L. de Broglie waves: a particular solution

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ABSTRACT. L. de Broglie wave singular solutions used in order to describe particles' motions are investigated from a series of 3 papers by M. Planck aimed to reconcile wave mechanics and corpuscular mechanics. If one continues Planck's reasoning, it seems possible to introduce a non-linear wave equation to describe the motion of the particles.

RÉSUMÉ. Les solutions d'ondes à singularité proposées par L. de Broglie pour décrire le mouvement des particles sont examinées à partir d'une série de 3 articles de M. Planck visant à réconcilier la mécanique ondulatoire avec la mécanique des corpuscules. Si l'on poursuit le raisonnement de Planck, il semble qu'il soit possible d'introduire une équation d'onde non-linéaire pour décrire le mouvement des particules.

1 Preliminary remarks

In symmetry with Einstein's view on duality between electromagnetic waves and light quanta, de Broglie[1, 2] constructed wave-particle duality contemplating the link between wave optics and mechanics within the limit of geometrical optics. In that picture, rays associated with a wave can be viewed as particle trajectories. To go further and obtain a complete wave description one has to describe these rays as regions of space-time where the density of the wave is strongly localized, as in Einstein "nadelstrahlung". The quest for the corresponding description for the particle motion, or for description of the particle itself, is still timely, either by singularities or by non linear wave equations. At that time also, Hadamard had extended the theory of the characteristics of the wave

equations[3]. The synthesis of de Broglie inserts quantum conditions in what will be the framework of geometrical dynamics or of semiclassical dynamics. The best guess for the wave function following de Broglie and Schroedinger is to write the wave function as $\Psi(x,t) = a(x,t)\, e^{i\,S(x,t)/\hbar}$ both $a(x,t)$ and $S(x,t)$ being real functions. This leads to non-linear equations and/or wave packet descriptions of particles. But another non linear choice, closer to that of the Schroedinger original papers[4], used by Brillouin[5], is to write the wave function associated with the particle's motion $\Psi(x,t) = e^{i\,S(x,t)/\hbar}$. These ideas have proved to be fruitful allowing Keller[6] to give a semiclassical form of the Rubinowicz reconciliation of Fresnel and Young views on diffraction. In the years 1940-1941, this choice was made by an 82 years old Max Planck in an attempt to reconcile wave and corpuscular dynamics *"Versuch einer synthese zwischen Wellenmechanik und Korpuscularmechanik"* in a series of three remarkable papers[7, 8, 9] , that do not seem to have been widely discussed. Following Planck's terminology, these papers do not use the general wave mechanics (GWM) theoretical framework in the usual way and *propose a modified wave mechanics (MWM) whose main characteristic is that it does not use wave packets description for particles.* The method is closer to the old quantum mechanics but makes explicit use of Schroedinger wave equation and de Broglie waves concept. The first part of this paper reviews Planck's results and the second part is a first trial to make them cope with the general wave mechanics framework and the way it can be consistent with singularity waves.
The results that are obtained by Planck deserve a renewed inspection as they can provide insight to improve attempts to reformulate quantum mechanics by non-linear wave equations. One can see it as Planck's insight on de Broglie waves and Schroedinger wave mechanics.

2 The two initial choices for wave mechanics:

One can write any wave function as a combination of elementary solutions:

$$\Psi(x,t) = \sum_k \Psi_k(x,t) = \sum_k a_k\, e^{i\,S_k(x,t)/\hbar} \tag{1}$$

And there are two initial choices for the elementary wave function to be associated with a dynamical system, say a structureless particle in space (x) and time (t) :

1. The one made by de Broglie following optics takes the wave func-

tion as:

$$\Psi(x,t) = a(x,t)\, e^{i\, S(x,t)/\hbar} \tag{2}$$

where $a(x,t)$ and $S(x,t)$ are real functions and the amplitude part $a(x,t)$ is a meaningful part of these description.

2. The one made by Schroedinger [4] following analytical dynamics considerations takes the wave function as:

$$\Psi(x,t) = e^{KS(x,t)} \tag{3}$$

where K is found convenient to be $i/\hbar$ and $S(x,t)$ is *a priori* a real function for Schroedinger, but this restriction is left open for Planck.

The Schroedinger equation for one particle of mass m in motion under a field of force given by the gradient of a potential function $U(x)$, where $\triangle$ stands for $div\,\overrightarrow{grad}$ is given by :

$$\triangle\Psi(x,t) - \frac{2\,m}{\hbar^2}U(x,t)\Psi(x,t) + \frac{2\,i\,m}{\hbar}\partial_t\Psi(x,t) = 0 \tag{4}$$

The choices 1 and 2 lead to two different systems of equations separating the real and the imaginary part terms:

For choice 1 one gets:

$$\begin{array}{c} \frac{\partial S(x,t)}{\partial t} + \frac{1}{2m}\left|\vec{\nabla} S(x,t)\right|^2 + U(x,t) - \frac{\hbar^2}{2m}\frac{\triangle a(x,t)}{a(x,t)} = 0 \\ \frac{\partial a(x,t)^2}{\partial t} + div\left[a(x,t)^2\frac{\vec{\nabla} S(x,t)}{m}\right] = 0 \end{array} \tag{5}$$

For choice 2 one gets:

$$\begin{array}{c} \frac{\partial S(x,t)}{\partial t} + \frac{1}{2m}\left|\frac{\partial S(x,t)}{\partial x}\right|^2 + U(x,t) = 0 \\ \frac{i\hbar}{2m}\triangle S(x,t) = 0 \end{array} \tag{6}$$

Choice 1 results in a modified Hamilton-Jacobi equation with a quantum potential term that is supposed to give some "randomness" to the pure classical motion. It presents the advantage to give the conservation of flux equation when $a(x,t)^2$ is considered as representing the particle density.

Choice 2 results in a standard Hamilton-Jacobi equation "clamped" to the classical motion. There is no argument for the particle density as we are concerned by pure point-like particles. In order to fulfill flux conservation conditions, a delta function product that imposes the value of relevant dynamical parameters can be taken implicit.

Both choices refer to two different elementary wave description of a quantum system. In choice 1 the amplitudes $a_k(x,t)$ enter as a dynamical part in the equation of motion and depend on boundary conditions, in choice 2 the amplitudes a_k do not enter as a dynamical part and are determined by boundary conditions. Choice 2 is very close to the solutions found by the Einstein Brillouin Keller (EBK)[10, 5, 6] method, as demonstrated by Keller[6, 14].

3 Planck's papers main results "Die modifizierte Wellenmechanik":

Following Planck who follows Schroedinger choice, particle's motion in space (x) time (t) is described (can be inferred), by (from) "ondes de phase" representation of de Broglie waves:

$$\Psi(x,t) = e^{i\, S(x,t)/\hbar} \tag{7}$$

where $S(x,t)$ is a continuous, function that can be multiple-valued and where quantum conditions make $\Psi(x,t)$ to be single valued. In the non-relativistic case, these waves have to obey the non relativistic Schroedinger's equation. As previously, for one particle of mass m in motion under a field of force given by the gradient of a potential function $U(x)$, one has:

$$\triangle\Psi(x,t) - \frac{2\,m}{\hbar^2}U(x,t)\Psi(x,t) + \frac{2\,i\,m}{\hbar}\partial_t\Psi(x,t) = 0 \tag{8}$$

And for "ondes de phase" solutions one gets:

$$\frac{\partial S(x,t)}{\partial t} + \frac{1}{2m}\left(\frac{\partial S(x,t)}{\partial x}\right)^2 + U(x,t) - \frac{i\hbar}{2m}\triangle S(x,t) = 0 \tag{9}$$

One can recognize that the real part of this expression corresponds to the Hamilton-Jacobi equation associated to the motion. The imaginary part is proportional to $\hbar$ and vanishes (or is small with respect to the real part) when $\hbar \to 0$ or when $\triangle S(x,t) \to 0$.

Following Planck, to define the classical limit, we focus on the limit $\hbar \to 0$ and get the condition for particle-like motion:

$$\frac{\hbar\triangle S(x,t)}{\left(\frac{\partial S(x,t)}{\partial x}\right)^2} \xrightarrow[\hbar\to 0]{} 0 \tag{10}$$

Then in the limit $\hbar \to 0$, in the full configuration space $S(x,t)$ obeys the Hamilton-Jacobi differential equation:

$$\frac{\partial S(x,t)}{\partial t} + \frac{1}{2m}\left(\frac{\partial S(x,t)}{\partial x}\right)^2 + U(x,t) = 0 \tag{11}$$

One can also express the condition (10) "backwards" with the wave function:

$$S(x,t) = -i\,\hbar\, Log\, \Psi(x,t) \tag{12}$$

and one gets *the condition for corpuscular-wave motion*:

$$\frac{\Psi(x,t).\,\triangle\Psi(x,t)}{\left(\frac{\partial\Psi(x,t)}{\partial x}\right)^2} \underset{\hbar\to 0}{\longrightarrow} 1 \tag{13}$$

Note that one can use directly the condition ($\triangle S(x,t) \to 0$) in (9) and obtain the same relation (13) without need of the constraint $\hbar \to 0$.

This expression (13) allows one to examine the classical character of the solutions using directly the wave functions. Those satisfying (13) will match the modified wave mechanics (MWM) or " modifizierte Wellenmechanik" character. If the quotient in (13) has the value 1, without considering the limit $\hbar \to 0$, the corresponding wave function, is related to a classical or particle motion.

In other choices as in general wave mechanics (GWM) or "gegenwaertigen wellenmechanik", following the rules of classical field theory, the wave function is mainly constrained by the boundary conditions and generally does not satisfy the condition (13).

As expressed in the third paper, this point is a crucial one for Planck who, on pure physical arguments, wanted to avoid, not to say reject, the need of wave packet representations for the description of the motion of one corpuscular particle. If one goes further using (8) in (13) taken as an equality one gets:

$$\left(\frac{\partial\Psi(x,t)}{\partial x}\right)^2 = \Psi(x,t).\,\triangle\Psi(x,t)$$

$$= \Psi(x,t).(+\frac{2\,m}{\hbar^2}U(x,t)\Psi(x,t) - \frac{2\,i\,m\partial_t}{\hbar}\Psi(x,t)) \tag{14}$$

$$\left(\frac{\partial\Psi(x,t)}{\partial x}\right)^2 = (\frac{2\,m}{\hbar^2}U(x,t) - \frac{2\,i\,m\partial_t}{\hbar})\Psi(x,t)^2 \tag{15}$$

Differential equation (15), resulting from Planck's condition (13) is *a first order non-linear differential equation* that is the replica of (11) and is a come back to the first step of Schroedinger's original derivation.

Indeed the step before Schroedinger's stationary wave equation was:

$$S(q,t) = -E\,t + W(q) = -E\,t + -i\,\hbar Log\,\psi(q))$$

$$H(q, \frac{\partial W}{\partial q}) = H(q, \frac{-i\hbar}{\psi}\frac{\partial \psi}{\partial q}) = E$$

where $H(q, \frac{\partial S}{\partial q}) = E$ is the stationary Hamilton Jacobi equation. Note that $p = \frac{-i\hbar}{\psi}\frac{\partial}{\partial q}\psi$ and $q = \frac{1}{\psi}q\psi$ are close to Eckart's form of quantization[12] . The values of E has to be chosen either as eigenvalues of (17) either using EBK quantum conditions. Finally one recovers the stationary form of equation (15):

$$\left(\frac{\partial \psi(q)}{\partial q}\right)^2 = \frac{2\,m}{\hbar^2}(U(q) - E))\psi(q)^2 \tag{16}$$

Important point, this Hamilton Jacobi wave equation is a non-linear equation, and in general, the sum of two solutions is not a solution.

4 Use of GWM and MWM wave functions

The quantum conditions fixing the values of E are obtained through the associated variational problem and the search of the eigenvalues of the stationary form of (8). To understand the link between the two procedures, one has to study how the eigenfunctions of equation (8) corresponding to GWM are related to the solutions of (15) for particle motion corresponding to MWM. The wave functions obtained in both methods may differ because they are not subject to the same equation and to the same constraints. As the motion characteristics are obtained through different procedures they might not be in a one-one correspondence. The GWM solutions have to be built with normalized eigenwave functions of (8) subject to boundary conditions. There is a "statistical character" in these solutions, from which average values and fluctuations of dynamical operators are obtained using the very rules of statistical mechanics. The MWM solutions are built to represent one (and only one) particle motion, and can be taken as normalized wave functions but the way they are subject to boundary conditions, within their (eventually piecewise) domain of definition, has to be scrutinized. From these MWM wave functions one gets the standard parameters of the motion by the usual analytical mechanics formula of Hamilton Jacobi theory. The values are not subject to averaging processes or fluctuations. There is no wave packets, no quantum potential as in the pilot wave formulation of de Broglie double solution theory. From the preceding remarks it appears that the search for a more non linear wave equation to describe particles as small singularities in a wave, should take equation (15), or its more fundamental relativistic form, as a departure point, in association with EBK quantum conditions.

5 Cases studies

5.1 The "initial paradigm": Plane waves in one dimension and associated free particles motions:

Taking into account the two possible directions for the motions, the corresponding wave functions can be written:

$$\Psi_s(x,t) = a_s\, e^{i\,\Phi_s(x,t)/\hbar} \tag{17}$$

where $s = r$(ight) or $s = \ell$(eft) stands for the direction of motion of the associated particle and the a_s are real constants and $\Phi_s(x,t)$, the phases, are real functions. The corresponding wave equation is:

$$\triangle\Psi_s(x,t) + \frac{2\,i\,m}{\hbar}\frac{\partial\Psi_s(x,t)}{\partial t} = 0 \tag{18}$$

This gives in terms of the phases $\Phi_s(x,t)$:

$$\frac{\partial\Phi_s(x,t)}{\partial t} + \frac{1}{2m}\left(\frac{\partial\Phi_s(x,t)}{\partial x}\right)^2 - \frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\Phi_s(x,t) = 0 \tag{19}$$

The real part corresponds to the Hamilton-Jacobi equation whose principal function can be chosen in the form:

$$\Phi_s(x_0,t_0;\,x,t) = -\alpha\,(t-t_0) + \beta\,(x-x_0) \tag{20}$$

inserting (20) in (19) one gets:

$$-\alpha + \frac{1}{2m}\beta^2 - \frac{i\hbar}{2m}0 = 0 \tag{21}$$

As a first result in (21) the imaginary part vanishes, this means that the criteria (10) or (13) is fulfilled, for any value of $\hbar$, by the corresponding wave function which thus corresponds to classical motions. The second result is to write (20) as a function of only one parameter as $\alpha = \frac{1}{2m}\beta^2$. If we choose this parameter as β:

$$\beta \to \beta_\pm = \beta_s = \pm\sqrt{2m\alpha} \tag{22}$$

then the right or left character of the solution follows immediately:

$$\Phi_r(x_0, t_0; \alpha;\ x, t) = -\alpha\,(t - t_0) + \sqrt{2m\alpha}\,(x - x_0) \tag{23}$$

$$\Phi_\ell(x_0, t_0; \alpha;\ x, t) = -\alpha\,(t - t_0) - \sqrt{2m\alpha}\,(x - x_0) \tag{24}$$

¿From these two principal functions the parameters corresponding to the possible motions are found using the standard rules where the constant term is taken to be 0.

$$\begin{gathered} -\frac{\partial\Phi_s(x_0,t_0;\alpha;\,x,t)}{\partial t} = \alpha \equiv\ W \\ \text{Energy} \end{gathered} \tag{25}$$

$$\begin{gathered} \frac{\partial\Phi_s(x_0,t_0;\alpha;\,x,t)}{\partial x} = \pm\sqrt{2m\alpha} \equiv p_s \\ right(+)\ or\ left(-)\ Momentum \end{gathered} \tag{26}$$

$$\begin{gathered} \frac{\partial\Phi_s(x_0,t_0;\alpha;\,x,t)}{\partial\alpha} = -(t - t_0) \pm \frac{m}{\sqrt{2m\alpha}}(x - x_0) = 0 \\ \pm \text{ trajectory with initial condition } (x_0, t_0) \end{gathered} \tag{27}$$

$$x_\pm(t) = x_0 \pm \frac{\sqrt{2m\alpha}}{m}(t - t_0) \tag{28}$$

and $v_\pm = \frac{\sqrt{2m\alpha}}{m}$ is the particle velocity. These results allows one to write the wave functions corresponding to each of these motions defined by the parameters $(x_0, t_0; \alpha, \pm)$ as:

$$\Psi_\pm(x_0, t_0; \alpha;\ x, t) = a_\pm\, e^{i\,\Phi_\pm(x_0,t_0;\alpha;\,x,t)/\hbar} = a_\pm\, e^{i\,[-\alpha\,(t-t_0)\pm\sqrt{2m\alpha}\,(x-x_0)]/\hbar} \tag{29}$$

Even if it seems clear how the $\pm$ character acts in this problem, in this derivation, it is nevertheless put "by hand" and is not the result of an operation acting on the Hamilton-Jacobi Principal Function. The general solution of the GWM (8) is a sum of any (r, ℓ) solutions,

$$\Psi(\alpha;\ x, t) = a_\ell\, e^{i\,\Phi_\ell(x_{0\ell},t_{0\ell};\alpha;\,x,t)/\hbar} + a_r\, e^{i\,\Phi_r(x_{0r},t_{0r};\alpha;\,x,t)/\hbar}$$

but it is not a solution of (16) except when

$$\left(a_\ell\, e^{i\,\Phi_\ell(\alpha;\,x,t)/\hbar} + a_r\, e^{i\,\Phi_r(\alpha;\,x,t)/\hbar}\right) = \left(a_\ell\, e^{i\,\Phi_\ell(\alpha;\,x,t)/\hbar} - a_r\, e^{i\,\Phi_r(\alpha;\,x,t)/\hbar}\right)$$

or

$$a_r \, a_\ell = 0 \tag{30}$$

Condition (30) related to "decoherence between plane waves" can not be achieved without further argument. In conclusion a plane wave is both a solution of Schroedinger equation (8)and MWM equation (15), a superposition of plane waves remains solution of Schroedinger equation (8) but is not a solution of the MWM equation (15).

5.2 Schroedinger's GWM plane wave solution as a combination of Planck's MWM individual plane wave solutions:

To go further we have to combine the individual motions and the associated wave functions. First we write them in a form that makes it explicit the individual parameters $(x_0, t_0; \alpha, \pm)$ or $(x_0, t_0; \alpha, s)$:

$$\begin{aligned} \Psi(x_0, t_0; \alpha, \pm; \, x, t) &= a(x_0, t_0; \alpha, \pm) \, e^{i \, \Phi_\pm((x, t_0; \alpha, \pm; \, x, t)/\hbar} \\ &= a(x_0, t_0; \alpha, \pm) \, e^{i \, [-\alpha \, (t - t_0) \pm \sqrt{2m\alpha} \, (x - x_0)]/\hbar} \end{aligned} \tag{31}$$

In relation with the description of a beam of particles, it is assumed that a $(\alpha, \pm)$ plane wave solution describes the "average" motion corresponding to an ensemble of particles randomly distributed with respect to initial conditions. For this purpose one can use the Epsilon distribution (Vallée[13])) which is obtained from the Gaussian distribution by making the limit of an infinite width, this distribution is in a sense the "inverse" of the Dirac distribution which focusses on one position. This distribution has everywhere a vanishing value, but its integral over the full range of variation is unity. It is well fitted to describe uniform (centered) random distribution of values. Let first take $a(X_0, T_0; x_0, t_0; \alpha, \pm)$ to be a Gaussian distribution of values for x_0 and t_0 over the averaged position and time X_0 and T_0 with dispersions σ_{x0} and σ_{t0}:

$$a(X_0, T_0; \, x_0, t_0; \alpha, \pm) = A(\alpha, \pm) \frac{1}{\sqrt{2\pi}\sigma_{x0}} e^{-\frac{(x_0 - X_0)^2}{2\sigma_{x0}^2}} \frac{1}{\sqrt{2\pi}\sigma_{t0}} e^{-\frac{(t_0 - T_0)^2}{2\sigma_{t0}^2}} \tag{32}$$

Whatever be σ_{x0} and σ_{t0} , vanishing or expanding one has:

$$\int_{-\infty}^{+\infty} dt_0 \int_{-\infty}^{+\infty} dx_0 \, a(X_0, T_0; \, x_0, t_0; \alpha, \pm) = A(\alpha, \pm) \tag{33}$$

For the description of one particle in motion with energy α in the $\pm$ direction, $A(\alpha, \pm) = 1$. The beam wave function is given by the averaging procedure:

$$\begin{aligned}&\Psi_{\pm}(\sigma_{x0},\sigma_{t0};X_0,T_0;\alpha,\pm;\,x,t)\\&=\int_{-\infty}^{+\infty}dt_0\int_{-\infty}^{+\infty}dx_0\;a(X_0,T_0;\,x_0,t_0;\alpha,\pm)e^{i\,[-\alpha\,(t-t_0)\pm\sqrt{2m\alpha}\,(x-x_0)]/\hbar}\end{aligned} \tag{34}$$

$$\begin{aligned}&\Psi(\sigma_{x0},\sigma_{t0};X_0,T_0;\alpha,\pm;\,x,t)\\&=A(\alpha,\pm)\,e^{-\alpha(2m\sigma_{x0}^2+\alpha\sigma_{t0}^2)/2\hbar^2}e^{i\,[-\alpha\,(t-T_0)\pm\sqrt{2m\alpha}\,(x-X_0)]/\hbar}\end{aligned} \tag{35}$$

Taking the infinite limit for σ_{x0} and σ_{t0}:

$$\begin{aligned}&\Psi(Epsilon;X_0,T_0;\alpha,\pm;\,x,t)\\&=\lim_{\sigma_{x_0},\sigma_{t_0}\to\infty}A(\alpha,\pm)\,e^{-\alpha(2m\sigma_{x0}^2+\alpha\sigma_{t0}^2)/2\hbar^2}e^{i\,[-\alpha\,(t-T_0)\pm\sqrt{2m\alpha}\,(x-X_0)]/\hbar}\end{aligned}$$

and taking the vanishing limit for σ_{x0} and σ_{t0}:

$$\begin{aligned}&\Psi(Dirac;X_0,T_0;\alpha,\pm;\,x,t)\\&=A(\alpha,\pm)\,e^{i\,[-\alpha\,(t-T_0)\pm\sqrt{2m\alpha}\,(x-X_0)]/\hbar}\end{aligned}$$

leads to different values for the probability amplitudes $|\Psi(x,t)|^2$, the *Epsilon* one is vanishing everywhere, the *Dirac* one is unity everywhere. Note that the sum of two $\Psi(Epsilon;X_0,T_0;\alpha,\pm;\,x,t)$ functions can be a solution of the non linear MWM equation as

$$a_r\,a_\ell=\lim_{\sigma_{x_0},\sigma_{t_0}\to\infty}A(\alpha,r)A(\alpha,\ell)\,e^{-\alpha(2m\sigma_{x0}^2+\alpha\sigma_{t0}^2)/\hbar}=0 \tag{36}$$

This is the further argument that makes it possible to "decorrelate" the different plane waves so they can represent different particles.

5.3 de Broglie's singular solutions :

In his 1927's paper de Broglie [14] developed an alternative form of his wave mechanics, based mainly on relativistic concepts. These singular solutions, extending the "ondes de phase" concept, were built in order to describe each particle by a singular wavefunction that allows to enlarge the point particle concept. The motion of the singularity has to match as much as possible the motion deduced from the Principal Function solution of the Hamilton-Jacobi equation. In order to fulfill that guess, for a free one dimensional motion, the singular wave function writes:

$$u_s(x,t)=f_s(x,t)e^{i\,\Phi_s(x,t)/\hbar} \tag{37}$$

where the phase term $\Phi_{s=\pm}(x,t) = -\alpha\,(t-t_0) \pm \sqrt{2m\alpha}\,(x-x_0)$ is the same as in (17) in order to fulfill the Hamilton-Jacobi equation. These singular waves have an amplitude term $f_s(x,t)$ that makes them different from the pure phase wave solutions considered before in (1)-(3) parts. As the described particle has a group velocity $v_\pm = \pm\frac{\sqrt{2m\alpha}}{m} = \pm\sqrt{\frac{2\alpha}{m}}$, and phase velocity $V_\pm = \pm\frac{1}{2}\frac{\sqrt{2m\alpha}}{m} = \pm\sqrt{\frac{\alpha}{2m}}$, in the non relativist limit two possibilities are thus to be considered:

$$u_{g\pm}(x_0,t_0;\alpha;\,x,t) = \frac{b}{\sqrt{b^2+[(x-x_0)\mp\sqrt{\frac{2\alpha}{m}}(t-t_0)]^2}}e^{i\,[-\alpha\,(t-t_0)\pm\sqrt{2m\alpha}\,(x-x_0))]/\hbar} \tag{38}$$

the de Broglie's one by relativistic considerations, or a tentative one

$$u_{\phi\pm}(x_0,t_0;\alpha;\,x,t) = \frac{b}{\sqrt{b^2+[(x-x_0)\mp\sqrt{\frac{\alpha}{2m}}(t-t_0)]^2}}e^{i\,[-\alpha\,(t-t_0)\pm\sqrt{2m\alpha}\,(x-x_0))]/\hbar} \tag{39}$$

where b is a regularization term to be chosen. Singularities happen when $b\to 0$. The indices u and ϕ refer to the choice of group or phase velocities for the singularities. First we notice that:

$$u_{g\pm}(x_0,t_0;\alpha;\,x = x_0 \pm \sqrt{\frac{2\alpha}{m}}(t-t_0),t) = e^{i\,[\alpha\,(t-t_0)]/\hbar}$$

$$u_{\phi\pm}(x_0,t_0;\alpha;\,x = x_0 \pm \sqrt{\frac{\alpha}{2m}}(t-t_0),t) = 1$$

We test these wave functions with Planck's modified wave mechanics criteria $QC(\Psi)\to 1$:

$$QC(u_{g\pm}) = 1 + O(\hbar^2)$$

$$QC(u_{\phi\pm}) = 1 + O(\hbar^2)$$

is fulfilled up to the second order in $\hbar$.

As a result de Broglie's singular solutions, which were not studied in Planck's papers, satisfy Planck's criterium for particle description. The particle velocity is not defined (calculated) as a group velocity so the amplitude pre-factor follows Planck's insight that the particle should not be a wave-packet. Here also further use of Epsilon distributions on initial conditions and wave parameters could be of interest to implement decoherence between individual waves.

6 Wave functions and boundary conditions

As stated by Einstein and used by Brillouin, the quantum conditions make $\Psi(x,t)$ to be a single valued function. This scheme works satisfactorily if $\Psi(x,t)$ describes a plane wave, that is a bunch of parallel rays, each ray being related to uniform motions differing by initial departure time and/or positions. Following analytical mechanics rules these motions are deduced from $S(x,t)$ where they are imbedded. But this picture appears to be less efficient when one is concerned with motion in a box or a varying index medium because of the introduction of boundary conditions that prevent pure plane wave solutions. In the case of one dimensional motion in a box, the solutions can be developed in a set of 2 plane waves, $\Psi_l(x,t) = e^{i\,S_l(x,t)/\hbar}$ (motion to the left) and $\Psi_r(x,t) = e^{i\,S_r(x,t)/\hbar}$ which are not solutions of the complete wave problem because the coefficients of the expansion are determined by the boundary conditions. Then a rule must be added such that one recovers from the motions described by $S_l(x,t)$ and $S_r(x,t)$ the motion of one particle, say p, that is a set of initial conditions and a switching rule to indicate at which position/time the particle has to be described by $S_{l,p}(x,t)$ or $S_{r,p}(x,t)$.

One important feature of the quantum theory describes how the wave function is related to the dynamical properties of the dynamical system under study. By the study of collision processes, M. Born [15] assumed that the square of the wave function is related to the probability density to find one particle. In that case it is possible to write the wave function as $\Psi(x,t) = a(x,t)\,e^{i\,S(x,t)/\hbar}$ both $a(x,t)$ and $S(x,t)$ being real functions. This $|\Psi(x,t)|^2 = a(x,t)^2$ argument is a very strong and fruitful one that makes it difficult to make another use of the very wave function. Nevertheless the spirit of the "semi-classical" arguments is always actual in classical wave mechanics or classical wave theory and it shows that Born's interpretation has to be used with much caution. For instance even for the 1 dimensional problem of the reflection an incident harmonic wave between two regions of space with different indices or wave velocities c_1 and c_2 one writes:

$$\Psi(x,t) = \begin{cases} A_i e^{i\,\omega(x-c_1t)} + A_r e^{i\,\omega(x+c_1t)} \text{ in region 1} \\ A_t e^{i\,\omega(x-c_2t)} \text{ in region 2} \end{cases}$$

and the (complex) amplitudes A_i, A_r and A_t are related by the well known formula $A_r = r\,A_i$ and $A_t = t\,A_i$. The usual interpretation, to be found in any textbook say on wave propagation, separates the contributions $A_i e^{i\,\omega(x-c_1 t)}$, $A_r e^{i\,\omega(x+c_1 t)}$ and $A_t e^{i\,\omega(x-c_2 t)}$. Now we apply the quantum mechanical $|\Psi(x,t)|^2$ condition. This formula, applied in region 2 to the transmitted wave, makes no problem. There is only one contribution and the usual particle probabilistic interpretation can be that, for one incident particle (*i.e.* $|A_i|^2 = 1$), there a probability $|t|^2$ to find a transmitted particle through the barrier. In region 1 the classical interpretation is that there exist an incident wave and a reflected wave. Each particle of the incident wave has a definite probability $|t|^2$ to go from region 1 to region 2 and a probability $|r|^2 = 1 - |t|^2$ to be reflected and to remain in region 1. If one considers the sum of these two waves as a quasi stationary wave the probability of finding a particle presents nodes and bump, because one does not separate the propagation directions that makes these two waves correspond to two separate classical motions, this is allowed because of the linearity of (8) but is not granted for the non-linear (15), except if one uses also Epsilon distribution for the wave initial conditions and parameters. The point is to build solutions that describe the various motions of the particles which either bounce or go straightforwardly at the boundary. In Dirac's book [17] it is the classical interpretation which is given for a polarized beam of light in a polarizer, or for the Stern Gerlach effect, the same kind of explanation is given, because the incident wave separate in two emergent waves with the same principal direction of propagation. In these cases the $|\Psi(x,t)|^2$ interpretation can be misleading for the region where the wave is separated in two sub-components. Nevertheless for the very case of wall refection Born [16, 18] has made a very refined study using either classical mechanics or wave mechanics that supports the statistical character of the classical motions, because of the loss of initial conditions, that are associated with the GWM description. Note that if the point is a very strong one, it presents nevertheless the flaw that it does not use the more accurate wave description for particles that can be found in MWM.

7 Conclusion

The novelty in this work is the emphasis on Planck's Modified Wave Mechanics that allows for a non linear wave equation to describe particles, this equation being nothing else than a modified form of the Hamilton Jacobi equation where quantum constraints have to be set. These solutions avoid the wave packet description for one particle motion problems, indeed they seem also to fit, for the simpler cases, with Wigner's group theory classification of particles [19]. The GWM solutions, whose constraint is to fulfill boundary conditions, might be not accurate enough to describe particle motion because, for instance, they do not separate the directions of motion, and the trajectories that they construct correspond to a statistical (with uniform probability) averaging of (MWM) waves corresponding to each motion. This uniform probability distribution "ansatz" is used in Keller's approach of diffraction by an edge [6]. The use of Epsilon distributions can be of great help to solve coherence-decoherence issues with waves. This is another way of understanding Born's remarks [16, 18]. Planck third paper, presents the study of an harmonic oscillator, restricted to the $\hbar = 0$ limit. This crucial case has to be studied also in the case where $\hbar$ is undefined. As evanescent waves solutions arise in this problem, that challenges the use of Hamilton's Jacobi method to recover parameters of the motion, and the comparison between GWM and MWM solutions will offer new horizons. In [20] , p. 95 de Broglie presents almost the same general ideas than the one used in this note, except for the care on the supperposition principle for non-linear waves. Further in p. 131 he presents also an objection rised by Einstein against the guiding formula, based on the case of the motion in between two walls, where stationary waves occur in GWM, which is not compatible with a classical particle motion. This objection appears for one of the simpler problem to be theoreticaly tackled. The MWM non linear solutions that separate the motions and are not compatible with standing waves could resolve this issue.

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