The flow of charges is maintained by the electromotive force in the battery so that the charge distribution on the poles remains unchanged, provided the internal resistance of the battery is small compared to that of the wire loop. According to (2) the divergence of the current density vanishes so that from (7) follows at constant resistivity:

$$\nabla \cdot \vec{E} = \eta \, \nabla \cdot \vec{j} = 0 \tag{10}$$

Substituting this into (9) we find that there is no additional charge density on the wire. It remains neutral as it was before its connection to the battery. With the charge distribution unaltered the electric field should be unchanged according to (9), but this is definitely not the case which may be easily verified by measuring the voltage between the wire pieces.

Equation (10) does not take into account any surface charges which could be responsible for a perpendicular field component between the wires. As it is difficult to find the appropriate surface charge distribution from the boundary conditions, we choose a simpler geometry which is sketched in Figure 2. In a cylindrical solid conductor (region 1) flows a current in z-direction. The current returns in a concentric hollow cylinder (region 3). The end, where the battery is connected, is assumed very far away, so that the dipole field is negligible in the region of interest where we want to calculate the electric field. It may be derived from a potential and is with (7) in region 1:

$$E_z = -\frac{\partial \phi}{\partial z} = \eta \, j_z = \frac{\eta \, I}{\pi \, a^2} \tag{11}$$

where I is the total current. Similarly, in region 3 we have a negative field along the conductor:

$$E_z = -\frac{\partial \phi}{\partial z} = -\frac{\eta \, I}{\pi \, (c^2 - b^2)} \tag{12}$$

In the vacuum region 2 Laplace's equation must hold:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{13}$$

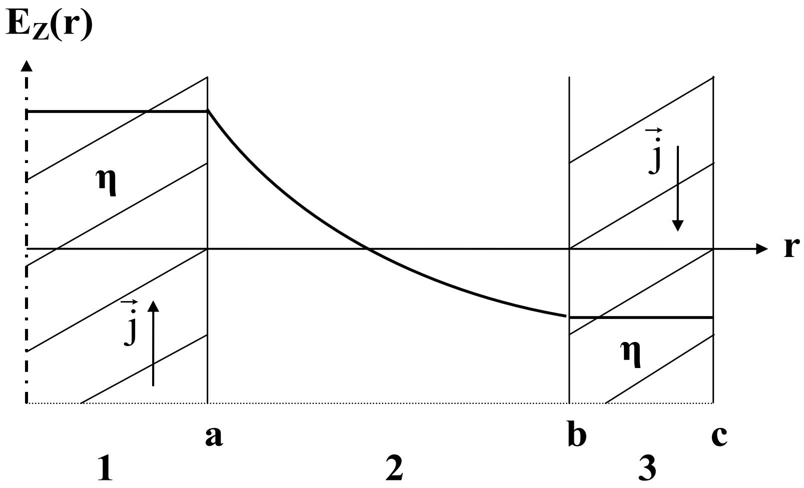

Figure 2 Electric field component in a long coaxial conductor

which is satisfied by the sum of two particular solutions:

$$\phi = z\ (A\ \ln r + B) \tag{14}$$

so that the electric field component in z-direction becomes $E_z = -A\ \ln r - B$ in region 2 as depicted in Fig. 2. Since the z-component of the electric field is continuous at the boundaries a and b we obtain from (11), (12), and (14):

$$\frac{\eta\, I}{\pi\, a^2} = -A\ \ \ln a - B \tag{15}$$

$$-\frac{\eta\, I}{\pi\ (c^2 - b^2)} = -A\ \ln b - B \tag{16}$$

which yields for the constants:

$$A\ \ln\frac{b}{a} = \frac{\eta\, I}{\pi}\left(\frac{1}{a^2} + \frac{1}{c^2 - b^2}\right) \tag{17}$$

$$B\ \ln\frac{b}{a} = -\frac{\eta\, I}{\pi}\left(\frac{\ln b}{a^2} + \frac{\ln a}{c^2 - b^2}\right) \tag{18}$$

The radial electric field $E_r = -\partial\phi/\partial r$ vanishes in region 1 and 3 according to (7), so that we must postulate surface charges sitting at the boundaries due to the discontinuity of the normal component of the electric field:

$$\sigma\,(a,\,z) = \epsilon\,E_r\,(a) = -\epsilon\,A\,\frac{z}{a}\quad,\quad \sigma\,(b,\,z) = -\epsilon\,E_r\,(b) = \epsilon\,A\,\frac{z}{b} \tag{19}$$

The potential produced by this surface charge distribution is then:

$$\phi = \frac{A}{4\pi}\int_0^{2\pi}\int_0^{L}\left[\frac{-z'}{d\,(a)} + \frac{z'}{d\,(b)}\right]dz'd\varphi \tag{20}$$

$$d\,(x) = \sqrt{x^2 + r^2 - 2\,x\,r\cos\varphi + (z - z')^2}$$

where L is the length of the conductor. Carrying out the integration over z' and choosing $L \gg z > 0$ we obtain:

$$\phi = \frac{A}{4\pi}\int_0^{2\pi}\left[d\,(a) - d\,(b) + z\,\ln\left(d\,(a) - z\right) - z\,\ln\left(d\,(b) - z\right)\right]d\varphi \tag{21}$$

Assuming furthermore $z \gg r$ yields:

$$\phi = \frac{A\,z}{4\,\pi}\int_0^{2\,\pi}\left[\ln\left(a^2 + r^2 - 2\,a\,r\cos\varphi\right) - \ln\left(b^2 + r^2 - 2\,b\,r\cos\varphi\right)\right]d\varphi \tag{22}$$

In the regions 1 and 3 the integral is independent[1] of r and vanishes for $r \geq b$. For $0 \leq r \leq a$ it becomes $\phi = -A\,z/\ln(b/a)$, or – with (11) – $E_z = A/\ln(b/a) = (\eta\,I)/\left(\pi\,a^2\right)$. This result is, however, incompatible with (17), since the constant A is independent of the boundary radius c according to (22) and (11), but depends on it according to (17). Another discrepancy arises at the boundary $r = b$: Since the potential vanishes for $r \geq b$, there would not be any electric field in the return conductor in contrast to (12). Hence, we must conclude that (7) as it stands is incompatible with (9) and (2) in the static case.

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[1]This result is most easily verified by numerical integration, or by expanding the integrand of (22) in powers of r or r^{-1}.

Sommerfeld [4] has also treated static current flow in a coaxial cylinder, but he chose $c \to \infty$ so that the electric field in region 3 vanishes in agreement with our result, and the constant A may be uniquely determined from either (17) or (22). Stumpf and Schuler [5] assumed c finite and obtained the surface charge density as given by (19). However, they did not evaluate the integral (20). Thus, they neither realized that the constant A as given by (17) is in disagreement with the constant as given by (22) and (11), nor that (22) predicts with (11) a vanishing field strength beyond $r = b$.

Our result is not very surprising when we realize that the task is to solve a Poisson equation for the potential in the whole conductor where the potential at the boundary $r = c$ is prescribed by (12) and, in addition, the normal derivative $\partial\phi/\partial r$ must vanish on the same boundary. This amounts to Cauchy type boundary conditions imposed on an elliptic differential equation. It is well known that this problem has no physical solution in general.

The total charge of the conductor vanishes according to (19) so that it appears neutral from outside without creating a finite scalar potential there. This implies that the z-component of the electric field vanishes outside of the conductor, but it is finite inside according to (12). A discontinuity of the tangential electric field, however, is excluded by Faraday's law (3) in steady state. The problem may also be analyzed in terms of the magnetic field created by a stationary current in a conductor with variable cross section. This is dealt with in the Appendix.

3 A slowly varying electric field and Ohm's law (skin effect)

We consider a case with a closed, slowly varying current so that $\partial\vec{D}/\partial t$ may be neglected [3], but not $\partial\vec{B}/\partial t$. The gradient part vanishes according to (1) or (9) as there are no free charges in the neutral conductor. This way we are left with (8) in the form:

$$\nabla \times \nabla \times \vec{E} = -\frac{\mu}{\eta}\frac{\partial \vec{E}}{\partial t} \qquad (23)$$

where we have substituted (7). Similar diffusion equations may be derived for the current density and the magnetic field. In the geometry of Figure 3 we have:

$$\frac{\mu}{\eta}\frac{\partial j_z}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial j_z}{\partial r}\right) \tag{24}$$

$$\frac{\mu}{\eta}\frac{\partial B_\varphi}{\partial t} = \frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial\,(r\,B_\varphi)}{\partial r}\right) \tag{25}$$

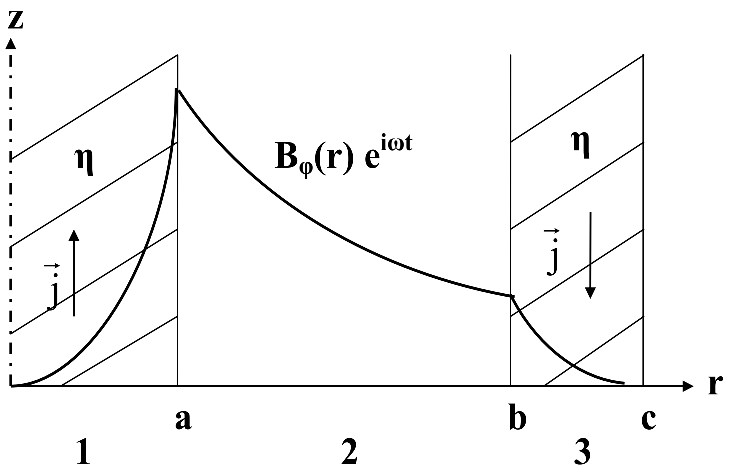

Figure 3 Oscillating magnetic field in a long coaxial conductor

According to (2) and (6) the magnetic field in region 2 of Fig. 3 is determined by the total oscillating current in the central conductor:

$$B_\varphi = \frac{\mu\,I\,e^{i\,\omega\,t}}{2\,\pi\,r} \tag{26}$$

With the ansatz $B_\varphi = B\,e^{i\,\omega\,t}$ and the new variable $x = k\,r$, $k^2 = i\,\omega\,\mu/\eta$ equation (25) becomes:

$$B = \frac{\partial}{\partial x}\left(\frac{1}{x}\,\frac{\partial\,(x\,B)}{\partial x}\right) \tag{27}$$

It is solved by modified Bessel functions:

$$B = C_1\,I_1\,(x) + C_2\,K_1\,(x) \tag{28}$$

In region 1 the constant $C_2^{(1)}$ must vanish in order to avoid a singularity on the axis. The other constant may be determined at the boundary $r = a$ with (26):

$$\frac{\mu\, I}{2\,\pi\, a} = C_1^{(1)}\, I_1\,(x_a) \tag{29}$$

At the inner boundary of region 3 we have similarly:

$$\frac{\mu\, I}{2\,\pi\, b} = C_1^{(3)}\, I_1\,(x_b) + C_2^{(3)}\, K_1\,(x_b) \tag{30}$$

and at $r = c$ the magnetic field must vanish, as there is no net current in the coaxial conductor:

$$0 = C_1^{(3)}\, I_1\,(x_c) + C_2^{(3)}\, K_1\,(x_c) \tag{31}$$

The resulting magnetic field is sketched in Fig. 3. It is concentrated near the boundaries due to the 'skin-effect' as described by the diffusion equation (25). The current density may be obtained from (2) and (6) by substituting (28):

$$j = \frac{k}{\mu}\left(\frac{1}{x}\,\frac{\partial\,(x\,B)}{\partial x}\right) = \frac{k}{\mu}\,\left(C_1\, I_0\,(x) - C_2\, K_0\,(x)\right) \tag{32}$$

This expression is a solution of (24) putting $j_z = j\, e^{i\,\omega t}$. The constants in regions 1 and 3 may be taken from (29 – 31).
In the vacuum region 2 we have from (3):

$$\frac{\partial B_\varphi}{\partial t} = \frac{\partial E_z}{\partial r} \tag{33}$$

which, upon integration over the radius, yields with (26):

$$E_z\,(x) = E_z\,(x_a) + \frac{i\,\omega\,\mu\, I}{2\,\pi}\;\ln\left(\frac{x}{x_a}\right) \tag{34}$$

Together with (7) we obtain a further boundary condition to be imposed on the current density:

$$j\;(x_b) = j\;(x_a) + \frac{I}{2\,\pi}\;k^2\;\ln\left(\frac{x_b}{x_a}\right) \tag{35}$$

which yields with (32):

$$C_1^{(3)}\, I_0\,(x_b) - C_2^{(3)}\, K_0\,(x_b) = C_1^{(1)}\, I_0\,(x_a) + \frac{\mu\, I}{2\,\pi}\,k\,\ln\left(\frac{x_b}{x_a}\right) \tag{36}$$

This condition over-determines the problem, since the constants are already determined by (29 – 31). As in Sec. 2 we must conclude that Ohm's law (7) is, in general, incompatible with relation (8) following from Maxwell's equations.

Our result is somewhat surprising, because the 'skin effect' equations do have unique solutions in special cases and predict – like solution (28) – that the magnetic field penetrates into the conductor only within a 'skin depth' of the order $\sqrt{\eta/\omega\,\mu}$. This is also confirmed by experiment to some extent. Joos [6] has calculated the case we have treated above, but he contented himself with the result (28) in region 1. The return conductor was left out of the consideration. In view of the over-determination represented by (36) we infer, however, that equations (1 – 7) require a modification which should make them compatible in general.

4 Discussion

From our analysis it appears that the electric field in Ohm's law (7) must be something else than the electric field defined by Maxwell's equations (1 – 6). In particular, it should not be confused with the static electric field as defined by (1) and (5) which is very obvious from our result in Sec. 2.

The microscopic picture of current flow suggests that electric currents should be conceived as moving electrons which are accelerated by an average field and stopped again by collisions. Each electron produces a magnetic field according to (2) and (6) which varies in time. As a result fluctuating electric fields are induced according to (3). On average these fields seem to add up to a quasistatic macroscopic field which ultimately enters into Ohm's law and may be expressed as the gradient of a potential. This is, however, not the electrostatic potential which is produced by a charge density according to (1) and (5). The divergence of the average macroscopic field in Ohm's law should, of course, vanish – in agreement with (10) – as it is a rotational field created microscopically by induction.

The situation is similar as in hydrodynamic flow. When water is driven through a pipe against viscosity, a pressure gradient along the pipe must be maintained. The pressure enters formally into the hydrodynamic equation of motion like a gravitational potential, but its origin is obviously of dynamical nature. It comes about by averaging over collisions between interacting atoms of the fluid. Due to the induced electric fields, the force range between interacting particles is in case of electric currents only longer than in case of colliding neutral atoms, but the resulting 'electric potential' is also of dynamical nature like the 'pressure potential'.

There is, of course, an important difference between hydrodynamic flow and electric current flow: The pressure gradient is confined to the interior of the pipe, whereas the gradient of the electric potential does not vanish also outside of the wire, as we have indicated in Figure 1b. To confirm this qualitative view it seems, however, worthwhile to perform local electrostatic measurements of the electric field in the space between current carrying conductors. In this way one would obtain a more detailed picture than if one measures just the voltage between the wire pieces by connecting galvanically a resistive voltmeter. This method distorts the electric field as it 'conducts' the electric potential to the instrument by the connecting wire like the pressure would be 'conducted' to a pressure meter when it is plumbed to a pipe which carries a water flow.

Our result in Sec. 3 poses a different problem. In the case considered we have only induced electric fields both inside and outside of the conductor. Although equations (23 – 25) predict what roughly seems to be observed, they still cannot be the correct description of what is actually going on, since they lead to contradicting solutions which do not satisfy all the boundary conditions to be imposed on the magnetic field and on the current density. The boundary conditions are, however, a consequence of Maxwell's equations and Ohm's law and cannot be chosen freely in most practical cases. We have no ready recipe how to repair this deficiency of the standard description of the skin effect.

As a first step to clarify the problem one should carry out careful measurements of the fields both outside and inside of conductors. The electric field should be measured not only with highly resistive wire loops, which permit to determine the induced rotational field, but also with electrometers which measure the irrotational field produced by charge distributions. In this way one could gain insight into the possible existence of surface charges. The magnetic field is obtainable from Hall probe measurements. In order to measure also inside of conductors, liquid materials should be used including ferromagnetic fluids. In this case the skin depth is strongly reduced and it is possible to work at relatively low frequencies.

Instead of modelling the experiments with the differential equations (1 – 7) one could take resort to Kirchhoff's formula:

$$U_n = R_n\, I_n + L_n \frac{dI_n}{dt} + \sum_m M_{mn} \frac{dI_m}{dt} \tag{37}$$

which holds for coupled current loops of small cross section. The real conductor may be modelled by many closely spaced loops which are subjected to an externally applied voltage, either induced or produced by an ac generator. The linear system of ordinary differential equations (37) has always a unique solution – provided the U_n's are given as 'boundary conditions' – which, however, is not the case for the system of differential equations (1– 7) as we have shown. Depending on the agreement of the model (37) with the field measurements, one could try to find out how the skin effect should be modelled in differential form and how the boundary conditions must be formulated such as to avoid the discrepancy which surfaced in equation (36).

5 Conclusion

We have shown that the electric potential produced by free charges according to Maxwell's equations is to be distinguished from the 'dynamic' electric potential which arises in a neutral conductor with static current flow due to collisions between electrons and lattice. Furthermore, it was found that the quasistatic diffusion equations for the magnetic field and the current density as derived from Maxwell's equations and Ohm's law have, in general, no consistent solution. Thus, we conclude that the electric field in Ohm's law cannot be simply identified with the electric field as defined by Maxwell's equations in terms of charge and current density. We propose detailed experiments to clarify the situation with the aim of finding a correct description of the skin effect in form of differential equations and boundary conditions.

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Appendix

Let us assume that a stationary current flows in a cylindrical conductor with uniform resistivity and varying cross section as sketched in Fig. 4. Taking the rotation of (7) and substituting (2), (3), and (6) we have $\nabla \times \nabla \times \vec{B} = 0$, or:

$$\Delta B_\varphi - \frac{B_\varphi}{r^2} = 0 \tag{A 1}$$

This Laplace equation in cylindrical coordinates has the solution

$$B_\varphi = \frac{\mu\, j_z\, r}{2} \tag{A 2}$$

in the region $z \geq h$ where the current density j_z is constant and the radial component of the current density vanishes. At the boundary $z = h$ the magnetic field must be continuous which yields the boundary condition for the region $z \leq h$:

$$B_\varphi = \frac{\mu\, j_z\, r}{2} \quad , \qquad z = h \;\; , \;\; 0 \leq r \leq r_1 \tag{A 3}$$

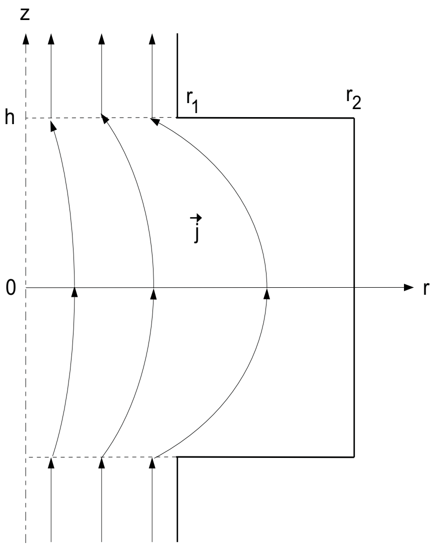

Figure 4 Current flow in a conductor with varying cross section

The other boundary conditions in this region are:

$$B_\varphi = \frac{\mu\, j_z\, r_1^2}{2\,r}\,, \qquad z = h\,,\;\; r_1 \le r \le r_2$$
$$B_\varphi = 0\,,\;\; r = 0\,, \qquad B_\varphi = \frac{\mu\, j_z\, r_1^2}{2\,r_2}\,,\;\; r = r_2 \qquad \text{(A 4)}$$
$$\frac{\partial B_\varphi}{\partial z} = -\mu\, j_r = 0\,,\quad z = 0\,,\;\; 0 \le r \le r_2$$

When (A1) is solved in the region $0 \le z \le h$ with the boundary conditions (A3) and (A4) a finite normal derivative will arise at the boundary $z = h$

$$\frac{\partial B_\varphi}{\partial z} = -\mu\, j_r\,,\quad z = h\,,\quad 0 \le r \le r_1 \qquad \text{(A 5)}$$

so that the radial component of the current density becomes discontinuous across the boundary as sketched in Fig. 4. This means with (7) that the radial component of the electric field must also become discontinuous in obvious contradiction to (3).

Rather than using the Diriclet condition (A3) one could solve equation (A1) with the Neumann condition

$$\frac{\partial B_\varphi}{\partial z} = 0\,, \quad z = h\,, \quad 0 \leq r \leq r_1 \tag{A 6}$$

in order to avoid the discrepancy. However, in this case the magnetic field would become a function of the radius at $z = h$ which is different from (A3). As a result the magnetic field would become discontinuous across the boundary which is only possible in presence of an infinite radial surface current density at $z = h$ because of (2) and (6). This is, of course, in contradiction to (A6).

Clearly, the elliptic equation (A1) cannot be solved by imposing both Neumann and Diriclet conditions on the same portion of the boundary. Hence, equations (1–7) imply an overspecification of the problem and do not permit a stationary solution.

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