L. de Broglie's double solution and self-gravitation

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ABSTRACT. We study the Schrödinger-Newton equation (a generalisation of the linear Schrödinger equation, which contains a self-focusing non-linear interaction of gravitational nature) in the light of de Broglie's double solution program. In particular we consider solutions of the Schrödinger-Newton equation which obey the so-called factorisation ansatz [24] according to which the full wave function is the product of a smoothly varying function with a peaked self-collapsed soliton. We show that these solitons obey a generalized de Broglie-Bohm guidance equation where the smooth function plays the role of the pilot-wave. We derive an Ehrenfest-like theorem for the Schrödinger-Newton equation in virute of which the trajectory of the soliton is classical, which brings us to conjecture the existence of a stochastic subquantum medium in order to explain the emergence of (non-classical) trajectories *à la* deBroglie-Bohm.

RÉSUMÉ. Nous étudions dans le présent travail l'équation de Schrödinger-Newton dans l'approche dite de la double solution proposée par Louis de Broglie en 1927. Cette équation constitue une généralisation de l'équation linéaire de Schrödinger et contient une interaction non-linéaire d'origine gravitationnelle. En particulier nous considérons des solutions satisfaisant l'ansatz de factorisation selon lequel la fonction d'onde du système est égale au produit d'une fonction à variation lente dans le temps et l'espace (solution de l'équation de Schrödinger linéaire que nous identifions avec l'onde pilote) avec une solution extrêmement piquée de type soliton que nous associons à la particule quantique. Nous dérivons un théorême d'Ehrenfest généralisé pour l'équation de Schrödinger-Newton en vertu duquel la trajectoire du soliton est classique, ce qui nous amène à conjecturer, afin de retrouver la dynamique de de Broglie-Bohm, l'existence d' un mileu stochastique subquantique.

1 Introduction

Self-gravitational interaction.

The so-called Schrödinger-Newton (S-N) equation[1] [36] reads

$$i\hbar\frac{\partial\Psi(t,\mathbf{x})}{\partial t} = -\hbar^2\frac{\Delta\Psi(t,\mathbf{x})}{2m} - Gm^2\int d^3x'(\frac{|\Psi(t,\mathbf{x}')|^2}{|\mathbf{x}-x'|})\Psi(t,\mathbf{x}), \qquad (1)$$

where G represents the Newton gravitational constant and m the mass of a quantum object. It has been intensively studied in the past, due to the self-focusing character of the non-linear potential which could possibly explain the wave function collapse, seen in this context as a self-localisation process [22, 42].

Our prior motivation is to study the manifestations of self-gravitation in presence of an external potential V^L where V^L represents the external potentials that are commonly considered when solving the linear Schrödinger equation (for instance electro-magnetic potentials). Contrary to the self-gravitational interaction which non-linearily depends on Ψ, V^L does not depend on Ψ. It is thus represented by a self-adjoint operator, linearily acting on the Hilbert space, as usually.

We shall thus assume that at the same time an external potential V^L acts on the particle together with a non-linear self-focusing potential V^{NL} of gravitational nature:

$$i\hbar\frac{\partial\Psi(t,\mathbf{x})}{\partial t} = -\hbar^2\frac{\Delta\Psi(t,\mathbf{x})}{2m} + V^L(t,\mathbf{x})\Psi(t,\mathbf{x}) + V^{NL}(\Psi)\Psi(t,\mathbf{x}), \qquad (2)$$

with $V^{NL}(\Psi) = -Gm^2\int d^3x'(\frac{|\Psi(t,\mathbf{x}')|^2}{|\mathbf{x}-\mathbf{x}'|})$, in the case of elementary particles[2].

[1] This equation is also often referred to as the (attractive) Schrödinger-Poisson equation [11, 35, 4] or the gravitational Schrödinger equation [36].

[2] It is worth noting that other choices of non-linear self-interaction are possible [28, 23, 14], which lead to essentially the same results as those derived in the present paper. For instance, in the case of an homogeneous sphere, whenever the mean width of the center-of-mass wave function is small enough in comparison to the size of the sphere, the gravitational self-interaction reduces, in a first approximation, to a non-linear harmonic potential (see [22, 51, 14]). Indeed, $-Gm^2/|\mathbf{x}-\mathbf{x}'|$ must be replaced in this case by

$$-G(\frac{M}{\frac{4\pi R^3}{3}})^2\int_{|\tilde{x}|\leq R,|\tilde{x}'|\leq R} d^3\tilde{x}d^3\tilde{x}'\frac{1}{|\mathbf{x}_{CM}+\tilde{\mathbf{x}}-(\mathbf{x}'_{CM}+\tilde{\mathbf{x}}')|}$$
$$\approx \frac{GM^2}{R}(-\frac{6}{5}+\frac{1}{2}(\frac{|\mathbf{x}_{CM}-\mathbf{x}'_{CM}|}{R})^2+\mathcal{O}((\frac{|\mathbf{x}_{CM}-\mathbf{x}'_{CM}|}{R})^3)).$$

If, for instance, we consider an electron, the self-gravitational potential is usually considered to be very weak and treated as a perturbation. The self-collapsed ground state (24) of (1) is usually predicted (see details in appendix) to be normalized to unity, in which case its size of the order of $\hbar^2/Gm^3$ (more or less 10^{32} meter in the case of an electron). The corresponding ground state energy (25) is of the order of $G^2m^5/\hbar^2$ which is very small (for instance, in the case of an electron it is very small compared to the usual energies of electronic orbitals in an atom). Now, this result is obtained by assuming that the norm of the ground state is equal to unity, which, following de Broglie, we consider to be a superfetatory condition (as discussed with more detail in the last section). Making use of the well-known scaling properties of equation (1) (see appendix C in the review paper by S. Colin, T.D. and R. Willox (same issue) and also [14, 48]) , it is actually possible to reduce arbitrarily the size of the self-collapsed ground state; at the same time, its norm will increase to plus infinity and its energy (which can be shown by a viriel-like reasoning to scale like the cube of the L_2 norm of the wave function, see e.g. section 5.1 in appendix) will tend to minus infinity.

Non-standard normalisation.

The first non-standard ingredient of our work is that we choose not to normalize to unity the L_2 norm of the wave function. Instead, we impose to begin with that the size of the self-collapsed ground state is very small (of the order of the Schwarzschild radius Gm/c^2, thus of the order of 10^{-57} meter in the case of an electron[3]). The ground state energy is then of the order of $-mc^2$. Contrary to the usual approach, in which external potentials are strong compared to the self-gravitational potential, in our case, the external potentials are supposedly weak, while the self-gravitational potential is strong and we treat the former as a perturbation. In particular the ground state will supposedly be very stable, because the stability analysis of the S-N equation in absence of external potential (1) shows that the non-linearity will inhibit the spreading unless the kinetic energy is at least of the order of the ground state energy [4, 48, 14]. We shall take for granted, without demonstration, that this is still approximately true in presence of an external potential V^L. Therefore we expect that in first approximation the state of the particle

[3]We were led to this choice by studying certain implications of our work that go beyond the scope of the present paper [24]. Making use of the scaling properties of (1) [14], the norm of the ground state is of the order of $(mc^2\hbar^2/G^2m^5)^{1/3} \approx 10^{30}$ in the case of an electron.

is, up to galilean boosts and translations, the static ground state, solution of (24). Moreover, the spectrum of negative energy solutions of (24) is discrete (see [6] and appendix), so that at usual temperatures, the transitions to excited static self-collapsed states are frozen.

What we are looking for is thus a solution of (2) in the form of a soliton (solitary wave) which, in first approximation, looks like the self-collapsed ground state of (1). In the rest of the paper we shall identify this highly concentrated peak of energy with the quantum particle itself, which is consistent with the aforementioned stability criterion: stability is menaced whenever an energy of the order of mc^2 is communicated to the system. It is worth noting that in our approach, contrary to the mainstream approach to self-localisation [22, 42, 14, 15], the system is supposedly collapsed to begin with, since arbitrary long times. This picture is reminiscent of the so-called de Broglie-Bohm (dB-B) causal interpretation of quantum mechanics. In particular, the fact that the solution is expected to have a very large norm and amplitude is reminiscent of de Broglie's double solution program[4], who wrote [20]

" ... a set of two coupled solutions of the wave equation: one, the Ψ wave, definite in phase, but, because of the continuous character of its amplitude, having only a statistical and subjective meaning; the other, the u wave of the same phase as the Ψ wave but with an amplitude having very large values around a point in space and which $(\cdots)$ can be used to describe the particle objectively."

The fact that in our approach the particle, represented by ϕ_{NL}, has a very small size is reminiscent of Bohm's description of particles as material points.

Our approach is also reminiscent of Poincaré's attempts [43] to explain the stability of the electron in terms of an internal self-attraction (the so-called Poincaré pressure), aimed at counterbalancing Coulomb self-repulsion. In our model, self-gravitation plays the role of the

[4]Louis de Broglie proposed in 1927 a realistic interpretation of the quantum theory in which particles are guided by the solution of the linear Schrödinger equation (Ψ_L), in accordance with the so-called guidance equation [20, 21]. The theory was generalised by David Bohm in 1952 [8, 9]. Certain ingredients of de Broglie's original idea disappeared in Bohm's formulation, in particular the double solution program, according to which the particle is associated to a wave u distinct from the pilot-wave Ψ_L. This program was never fully achieved, u being sometimes treated as a moving singularity [49], and sometimes as a solution ϕ_{NL} of a non-linear equation (see [21, 28] and the papers Fargue, and of Colin, Durt and Willox, same issue).

Poincaré pressure, and it counterbalances the spread of the soliton that we identify with the quantum particle[5].

Factorization ansatz.

In a first step, we tried to find a double solution *à la* de Broglie in the form of the sum of a wave function Ψ_L (where Ψ_L is a solution of the linear Schrödinger equation (4)) and of a soliton ϕ_{NL}. However, due to the intrinsic non-linearity of (2), we did not manage to derive interesting results.

This brings us to the second non-standard ingredient of our paper which is that we tried to solve (2) with an ansatz solution Ψ which factorizes (3) into the product of two functions Ψ_L and ϕ_{NL}:

$$\Psi(t, \mathbf{x}) = \Psi_L(t, \mathbf{x}) \cdot \phi_{NL}(t, \mathbf{x}), \tag{3}$$

for which we imposed that Ψ_L, the linear wave, is a solution of the linear Schrödinger equation (4):

$$i\hbar \cdot \frac{\partial \Psi_L(t, \mathbf{x})}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi_L(t, \mathbf{x}) + V^L(t, \mathbf{x}) \Psi_L(t, \mathbf{x}), \tag{4}$$

The factorization ansatz results from the recognition that, due to the fundamental non-linearity of the wave dynamics, a linear partition of the type $\Psi(t, \mathbf{x}) = \Psi_L(t, \mathbf{x}) + \phi_{NL}(t, \mathbf{x})$ is irrelevant. From this point of view the factorization ansatz incorporates non-linearity from the beginning.

Originally, this ansatz has been introduced by us [24] in order to describe the phenomenology of "walkers" (also called bouncing oil droplets[6]). In the case of droplets, our basic motivation for imposing

[5]Several physicists of the de Broglie school, Fer, Lochak, Andrade e Silva, Lochak and others developed in the past models mixing Poincaré and de Broglie views on the stability of particles (see for instance [21, 28, 29] and references therein as well as the papers of Fargue, Drezet and Colin, Durt and Willox, same issue)). These ideas are in a sense unavoidable whenever we try and describing particles as localised waves.

[6]These are macroscopic objects that exhibit certain quantum-like features. In particular their average trajectories seemingly obey a pseudo dB-B dynamics. In ref.[24], we simulated the properties of bouncing oil droplets by representing through $\Psi_L(t, \mathbf{x})$ the medium (oil bath) on which droplets propagate and through $\phi_{NL}(t, \mathbf{x})$ the droplets themselves. We derived in that paper an expression for the pseudo-gravitational interaction between two droplets, assuming from the beginning that dB-B guidance equation (12) was satisfied. In the present paper, we focus on the single particle (droplet) case. We aim here at deriving the dB-B guidance equation from the ansatz (3).

the factorization ansatz is that walkers prepared at different positions and represented by $\phi^i_{NL}(t,\mathbf{x})(i=1,2...)$ always see the same bath (environment) represented by $\Psi_L(t,\mathbf{x})$. In the same paper [24], we extended this idea to arbitrary quantum systems, for instance to elementary particles and/or atoms molecules and so on.

In our (wave monistic) approach, $\Psi(t,\mathbf{x})$ is assumed to represent the full reality of the quantum system. According to our factorization ansatz $\Psi(t,\mathbf{x})$ can be split into the particle represented by $\phi_{NL}(t,\mathbf{x})$ and in the "linear" wave represented by $\Psi_L(t,\mathbf{x})$. The spatial size of the particle being assumed to be extremely small (every particle is a tiny black hole in our approach [24]), the experimenter has supposedly no direct access to/control on their location. In our view, identical experimental preparations however result in the same value for $\Psi_L(t,\mathbf{x})$, which, considered so, does not represent the full information about the system but the information accessible to and controllable by the experimentalist. As we shall show, Ψ_L can be interpreted as a pilot-wave, while ϕ_{NL} behaves as a solitary wave moving, in good approximation, in accordance with a generalized dB-B guidance equation (12). Roughly summarized, our main results are the following:

Property 1

whenever ϕ_{NL} remains peaked throughout time in a sufficiently small region, its barycentre (from now on denoted $\mathbf{x_0}$) obeys, in good approximation, the generalized guidance equation

$$\begin{aligned}\mathbf{v}_{drift} &= \frac{\hbar}{m}\nabla\varphi_L(\mathbf{x_0}(t),t) + \frac{<\phi_{NL}|\frac{\hbar}{im}\nabla|\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>} \\ &= \mathbf{v}_{dB-B} + \mathbf{v}_{int.},\end{aligned} \tag{5}$$

which contains the well-known Madelung-de Broglie-Bohm contribution ($\mathbf{v}_{dB-B} = \frac{\hbar}{m}\nabla\varphi_L(\mathbf{x_0}(t),t)$) plus a new contribution due to the internal structure of the soliton ($\mathbf{v}_{int.} = \frac{<\phi_{NL}|\frac{\hbar}{im}\nabla|\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>}$).

Property 2

Denoting $\Psi_L = A_L e^{i\varphi_L}$, where A_L is a real amplitude and φ_L a real phase, and defining $\phi'_{NL} \equiv \phi_{NL}/A_L$ we find that the L_2 norm of ϕ'_{NL} remains constant throughout time in good approximation, while the solution Ψ obeys

$$\Psi(t, \mathbf{x}) \approx e^{i\varphi_L(t,\mathbf{x})}\phi'_{NL}(t, \mathbf{x}), \tag{6}$$

which confirms indirectly de Broglie's double solution program according to which the linear wave Ψ_L does not represent the particle. Here Ψ represents the particle and it is essentially equal to the product of the soliton ϕ'_{NL} (which plays the role of de Broglie's second solution here) with $e^{i\varphi_L}$, the phase of the "pilot-wave" Ψ_L. A_L, the amplitude of the linear wave function, plays here the role of an auxiliary computation tool.

What we shall not prove rigorously in the present paper is the stability of the soliton, in the sense that we assume from the beginning that the soliton remains peaked throughout time, due to the self-focusing nature of the non-linear self-interaction to which it is submitted. However, if this stability condition is satisfied, then the properties 1 and 2 can be established by lengthy but straightforward computations that we shall detail in the core of the paper. Actually, we independently established by numerical simulations, in a special case (homogeneous self-gravitating sphere in the case where the extent of the ground state is quite smaller than the radius of the sphere), that stability is de facto guaranteed while properties 1 and 2 are satisfied in very good approximation.

The paper is structured as follows.

In section 2 we derive the aforementioned properties 1 and 2, concerning the velocity of the barycentre of ϕ_{NL} and its scaling. In section 3.1 we present the confirmations of properties 1 and 2 obtained from numerical simulations. Those simulations also confirm the stability of the soliton, provided the self-focusing is strong enough. Moreover they show that in the classical, non-relativistic regime, the trajectories of the solitons are classical and do not obey the de Broglie guidance law. This property can be explained in terms of a generalized Ehrenfest's theorem. This leads us to formulate in section 3.2 a conjecture (main conjecture) according to which de Broglie guidance equation is valid "in average" due to the presence of an external stochastic field acting at the level of individual velocities. The last section is devoted to discussions and conclusions. In appendix (section 5.3), we attempt to generalize the previous results to Dirac's equation.

2 Factorisability ansatz, solitary waves and generalized dB-B guidance.

We now assume that at the same time an external potential acts on the particle together with a non-linear self-focusing potential of a gravitational nature. We simultaneously impose the factorisability ansatz. Therefore equations (2,3) are valid. Substituting (3) in (2) we get

$$\begin{aligned} & i\hbar \cdot ((\frac{\partial \Psi_L(t,\mathbf{x})}{\partial t})\phi_{NL}(t,\mathbf{x}) + \Psi_L(t,\mathbf{x}) \cdot (\frac{\partial \phi_{NL}(t,\mathbf{x})}{\partial t})) = \\ & -\frac{\hbar^2}{2m}\Delta\Psi_L(t,\mathbf{x}) \cdot \phi_{NL}(t,\mathbf{x}) \\ & -\frac{\hbar^2}{2m}(2\bigtriangledown\Psi_L(t,\mathbf{x}) \cdot \bigtriangledown\phi_{NL}(t,\mathbf{x}) + \Psi_L(t,\mathbf{x}) \cdot \Delta\phi_{NL}(t,\mathbf{x})) \\ + \quad & V^L\Psi(t,\mathbf{x}) + V^{NL}(\Psi)\Psi(t,\mathbf{x}), \end{aligned} \tag{7}$$

that, making use of the identity

$\bigtriangledown\Psi_L(t,\mathbf{x}) = (\bigtriangledown A_L(t,\mathbf{x}))e^{i\varphi_L(t,\mathbf{x})} + \Psi_L(t,\mathbf{x})i\bigtriangledown\varphi_L(t,\mathbf{x})$, we replace by a system of two equations[7]:

-the linear Schrödinger equation

$$i\hbar \cdot \frac{\partial \Psi_L(t,\mathbf{x})}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi_L(t,\mathbf{x}) + V^L(t,\mathbf{x})\Psi_L(t,\mathbf{x}),$$

and the non-linear equation

$$\begin{aligned} & i\hbar \cdot \frac{\partial \phi_{NL}(t,\mathbf{x})}{\partial t} = -\frac{\hbar^2}{2m} \cdot \Delta\phi_{NL}(t,\mathbf{x}) \\ - \quad & \frac{\hbar^2}{m} \cdot (i\bigtriangledown\varphi_L(t,\mathbf{x}) \cdot \bigtriangledown\phi_{NL}(t,\mathbf{x}) + \frac{\bigtriangledown A_L(t,\mathbf{x})}{A_L(t,\mathbf{x})} \cdot \bigtriangledown\phi_{NL}(t,\mathbf{x})) \\ + \quad & V^{NL}(\Psi)\phi_{NL}(t,\mathbf{x}) \end{aligned} \tag{8}$$

In order to solve the system of equations (4,8), it is worth noting that while the L_2 norm of the linear wave Ψ_L is preserved throughout time, because (4) is unitary, this is no longer true in the case of the non-linear wave ϕ_{NL}, because the terms mixing Ψ_L and ϕ_{NL} are not hermitian.

[7]This replacement is not one to one in the sense that there could exist solutions of equation (7) that do not fulfill the system (4,8). In any case, we focus on a particular class of solutions here.

By a straightforward but lengthy computation that we reproduce integrally in appendix, we established the following result:

The change of norm of ϕ_{NL} obeys

$$\frac{d < \phi_{NL}|\phi_{NL} >}{dt} \approx \frac{\hbar}{m}\Delta\varphi_L(t,\mathbf{x_0})\cdot < \phi_{NL}|\phi_{NL} >$$
$$-2\frac{\nabla A_L(t,\mathbf{x_0})}{A_L(t,\mathbf{x_0})}\cdot\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^*\frac{\hbar\nabla}{mi}\cdot\phi_{NL}(t,\mathbf{x}). \tag{9}$$

2.1 Property 1

Let us now consider the barycentre $\mathbf{x_0}$ of the soliton: $\mathbf{x_0} \equiv \frac{<\phi_{NL}|\mathbf{x}|\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>}$ in order to estimate its velocity $\mathbf{v}_{drift}$:

$$\mathbf{v}_{drift} \equiv \frac{d(\frac{<\phi_{NL}|\mathbf{x}|\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>})}{dt} \tag{10}$$

For instance, if we consider its z component:

$z_0 = \frac{<\phi_{NL}|z|\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>}$ and

$\frac{dz_0}{dt} = \frac{1}{<\phi_{NL}|\phi_{NL}>}\frac{d<\phi_{NL}|z|\phi_{NL}>}{dt} - \frac{z_0}{<\phi_{NL}|\phi_{NL}>}\frac{d<\phi_{NL}|\phi_{NL}>}{dt}$,

so that we find (making use of (9) as well as of results in appendix, section 5.2)

$$\begin{aligned}
\frac{dz_0}{dt} &= \frac{1}{< \phi_{NL}|\phi_{NL} >}\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^*\frac{\hbar\nabla_\mathbf{z}}{mi}\cdot\phi_{NL}(t,\mathbf{x}) \\
+ &\frac{1}{< \phi_{NL}|\phi_{NL} >}\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^*(\frac{\hbar\nabla_\mathbf{z}}{m}\cdot\varphi_L(t,\mathbf{x}))\phi_{NL}(t,\mathbf{x}) \\
+ &\frac{1}{< \phi_{NL}|\phi_{NL} >} < \phi_{NL}|(\frac{\hbar}{m}\Delta\varphi_L(t,\mathbf{x}))\cdot z|\phi_{NL} > \\
+ &\frac{1}{< \phi_{NL}|\phi_{NL} >}\frac{\hbar}{im}\int d^3\mathbf{x}\frac{\nabla A_L(t,\mathbf{x})}{A_L(t,\mathbf{x})}\cdot\nabla(\phi_{NL}(t,\mathbf{x}))^*\cdot z\cdot\phi_{NL}(t,\mathbf{x}) \\
- &\frac{1}{< \phi_{NL}|\phi_{NL} >}\frac{\hbar}{im}\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^*\cdot z\cdot\frac{\nabla A_L(t,\mathbf{x})}{A_L(t,\mathbf{x})}\cdot\nabla\phi_{NL}(t,\mathbf{x})) \\
- &\frac{z_0}{< \phi_{NL}|\phi_{NL} >}\cdot(\frac{\hbar}{m})\Delta\varphi_L(t,\mathbf{x_0})\cdot < \phi_{NL}|\phi_{NL} > \qquad (11) \\
+ &2\frac{z_0}{< \phi_{NL}|\phi_{NL} >}\frac{\nabla A_L(t,\mathbf{x_0})}{A_L(t,\mathbf{x_0})}\cdot\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^*\frac{\hbar\nabla}{mi}\cdot\phi_{NL}(t,\mathbf{x}))
\end{aligned}$$

Now, $\frac{\hbar}{im}\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^* \cdot z \cdot \frac{\nabla A_L(t,\mathbf{x})}{A_L(t,\mathbf{x})} \cdot \nabla\phi_{NL}(t,\mathbf{x})$
$\approx z_0 \frac{\nabla A_L(t,\mathbf{x_0})}{A_L(t,\mathbf{x_0})} \int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^* \frac{\hbar\nabla}{mi} \cdot \phi_{NL}(t,\mathbf{x}))$,
while
$< \phi_{NL}|(\frac{\hbar}{m})\Delta\varphi_L(t,\mathbf{x}) \cdot z|\phi_{NL} > \approx z_0 \cdot (\frac{\hbar}{m})\Delta\varphi_L(t,\mathbf{x_0}) \cdot < \phi_{NL}|\phi_{NL} >$
and so on so that finally only the two first lines of (11) survive. We get thus the generalized dB-B guidance equation (5), which constitutes the **Property 1**:

$$\begin{aligned}\mathbf{v}_{drift} &= \frac{\hbar}{m}\nabla\varphi_L(\mathbf{x_0}(t),t) + \frac{< \phi_{NL}|\frac{\hbar}{im}\nabla|\phi_{NL} >}{< \phi_{NL}|\phi_{NL} >} \\ &= \mathbf{v}_{dB-B} + \mathbf{v}_{int.}.\end{aligned}$$

$\mathbf{v}_{drift}$ contains the de Broglie-Bohm velocity

$$\mathbf{v}_{dB-B} \equiv \frac{\hbar}{m}\nabla\varphi_L(\mathbf{x_0}(t),t),, \tag{12}$$

and the internal velocity

$$\mathbf{v}_{int.} \equiv \frac{< \phi_{NL}|\frac{\hbar}{im}\nabla|\phi_{NL} >}{< \phi_{NL}|\phi_{NL} >}. \tag{13}$$

(12) is nothing else than de Broglie-Bohm's guidance equation [34], while $\mathbf{v}_{int.}$ can be considered as a contribution to the average velocity originating from the internal structure of the soliton. Both contributions to the drift are evaluated at the barycentre of the soliton, $\mathbf{x_0}$.

2.2 Property 2

Let us now consider the change of norm of ϕ_{NL}.

To do so, we introduce the total time derivative of A_L ($\frac{dA_L}{dt} = \frac{\partial A_L}{\partial t} + \mathbf{v}_{drift} \cdot \nabla A_L$) where $\mathbf{v}_{drift} = \frac{\frac{d<\phi_{NL}|\mathbf{x}|\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>}}{dt}$ obeys the generalized dB-B guidance equation (5).

By a direct computation, we find

$$\frac{\frac{dA_L}{dt}}{A_L} = \frac{1}{A_L}(\frac{\partial A_L}{\partial t} + \nabla A_L \cdot \frac{\hbar\nabla}{m} \cdot \varphi_L(t,\mathbf{x_0})) + \frac{1}{A_L}\nabla A_L \cdot \frac{1}{< \phi_{NL}|\phi_{NL} >}\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^* \frac{\hbar\nabla}{mi} \cdot \phi_{NL}(t,\mathbf{x}) \tag{14}$$

Making use of the conservation equation of the linear Schrödinger equation $\frac{\partial A_L^2}{\partial t} = -div(A^2 \frac{\hbar \bigtriangledown}{m} \cdot \varphi_L(t, \mathbf{x_0}))$ we find

$\frac{1}{A_L}(\frac{\partial A_L}{\partial t} + \bigtriangledown A_L \cdot \frac{\hbar \bigtriangledown}{m} \cdot \varphi_L(t, \mathbf{x_0})) = \frac{-1}{2} div(\frac{\hbar \bigtriangledown}{m} \cdot \varphi_L(t, \mathbf{x_0}))$ and we can rewrite (14) as follows:

$$\frac{\frac{dA_L}{dt}}{A_L} = \frac{-1}{2}\frac{\hbar}{m}\Delta\varphi_L(t, \mathbf{x_0})$$
$$+\frac{\bigtriangledown A_L}{A_L} \cdot \frac{1}{< \phi_{NL}|\phi_{NL} >} \int d^3\mathbf{x}(\phi_{NL}(t, \mathbf{x}))^* \frac{\hbar \bigtriangledown}{mi} \cdot \phi_{NL}(t, \mathbf{x}) \tag{15}$$

Making use of (9) (derived in appendix, section 5.2), we obtain at the end

$\frac{\frac{dA_L}{dt}}{A_L} = \frac{-1}{2} \frac{1}{<\phi_{NL}|\phi_{NL}>} \frac{d<\phi_{NL}|\phi_{NL}>}{dt}$ so that, finally,

$$\frac{\frac{d<\phi_{NL}|\phi_{NL}>}{dt}}{< \phi_{NL}|\phi_{NL} >} = -2\frac{\frac{dA_L}{dt}}{A_L}. \tag{16}$$

From the constraint (16) we infer the

Property 2

$$\frac{< \phi_{NL}|\phi_{NL} > (t)}{< \phi_{NL}|\phi_{NL} > (t = 0)} = \frac{A_L^2(t = 0)}{A_L^2(t)}, \tag{17}$$

where we evaluate $A_L^2(t)$ at the barycentre of ϕ_{NL}, which moves according to the generalized dB-B guidance equation (5). Let us rescale $\phi_{NL}(t, \mathbf{x})$ by defining ϕ'_{NL} through $\phi_{NL}(t, \mathbf{x}) \equiv \phi'_{NL}(t, \mathbf{x})/A_L$; we can thus predict in general that, if it exists and remains peaked during its evolution, the solution of (8) has the form

$$\Psi(x, y, z, t) \approx \phi'_{NL}(\mathbf{x}, t)e^{i\varphi_L(\mathbf{x},t)}, \tag{18}$$

where $\phi'_{NL}(\mathbf{x}, t)$ is centred in $\mathbf{x}_0(t = 0) + \int_0^t dt\mathbf{v}_{drift}$ and is of constant L_2 norm. Now, $V^{NL}(\Psi) = V^{NL}(\phi'_{NL}(t, \mathbf{x}))$, in virtue of (28) (appendix)

and (8) can then be cast in the form

$$
\begin{aligned}
& i\hbar \cdot \frac{\partial(\phi'_{NL}(t,\mathbf{x})/A_L(t,\mathbf{x_0}))}{\partial t} = \\
& -\frac{\hbar^2}{2m} \cdot \Delta(\phi'_{NL}(t,\mathbf{x})/A_L(t,\mathbf{x_0})) + V^{NL}(\phi'_{NL})(\phi'_{NL}(t,\mathbf{x})/A_L(t,\mathbf{x_0})) \\
& -\frac{\hbar^2}{m} \cdot i\bigtriangledown\varphi_L(t,\mathbf{x}) \cdot \bigtriangledown(\phi'_{NL}(t,\mathbf{x})/A_L(t,\mathbf{x_0})) \\
& -\frac{\hbar^2}{m} \cdot \frac{\bigtriangledown A_L(t,\mathbf{x})}{A_L(t,\mathbf{x})} \cdot \bigtriangledown(\phi'_{NL}(t,\mathbf{x})/A_L(t,\mathbf{x_0})). \qquad (19)
\end{aligned}
$$

In order to say more about $\phi'_{NL}(\mathbf{x},t)$ we must solve equation (19) which is a complicated problem, beyond the scope of our paper.

Remark.

As has been noted in [23], when there is no external potential (V^L), which means that the particle is only submitted to its self-interaction, we find the exact solution $\phi^0_{NL}(\mathbf{x}-\mathbf{v}\cdot t)e^{-i((E_0+\hbar\omega)\cdot t-\hbar\mathbf{k}\cdot\mathbf{x})/\hbar}$. This is a plane wave moving at velocity v_{dB-B}. As already noted by Fargue [28] many years ago, such solutions exist for a large class of different non-linearities (e.g. a non-linearity proportional to $|\Psi|^2$), and they all agree with the principle of phase concordance proposed by de Broglie in 1927. Actually, this class of solution is well-known and it can be generated from the static solution $\phi^0_{NL}(\mathbf{x})e^{-iE_0t/\hbar}$ by a Galilean boost. In the present case, this property is seen to be a direct consequence of the Galilean invariance of equation (1).

3 Main conjecture.

3.1 Numerical simulations.

Normalisation and Born rule.

One could object that in order to fit to the constraints required by our model, in particular in order to ensure that the size of the soliton is quite smaller than the size of the linear wave, the coupling constant ought to be huge. However this is not true. We are free to rescale the solitonic solutions of (24) without being constrained by the normalisation to unity of the wave function, which is a condition imposed by the Born rule. In our case, the Born rule is not postulated to begin with, it should rather be derived from the so-called equilibrium condition according to which the statistical distribution of the positions of the particles asymptotically

converges in time to the Born distribution in $|\Psi_L|^2$. In our eyes, the equilibrium condition ought to be derived from the generalized dB-B dynamics as is done in e.g. classical chaos theory[8]. It is well-known for instance from the study of deterministic chaotic systems that the sensitivity to initial conditions is an essential ingredient for generating stochasticity and impredictability. This ingredient is present in the dB-B [25] dynamics too.

Self-gravitating nanosphere in an external one-dimensional harmonic trap.

We shall now briefly mention some results obtained by us regarding the properties of a self-gravitating nanosphere placed in an harmonic trap, relaxing the constraint on the normalisation of the wave function. For convenience, we treat the system as an homogeneous sphere, in the limit where the extent of the wave function of its center of mass is quite smaller than its radius. In this case, $V^L = k^{ext.} x^2/2$ while (see [22, 51, 14] and discussion of footnote 2).

$$V^{NL}(x) = \frac{GM^2}{R}\int dx'|\Psi(t,x')|^2(-\frac{6}{5}+\frac{1}{2}(\frac{|x-x'|}{R})^2+\mathcal{O}((\frac{|x-x'|}{R})^3))$$
$$\approx N^2(\frac{-6GM^2}{5R}+\frac{GM^2}{2R^3}(<x^2>-<x>^2))+\frac{GM^2}{2R^3}N^2(x-<x>)^2, \quad (20)$$

with $N^2 = \int dx'|\Psi(t,x')|^2$,
$<x>= \int dx'|\Psi(t,x')|^2 x'/N^2$ and $<x^2>= \int dx'|\Psi(t,x')|^2 (x')^2/N^2$.

Denoting $k^{self} = \frac{GM^2}{2R^3}N^2$ and reexpressing the Hamiltonian up to irrelevant position-independent factors, we get the following evolution equation:

$$i\hbar\frac{\partial\Psi(t,x)}{\partial t} = -(\frac{\hbar^2}{2m})\frac{\partial^2\Psi(t,x)}{\partial x^2}+(\frac{k^{ext.}}{2}x^2+\frac{k^{self}}{2}(x-<x>)^2)\Psi(t,x). \quad (21)$$

Numerical simulations.

[8]There exist serious attempts to derive the onset of quantum equilibrium from the de Broglie-Bohm mechanics [47, 26, 13, 16, 17, 1], but this is a deep and complex problem, reminiscent of the H-theorem of Boltzmann, which opens the door to a Pandora box that we do not wish to open here (see papers of Colin, Durt and Willox, of Drezet and of Efthymiopoulos, same issue).

The main advantage of this simple model is that it is gaussian: initially gaussian states remain gaussian throughout their temporal evolution under (21). Therefore we were able to numerically solve it with high accuracy, which would be impossible with for instance the single particle N-S equation (1) [32].

N^2, the square of the L_2-norm of Ψ ($N^2 = \int dx' |\Psi(t,x')|^2$) is a conserved quantity under the evolution (21); it is however a free parameter in our approach (in agreement with the discussion of the previous paragraph), which means that we are essentially free to choose the value of the effective self-gravitating constant k^{self}. Varying this free parameter, we studied the behavior of gaussian solutions (denoted Ψ^{NL}) of the non-linear equation (21). We also decomposed the initial gaussian states $\Psi^{NL}(t, x = 0)$ according to the ansatz (3):

$$\Psi^{NL}(x, t = 0) = \Psi^L(x, t = 0) \cdot \phi_{NL}(x - < x >_0, t = 0),$$

where $\phi_{NL}(x - < x >_0, t = 0)$ is a gaussian real state, of which the shape is close to the shape of the ground state $\phi^0_{NL}(x)$ of the "free" equation $i\hbar \frac{\partial \Psi(t,x)}{\partial t} = -(\frac{\hbar^2}{2m}) \frac{\partial^2 \Psi(t,x)}{\partial x^2} + (\frac{k^{self}}{2}(x - < x >)^2)\Psi(t,x)$, which is [14] a real gaussian state centered in $< x >_0$ of extent $\sqrt{\frac{\hbar}{\sqrt{k^{self} \cdot m}}}$. $\Psi^L(x,t)$ and $\Psi^{NL}(x,t)$ were obtained by integrating respectively the linear equation $i\hbar \frac{\partial}{\partial t}\Psi^L(t,x) = -(\frac{\hbar^2}{2m}) \frac{\partial^2}{\partial x^2}\Psi^L(t,x) + (\frac{k^{ext.}}{2}x^2)\Psi^L(t,x)$ and equation (21). $< x >_0$ was chosen freely, excepted that we imposed that $\Psi^L(x, t = 0) = \Psi^{NL}(x, t = 0)/\phi^{NL}(x, t = 0)$ was a normalisable (gaussian) state. This allowed us to estimate the solitonic wave function $\phi^{NL}(x,t) = \Psi^{NL}(x,t)/\Psi^L(x,t)$ at all times. Our first result concerns stability: our criterion for stability is that $\phi^{NL}(x,t) = \Psi^{NL}(x,t)/\Psi^L(x,t)$ is a normalisable (gaussian) state. This means that the soliton remains peaked during time. We noticed that when $k^{self}/k^{ext.}$ was high enough (larger than 10 was obviously sufficient), stability was ensured. Not surprisingly, in this case, the self-gravitation forces the soliton to oscillate around its center, without ever escaping to the self-focusing gravitational potential $\frac{k^{self}}{2}(x - < x >)^2$.

The results of our numerical study are encapsulated in the figure 1 [40] which represents the velocity $v_{drift} \equiv \frac{d}{dt}(< x_{NL} >)$ (with $< x_{NL} >= \int d^3x' |\phi^{NL}(t,x')|^2 x'/N^2_{NL}$) of the centre of the soliton $\phi^{NL}(x,t)$, in function of time.

To derive the plot 1, we imposed the following constraints:

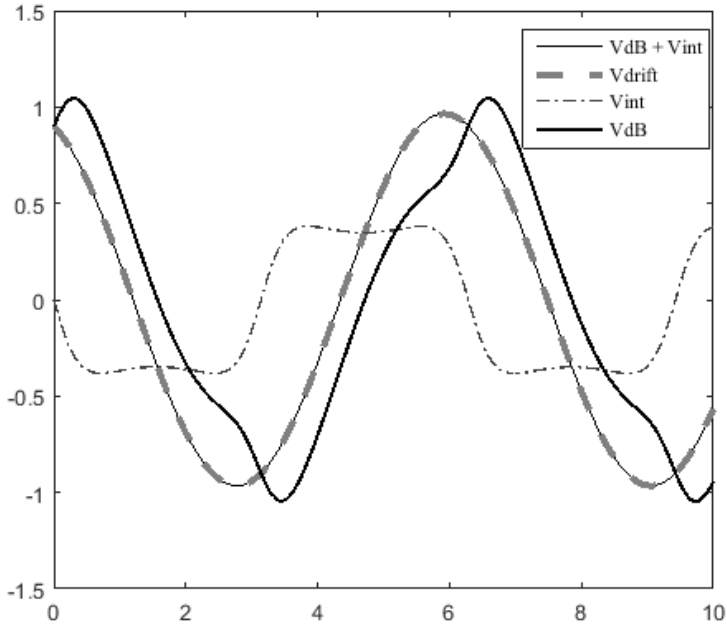

Figure 1: Plot of $v_{dBB}, v_{int.}, v_{dBB} + v_{int.}$ and $v_{drift} = \frac{d}{dt}(< x_{NL} >)$ with $< x_{NL} >= \int d^3x' |\phi^{NL}(t,x')|^2 x'/N_{NL}^2)$ in function of time; space and time were rescaled and are of the order of unity.

$k^{self}/k^{ext.} = 10^3$, $(< x_{NL}^2 > - < x_{NL} >^2)_{t=0} = \sqrt{\frac{\hbar}{\sqrt{(k^{ext}+k^{self})\cdot m}}}$,

and $(< x_{NL}^2 > - < x_{NL} >^2)_{t=0} = 10^{-3}(< x_L^2 > - < x_L >^2)_{t=0}$.

We observed that

(i) In the limit where $k^{self}/k^{ext.}$ is large enough, and provided at time $t = 0$ the extent of the soliton $(< x_{NL}^2 > - < x_{NL} >^2)_{t=0}$ is comparable to the extent of the self-focused ground state $(\sqrt{\frac{\hbar}{\sqrt{k^{self}\cdot m}}})$, and also quite smaller than the extent of the linear wave $((< x_{NL}^2 > - < x_{NL} >^2)_{t=0} << (< x_L^2 > - < x_L >^2)_{t=0})$, $\phi^{NL}(x,t)$ remains strongly peaked and oscillates around its central value throughout time.

(ii) In good approximation (the quality of the approximation being an increasing function of the ratio $k^{self}/k^{ext.}$), property 1 is confirmed by our numerical estimates. Indeed, as can be seen in figure 1, v_{drift} is indistinguishable from $v_{dBB} + v_{int.}$.

(iii) In these conditions, property 2 is automatically satisfied.

(iv) We also noted that in good approximation (the quality of the approximation being an increasing function of the ratio $k^{self}/k^{ext.}$) the soliton $\phi^{NL}(x,t)$, as well as $\Psi^{NL}(x,t)$, follow classical trajectories.

3.2 Main conjecture.

The numerical results presented in the previous chapter (in particular observation (iv)) brought us to demonstrate the

Generalized Ehrenfest theorem.

To do so, let us consider the evolution equation

$$i\hbar\frac{\partial\Psi(t,\mathbf{x})}{\partial t} = (-\hbar^2\frac{\Delta}{2m} + V^L(t,\mathbf{x}) + V^{NL}(t,\mathbf{x}))\Psi(t,\mathbf{x}), \qquad (22)$$

where $V^{NL}t,\mathbf{x}) = -Gm^2\int d^3x'|\Psi(t,\mathbf{x}')|^2F(|\mathbf{x}-\mathbf{x}'|)$, with F a real function of $|\mathbf{x}-\mathbf{x}'|$. Then the centre of its solution, denoted $< x >$ ($< x >= \int d^3x|\Psi(t,x)|^2x/N^2$ with $N^2 = \int d^3x'|\Psi(t,x')|^2$) obeys the laws of classical dynamics in the limit where $< x^2 > - < x >^2$ goes to zero.

To prove it, it suffices to note that N remains constant throughout time, while

$\frac{d<x>}{dt} =< \frac{i\hbar}{m}\nabla >$, and

$\frac{d^2<x>}{dt^2} = \frac{1}{m} < [\nabla, V^L + V^{NL}] >= \frac{1}{m} < \nabla V^L + \nabla V^{NL} >$.

Now, $< \nabla V^{NL} >$

$= \int d^3x'|\Psi(t,x')|^2\int d^3x|\Psi(t,x)|^2(\mathbf{x}-\mathbf{x}')\frac{dF}{du}_{|u=|\mathbf{x}-\mathbf{x}'|} = 0$, by symmetry so that

$\frac{d^2<x>}{dt^2} = \frac{1}{m} < \nabla V^L >$.

In the limit where $< x^2 > - < x >^2$ goes to zero, $< \nabla V^L >\approx \nabla V^L(< x >)$. Note that when V^L is a quadratic function of the position (harmonic oscillator), $< \nabla V^L >$ is always exactly equal to $\nabla V^L(< x >)$.

In other words, our generalized Ehrenfest theorem establishes that trajectories of peaked solitons are classical, and do only depend on external forces, not on the quantum potential: $\mathbf{v}_{drift} = \mathbf{v}_{classical}$.

At this level, we face a serious problem: our original goal [23] was to derive dB-B dynamics from the non-linear dynamics (2) and the factorization ansatz (3). However, the trajectory of the soliton departs from dB-B dynamics, due to the presence of the residual contribution $\mathbf{v}_{int.} = \mathbf{v}_{drift} - \mathbf{v}_{dB-B}$. Our simulations reveal that, in order to ensure the constraints imposed by Eherenfest's theorem (that is to say $\mathbf{v}_{drift} = \mathbf{v}_{classical}$), $\mathbf{v}_{int.}$ "conspires" in order to be equal to $\mathbf{v}_{classical} - \mathbf{v}_{dB-B}$. This compensation is fragile however in the sense that $\mathbf{v}_{classical} - \mathbf{v}_{dB-B}$ contains the memory of the initial preparation process that eventually took place a long time ago. We shall conjecture here that in the practice some stochasticity is present that will wash out this memory and decorrelate $\mathbf{v}_{int.}$ from $\mathbf{v}_{classical}$. Averaging over this stochastic contribution we expect thus (and this is our main conjecture) that $<< \mathbf{v}_{int.} >>= 0$ so that $<< \mathbf{v}_{drift} >>=<< \mathbf{v}_{dB-B} >>$, where the bracket $<<,>>$ represents an averaging over this extra-stochastic perturbation of the velocities that we conjecture to be present in nature.

Note that in the past Bohm, Vigier [10] and de Broglie [21] suspected the existence of a stochastic noise superposed to the quantum potential, necessary in their eyes in order to explain the irreversible in time convergence to quantum equilibrium. de Broglie wrote for instance in [21] (chapter XI: On the necessary introduction of a random element in the double solution theory. The hidden thermostat and the Brownian motion of the particle in its wave) the following sentences

...Finally, the particle's motion is the combination of a regular motion defined by the guidance formula, with a random motion of Brownian character... any particle, even isolated, has to be imagined as in continuous "energetic contact" with a hidden medium, which constitutes a concealed thermostat. This hypothesis was brought forward some fifteen years ago by Bohm and Vigier [10], who named this invisible thermostat the "subquantum medium"....If a hidden sub-quantum medium is assumed, knowledge of its nature would seem desirable...

Our conjecture is a re-expression of the subquantum medium hypothesis invoked by Bohm and Vigier. At this level its deep nature is not clear yet: it could be the manifestation of a relativistic effect similar to the zitterbewegung (footnote 13 in appendix) or it could be of a fundamental nature [45]. It is out of the scope of the present paper to study these possibilities. It is worth noting however that in the case of

bouncing oil droplets[9] [24] such a stochastic disturbance of velocities is always present, due to the periodical forcing imposed to the bath. This explains why dB-B trajectories are never observed directly at the level of droplets but result from the averaging of a large number of trajectories [18, 19, 12] (see also the contribution of C. Borghesi, same issue).

4 Discussion and Conclusions.

Discussion: experimental validation.

The overwhelming majority of experiments [44, 51, 15, 7, 31] proposed so far in order to reveal the existence of intrinsic non-linearities at the quantum level (like e.g. the self-gravity interaction) is a priori doomed to fail, for what concerns our model, because their conceivers systematically took for granted that the wave function was normalised to unity[10]. In the same line of thought, if our conjecture is correct, then trajectories of the self-collapsed solitons are very close to de Broglie-Bohm trajectories. Therefore we expect that our model cannot easily be falsified, to the same extent that the dB-B interpretation leads to the same predictions as the standard quantum theory (through the Born rule).

Even if our model is not relevant, and in the last resort its relevance ought to be confirmed or falsified by experiments, it could appear to be useful as a phenomenological tool.

For instance it was already applied by us [24] in the past[11] to the

[9] Of course we do not expect that Planck's constant plays a role in the case of walkers, because instead of Schrödinger equation, one should consider the classical wave equation that describes the dynamics of waves at the surface of the oil bath to begin with, which is outside of the scope of the present paper. We expect however that the techniques developed in the present paper are still valid in the classical regime in the case of droplets.

[10] Even in this case, the generalized Ehrenfest theorem demonstrated in section 3.2 is valid, which predicts that in the limit of massive enough objects trajectories become classical. This could open the way to new experimental tests for falsifying self-gravity. It also sheds a new light onto the problem of the classical limit (see paper of Matzkin same issue).

[11] In the present paper we essentially studied the feedback of the pilot wave Ψ_L on the soliton or particle ϕ_{NL}, postulating to begin with that the factorisation ansatz is satisfied and that a self-focusing non-linearity is present, that prohibits the spread of the soliton. In ref. [24], which is complementary to the present paper we studied the feedback of the dB-B solitons on the pilot wave. We predicted the appearance, when several droplets are present simultaneously, of an effective gravitational potential and formulated certain predictions that are likely to be tested in the laboratory but this is another story.

phenomenology of bouncing oil droplets [18, 19, 12]. It is at this level that we expect that the validity of the approximations and hypotheses of our model could be confirmed or falsified.

The dynamics outlined here might also be relevant in the field of cold atoms physics [38] where effective non-linear self-focusing equations properly describe collective excitations of the atomic density [3]. As far as we know, no de Broglie like trajectory has yet been observed during such experiments.

In the same order of ideas, it would be highly interesting to investigate whether de Broglie like trajectories are good tools for describing optical and/or rogue waves [2, 37]. After all our factorisability ansatz could be applied to classical non-linear wave equations too.

Conclusions.

We have studied particular solutions of the S-N equation, in presence of an external potential, that behave as peaked solitons, due to the self focusing nature of self-gravitation. We showed that, if they exist and are stable, their shape is close in first approximation to the shape of the self-collapsed ground state, solution of (24) and, provided they remain peaked throughout time, they move in accordance with a generalized dB-B dynamics. These predictions were confirmed by numerical simulations. These simulations also showed that in the non-relativistic limit, the generalized dB-B dynamics is equivalent to classical dynamics, which is confirmed by a generalized Ehrenfest theorem. We conjecture however that if we add a stochastic field to the velocity field, dB-B dynamics will be restored. In last resort, this field could be a manifestation of the relativistic zitter bewegung (see appendix). In our approach, the linear wave function plays the role of an auxiliary computation tool (a pilot wave), and does not represent the particle which is represented by the soliton, which fits well into the double solution program of de Broglie.

We hope that the rather simple models treated in this paper will convince the reader that de Broglie's ideas were maybe not that much surrealistic [27] and deprived of consistence. Our results indeed reinforces the dB-B picture according to which the particle non-locally and contextually explores its environment thanks to the nearly immaterial tentacles provided by the solution of the linear Schrödinger equation. This picture is not comfortable but it is maybe the price to pay to restore wave monism[12]. Our conjecture (section 3.2) also reemphasises

[12] An advantage of wave monism is that, contrary to the Bohmian dynamics formu-

the necessity to invoke the existence of a subquantum medium in order to add to the regular and deterministic dB-B trajectories a random motion of brownian character. The brief incursion in the relativistic domain sketched in appendix suggests that the origin of this brownian motion could be linked to zitterbewegung [33, 41]. It would be highly interesting to try and reveal the existence of this subquantum medium experimentally, which is seemingly not an impossible task as suggested in ref.[50].

Last but not least, our analysis also confirms several prophetic intuitions originally presented by Louis de Broglie during the Solvay conference of 1927 [5].

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lated in terms of material points, wave monism does not violate the No Singularity Principle formulated by Gouesbet [30]. Everything is continuous in our approach, even if very different spatial scales coexist.

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5 Appendices.

5.1 The Schrödinger-Newton equation.

One can look for a "ground-state" solution to (1) in the form

$$\psi(\mathbf{x},t) = e^{-\frac{iE_0 t}{\hbar}}\phi^0_{NL}(\mathbf{x})\,, \tag{23}$$

This leads to a stationary equation for ϕ^0_{NL}

$$\frac{-\hbar^2}{2M}\Delta\phi^0_{NL}(\mathbf{x}) + GM^2\int d^3y\,\frac{\left|\phi^0_{NL}(\mathbf{y})\right|^2}{|\mathbf{x}-\mathbf{y}|}\,\phi^0_{NL}(\mathbf{x}) = E_0\phi^0_{NL}(\mathbf{x})\,, \tag{24}$$

which was studied in astrophysics and is known under the name of the *Choquard* equation [39]. In [39], Lieb showed that the energy functional

$$E(\phi) = \frac{\hbar^2}{2M}\int d^3x\,\left|\nabla\phi(\mathbf{x})\right|^2 - \frac{GM^2}{2}\iint d^3x d^3y\,\frac{\left|\phi(\mathbf{y})\right|^2}{|\mathbf{x}-\mathbf{y}|}\,\left|\phi(\mathbf{x})\right|^2\,, \tag{25}$$

is minimized by a unique solution $\phi^0_{NL}(\mathbf{x})$ of the Choquard equation (24) for a given L_2 norm $N(\phi^0_{NL}(\mathbf{x}))$. However no analytical expression is known for this ground state. Numerical treatments established that this ground state has a quasi-gaussian shape, and that its extent is, in the case $N(\phi^0_{NL}(\mathbf{x})) = 1$, of the order of $\frac{\hbar^2}{GM^3}$. Then its energy is of the order of $-G^2M^5/\hbar^2$. As shown in [14] (see also appendix C of the review paper by S. Colin, T.D. and R. Willox, same issue) one can show through a viriel-like analysis that energy scales like $N^3(\phi^0_{NL}(\mathbf{x}))$. Actually, one can find rescaled solutions of the *Choquard* equation such that both kinetic energy and potential energy scale like $N^3(\phi^0_{NL}(\mathbf{x}))$. The full energy is then of the order of $-(G^2M^5/\hbar^2)(N(\phi^0_{NL}(\mathbf{x})))^3$.

Numerical evidence also suggests the existence of a discrete spectrum of values for the energy (25) of the bound states solutions of (24). They can be written in the form $-\mathrm{e}_n\frac{G^2M^5}{\hbar^2}$, described in terms of dimensionless constants e_n as :

$$\mathrm{e}_n = \frac{a}{(n+b)^c}\,, \quad n = 0,1,2,\ldots\,, \tag{26}$$

for approximated constants [6]

$$a = 0.096 \pm 0.01\,, \quad b = 0.76 \pm 0.01\,, \quad c = 2.00 \pm 0.01\,. \tag{27}$$

The "ground state" for the Choquard equation corresponds, at $N = 1$, to $n = 0$ (see [14] and references therein for more details).

In summary, the S-N potential exhibits several interesting features that play an important role in our paper: it is self-focusing and possess a localized ground state of the type $\phi^0_{NL}(\mathbf{x})e^{-iE_0t/\hbar}$ which behaves as a static bright soliton. The spectrum of negative energy eigenstates solutions of (24) is discrete. Moreover this potential scale like a Coulomb and/or Newtonian self-interaction in the sense that

$$V^{NL}(\lambda\Psi) = |\lambda|^2 V^{NL}(\Psi) \tag{28}$$

5.2 Change of norm

Let us denote H_L the linear part of the the full Hamiltonian in (8). It is not hermitian, so that $\sqrt{< \phi_{NL}|\phi_{NL} >}$, the L_2 norm of its solution $\phi_{NL}(t, \mathbf{x}))$ is not constant throughout time. The non-linear potentials considered by us preserve the L_2 norm however. We can thus evaluate the time derivative of $< \phi_{NL}|\phi_{NL} >$ by direct computation, either integrating by parts, or making use of the formula

$$\begin{aligned}
&\frac{d < \phi|O|\phi >}{dt} = \\
&< \phi|\frac{\partial O}{\partial t}|\phi > + \frac{1}{i\hbar} < \phi|OH_L - H_L^\dagger O|\phi > \\
&=< \phi|\frac{\partial O}{\partial t}|\phi > + \frac{1}{i\hbar}(< \phi|[O, Re.H_L]_-|\phi > \\
&+\frac{1}{\hbar}(< \phi|[O, Im.H_L]_+|\phi >),
\end{aligned} \tag{29}$$

where O is an arbitrary observable, described by a self-adjoint operator, while $Re.H_L$ and $Im.H_L$, the real and imaginary parts of H_L are self-adjoint operators defined through $2 \cdot Re.H_L = H_L + H_L^\dagger$ and $2i \cdot Im.H_L = H_L - H_L^\dagger$. Here, the symbol $[,]_-$ ($[,]_+$) represents the (anti)commutator.

We find by direct computation that

$$\begin{aligned}
&Re.(-\frac{\hbar^2}{m} i \bigtriangledown \varphi_L(t, \mathbf{x}) \cdot \bigtriangledown) \\
&= (-\frac{\hbar^2}{m} i \bigtriangledown \varphi_L(t, \mathbf{x}) \cdot \bigtriangledown) - (\frac{\hbar^2}{2m} i \Delta \varphi_L(t, \mathbf{x}))
\end{aligned} \tag{30}$$

and $Im.(-\frac{\hbar^2}{m}i\nabla\varphi_L(t,\mathbf{x})\cdot\nabla) = (\frac{\hbar^2}{2m}\Delta\varphi_L(t,\mathbf{x}))$.

Therefore the guidance potential contributes to $\frac{d<\phi_{NL}|\phi_{NL}>}{dt} = \frac{d<\phi_{NL}|1|\phi_{NL}>}{dt}$ by a quantity

$$<\phi_{NL}|(\frac{\hbar}{m}\Delta\varphi_L(t,\mathbf{x}))|\phi_{NL}>\approx(\frac{\hbar}{m}\Delta\varphi_L(t,\mathbf{x}))<\phi_{NL}|\phi_{NL}>),$$

due to the fact that, over the size of the soliton, $\varphi_L(t,\mathbf{x})$ and its derivatives are supposed to vary so slowly that we can consistently neglect their variation and put them in front of the L_2 integral.

The contribution of the $A_L-\phi_{NL}$ coupling to $\frac{d<\phi_{NL}|\phi_{NL}>}{dt}$ is $\frac{\hbar^2}{m}\frac{1}{i\hbar}\int d^3\mathbf{x}(\frac{\nabla A_L(t,\mathbf{x})}{A_L(t,\mathbf{x})}\cdot\nabla(\phi_{NL}(t,\mathbf{x}))^*\phi_{NL}(t,\mathbf{x})-(\phi_{NL}(t,\mathbf{x}))^*\frac{\nabla A_L(t,\mathbf{x})}{A_L(t,\mathbf{x})}\cdot\nabla\phi_{NL}(t,\mathbf{x}))$.

We now suppose that we are in right to neglect the variation of $\frac{\nabla A_L(t,\mathbf{x})}{A_L(t,\mathbf{x})}$ in the integral above and we find, integrating by parts, a contribution $-2\frac{\nabla A_L(t,\mathbf{x_0})}{A_L(t,\mathbf{x_0})}\cdot\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^*\frac{\hbar\nabla}{mi}\cdot\phi_{NL}(t,\mathbf{x})$

Putting all these results together, we find that

$$\frac{d<\phi_{NL}|\phi_{NL}>}{dt}\approx\frac{\hbar}{m}\Delta\varphi_L(t,\mathbf{x_0})\cdot<\phi_{NL}|\phi_{NL}>$$
$$-2\frac{\nabla A_L(t,\mathbf{x_0})}{A_L(t,\mathbf{x_0})}\cdot\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^*\frac{\hbar\nabla}{mi}\cdot\phi_{NL}(t,\mathbf{x}). \tag{31}$$

5.3 The Dirac spinor.

In our view the S-N equation is a good candidate for fulfilling the de Broglie double solution program of 1927, in the same sense that Schrödinger's equation was a good candidate for realizing de Broglie's wave mechanics program of 1925. In the same line of thought, it is natural to look for a non-linear relativistic equation that would be to the S-N equations what are the Klein-Gordon or Dirac equations to the non-relativistic Schrödinger equation.

To do so let us consider the non-linear Dirac equation

$$\beta i\hbar\partial_t\mathbf{\Psi}(t,\mathbf{x})-\beta\alpha c\frac{\hbar}{i}\nabla\mathbf{\Psi}(t,\mathbf{x})-(mc^2+V_L)\mathbf{\Psi}(t,\mathbf{x})-V_{NL}\mathbf{\Psi}(t,\mathbf{x})=0. \tag{32}$$

where α and β are the well-known Dirac matrices and

$$V_{NL}\mathbf{\Psi}(t,\mathbf{x}) = -Gm^2 \int d^3x'(\frac{|\Psi(t,\mathbf{x}')|_4^2}{|\mathbf{x}-\mathbf{x}'|})\Psi(t,\mathbf{x}). \tag{33}$$

Here, $|\Psi(t,\mathbf{x}')|_4^2$ represents the local Dirac density:

$|\Psi(t,\mathbf{x}')|_4^2$=$(\mathbf{\Psi}(t,\mathbf{x}))^\dagger \cdot_4 \mathbf{\Psi}(t,\mathbf{x}))$, where the lower index $_4$ refers to the spinorial in-product. Let us solve (32) imposing the ansatz

$$\mathbf{\Psi} = \begin{pmatrix} \Psi_0(t,\mathbf{x}) \\ \Psi_1(t,\mathbf{x}) \\ \Psi_2(t,\mathbf{x}) \\ \Psi_3(t,\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \Psi_0^L(t,\mathbf{x}) \\ \Psi_1^L(t,\mathbf{x}) \\ \Psi_2^L(t,\mathbf{x}) \\ \Psi_3^L(t,\mathbf{x}) \end{pmatrix} \phi_{NL}(t,\mathbf{x}) = \mathbf{\Psi_L}\phi_{NL} \tag{34}$$

where $\phi_{NL}(t,\mathbf{x})$ is a Lorentz scalar. We find

$$\begin{aligned}(\phi_{NL}(t,\mathbf{x}))(\beta i\hbar\partial_t\mathbf{\Psi^L}(t,\mathbf{x}) - \beta\alpha c\frac{\hbar}{i}\bigtriangledown\mathbf{\Psi^L}(t,\mathbf{x}) - (mc^2+V_L)\mathbf{\Psi^L}(t,\mathbf{x}))\\ +(\beta\mathbf{\Psi^L}(t,\mathbf{x})\frac{i\hbar\partial\phi_{NL}(t,\mathbf{x})}{\partial t} - \beta\alpha c\mathbf{\Psi^L}(t,\mathbf{x})\frac{\hbar}{i}\bigtriangledown\phi_{NL}(t,\mathbf{x})\\ +\mathbf{\Psi^L}(t,\mathbf{x})V_{NL}(|\mathbf{\Psi^L}\phi_{NL}|)\phi_{NL}(t,\mathbf{x})) = 0,\end{aligned} \tag{35}$$

that we solve in the same way as in the section 2 by requiring two constraints to be fulfilled:

$$\beta i\hbar\partial_t\mathbf{\Psi^L}(t,\mathbf{x}) - \beta\alpha c\frac{\hbar}{i}\bigtriangledown\mathbf{\Psi^L}(t,\mathbf{x}) - (mc^2+V_L)\mathbf{\Psi^L}(t,\mathbf{x}) = 0, \tag{36}$$

which is is the usual, linear, Dirac equation, and the non-linear constraint

$$\begin{aligned}\beta\mathbf{\Psi^L}(t,\mathbf{x})\frac{i\hbar\partial\phi_{NL}(t,\mathbf{x})}{\partial t} - \beta\alpha c\mathbf{\Psi^L}(t,\mathbf{x})\frac{\hbar}{i}\bigtriangledown\phi_{NL}(t,\mathbf{x})\\ +\mathbf{\Psi^L}(t,\mathbf{x})V_{NL}(|\Psi(t,\mathbf{x})|_4)\phi_{NL}(t,\mathbf{x})) = 0.\end{aligned} \tag{37}$$

In agreement with (28), V_{NL} supposedly scales like a Newtonian self interaction:

$$V_{NL}(\mathbf{\Psi^L}\phi_{NL}) = A_L^2 V_{NL}((\mathbf{\Psi^L}(t,\mathbf{x})/A_L)\cdot\phi_{NL}(t,\mathbf{x}))\phi_{NL}(t,\mathbf{x}) \tag{38}$$

where

$$A_L = \sqrt{|\Psi_0^L(t,\mathbf{x})|^2 + |\Psi_1^L(t,\mathbf{x})|^2 + |\Psi_2^L(t,\mathbf{x})|^2 + |\Psi_3^L(t,\mathbf{x})|^2}. \tag{39}$$

At this level, a serious problem appears: equation (37) consists of four equations. In the non-relativistic limit [41] and in absence of external magnetic field, spin decouples and can be factorized, but in general the four equations implicitly contained in equation (37) are NOT equivalent to each other.

A posteriori, we feel free to formally tackle the problem by proceeding as follows. We firstly decompose $\mathbf{\Psi}$ into its projection along $\mathbf{\Psi_L}$ and its projection along the 4-spinors orthogonal to $\mathbf{\Psi_L}$:

$$\mathbf{\Psi} = \mathbf{\Psi_L}(\mathbf{\Psi_L}^\dagger\mathbf{\Psi}) + (1 - \mathbf{\Psi_L}\cdot\mathbf{\Psi_L}^\dagger)\mathbf{\Psi} \tag{40}$$

Let us introduce $\phi_{NL}(t,\mathbf{x})$ through $\mathbf{\Psi_L}(\mathbf{\Psi_L}^\dagger\mathbf{\Psi}) = \mathbf{\Psi_L}\phi_{NL}(t,\mathbf{x})$, as well as the auxillary spinor $\delta\mathbf{\Psi}$=$(1 - \mathbf{\Psi_L}\cdot\mathbf{\Psi_L}^\dagger)\mathbf{\Psi}$, which allows us to rewrite (40) as follows:

$$\mathbf{\Psi} = \mathbf{\Psi_L}\phi_{NL} + \delta\mathbf{\Psi} \tag{41}$$

In a second time, let us simply neglect $\delta\mathbf{\Psi}$, in a grossly coarse-grained approach, in order to gain more insight about the physics of the problem, but keeping in mind however that there is a serious difficulty hidden under the rug at this level.

Retrospectively, our approximation justifies the ansatz (34). 4-multiplying (37) by $(\mathbf{\Psi^L}(t,\mathbf{x}))^\dagger\beta$ and dividing by $(\mathbf{\Psi^L}(t,\mathbf{x}))^\dagger\cdot_4\mathbf{\Psi^L}(t,\mathbf{x})) = A_L^2$ we get

$$\begin{aligned}\frac{i\hbar\partial\phi_{NL}(t,\mathbf{x})}{\partial t} &= \mathbf{v^L_{Dirac}}(t,\mathbf{x})\frac{\hbar}{i}\nabla\phi_{NL}(t,\mathbf{x}) \\ &+ <\beta>_4\cdot A_L^2\cdot V_{NL}(\phi_{NL})\phi_{NL}(t,\mathbf{x}).\end{aligned} \tag{42}$$

where $< \beta >_4 = (\mathbf{\Psi^L}(t,\mathbf{x}))^\dagger \beta \mathbf{\Psi^L}(t,\mathbf{x}))/(\mathbf{\Psi^L}(t,\mathbf{x}))^\dagger \cdot_4 \mathbf{\Psi^L}(t,\mathbf{x}))$ while $\mathbf{v^L_{Dirac}}$ obeys

$$\mathbf{v^L_{Dirac}} \equiv \frac{(\mathbf{\Psi^L})^\dagger \alpha c \mathbf{\Psi^L}}{(\mathbf{\Psi^L})^\dagger \mathbf{\Psi^L}} \tag{43}$$

As is well-known, the conservation equation associated to the linear Dirac equation is $\frac{\partial A_L^2}{\partial t} = -div(A_L^2 \cdot \mathbf{v_{Dirac}}(t,\mathbf{x_0}))$. The dB-B guidance equation is thus nothing else than (43) [46, 34]. Now, if we suppress in (8) the kinetic energy and the factor proportional to A_L, we find an equation which is formally equivalent to (42). Therefore, by repeating computations similar to those made in section 2, one can easily show that the dB-B guidance equation linked to Dirac's equation (43) is exactly satisfied. Furthermore, resorting to the conservation equation $\frac{\partial A_L^2}{\partial t} = -div(A_L^2 \cdot \mathbf{v_{Dirac}}(t,\mathbf{x_0}))$, it is straightforward to establish the validity of property 2 as in the non-relativistic case studied before. For instance $\frac{\frac{dA_L}{dt}}{A_L} = \frac{-1}{2} \frac{1}{<\phi_{NL}|\phi_{NL}>} \frac{d<\phi_{NL}|\phi_{NL}>}{dt}$

Consequently, we are entitled to look for peaked wave functions of the form

$$\mathbf{\Psi} = \begin{pmatrix} \Psi_0(t,\mathbf{x}) \\ \Psi_1(t,\mathbf{x}) \\ \Psi_2(t,\mathbf{x}) \\ \Psi_3(t,\mathbf{x}) \end{pmatrix} = \frac{1}{A_L} \cdot \begin{pmatrix} \Psi_0^L(t,\mathbf{x}) \\ \Psi_1^L(t,\mathbf{x}) \\ \Psi_2^L(t,\mathbf{x}) \\ \Psi_3^L(t,\mathbf{x}) \end{pmatrix} \cdot \phi'_{NL}(t,\mathbf{x}), \tag{44}$$

for which we know that

(Property 1) the barycentre of ϕ'_{NL} moves along the hydrodynamical flow lines of Dirac's linear equation, at velocity $\mathbf{v_{Dirac}}$.

(Property 2) the L_2 norm of $\phi'_{NL}(t,\mathbf{x})$ is constant throughout time.

As in the non-relativistic case, the amplitude A_L is an auxiliary function that disappears at the end of the computation of $\mathbf{\Psi}$. It is worth noting that we did not demonstrate the existence of a ground state of the type ϕ^0_{NL} in the relativistic case[13]. We also suspect that the self-gravitational potential V_{NL} proposed in (33) is not the right candidate

[13]Setting $V_L = 0$ in (32) an mutiplying it by $(\beta i\hbar\partial_t - \beta\alpha c\frac{\hbar}{i}\bigtriangledown - mc^2(1+\phi_G^{self}/c^2))$ where we denoted $\phi_G^{self} = -Gm\int d^3x'(\frac{|\Psi(t,\mathbf{x}')|^2}{|\mathbf{x}-\mathbf{x}'|})$, we get the non-linear Klein-

in a relativistic context because it is obviously not Lorentz covariant. In any case, it is beyond the scope of our paper to characterize in detail what "really" happens at the level of the soliton. We expect for instance [24] that in the case of an electron, the size of the soliton is of the order of 10^{-57} meter, well beyond the Planck scale. Our main aim in the present section was merely to suggest that hopefully properties 1 and 2 exhibit some structural invariance when we pass from the non-relativistic to the relativistic regime.

Gordon equation

$$\hbar^2 c^2 (\frac{1}{c^2}(\frac{\partial^2}{\partial t^2} - \Delta))\mathbf{\Psi}(t,\mathbf{x}) + m^2 c^4 (1 + \phi_G^{self}/c^2)^2 \mathbf{\Psi}(t,\mathbf{x}) = 0 \tag{45}$$

Then the static version of the non-linear Klein-Gordon equation reads

$$-\hbar^2 c^2 \Delta \mathbf{\Psi}(\mathbf{x}) + m^2 c^4 (1 + \phi_G^{self}/c^2)^2 \mathbf{\Psi}(\mathbf{x}) = \hbar^2 E^2 \mathbf{\Psi}(\mathbf{x}) \tag{46}$$

In the weak potential limit ($\phi_G^{self} << c^2$) the static solution of (46) also satisfies Choquard equation (24), up to rescaling. In the strong field limit, we have no guarantee about the existence of such a solution, but if it exists then the non-linear Klein-Gordon equation and its static counterpart, the non-linear Choquard equation constitute an interesting toy-model for self-gravity, even in the relativistic regime. We suspect however the appearance of difficulties in the same limit, related to zitter bewegung [41, 33], because in general the spin does not decouple from ϕ_{NL} in (37).