The theory of the Double Solution: Dynamical issues in quantum systems in the semiclassical regime

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ABSTRACT. The "*dynamical mismatch*" observed in quantum systems in the semiclassical regime challenge the Pilot wave model. Indeed the dynamics and properties of such systems depend on the trajectories of the classically equivalent system, whereas the de Broglie-Bohm trajectories are generically non-classical. In this work we examine the situation for the model favoured by de Broglie, the theory of the Double Solution (DS). We will see that the original DS model applied to semiclassical systems is also prone to the dynamical mismatch. However we will argue that the DS theory can be modified in order to yield propagation of the singularity in accord with the underlying classical dynamics of semiclassical systems.

1 Introduction

The de Broglie-Bohm theory of motion, often known as the Pilot-wave model, or the Bohmian model (BM), is undoubtedly attractive when compared to the plague of interpretational problems affecting the formalism of standard quantum mechanics. These problems arise because the theoretical entities of the formalism do not refer unambiguously to objects and properties of the observable universe [1]. In the Bohmian model instead [2, 3] the ontology is simple: the quantum world is made up of waves and particles pursuing deterministic trajectories. Waves and particles are taken to be real, allowing to unify the classical and quantum descriptions of nature: "*there is no need for a break or 'cut' in the* *way we regard reality between quantum and classical levels*" [4].

Nevertheless, the similarity of the Bohmian model relative to classical physics (be it classical waves or classical mechanics) is very superficial [5]. On the one hand, the pilot waves are not defined in our four dimensional physical space-time, but in a multidimensional configuration space. On the other hand, the particle trajectories are highly non-classical. This feature is readily understandable when needing to cope with entangled states of several particles (as is well-known [2], the BM trajectories are driven by a nonlocal quantum potential). The situation is perhaps less understandable when considering semiclassical systems – quantum systems in which the wavefunction evolves according to the semiclassical Feynman propagator, that is along classical trajectories [6]. Indeed in semiclassical systems the Bohmian trajectories remain non-classical although such systems display physical properties in correspondence with those of classically equivalent systems (crudely speaking, systems having the same Hamiltonian, previous to canonical quantization). These features constitute a serious problem in accounting for the quantum to classical transition within the Pilot wave model, as ad-hoc mechanisms involving decoherence need to be postulated.

It is well-known that de Broglie was the first to propose the Pilot wave theory [7], a quarter century before Bohm independently rediscovered essentially the same model, supplementing it with further developments [8]. It is less well-known that de Broglie originally intended to propose a more ambitious programme – the theory of the Double Solution – but gave presentations of the Pilot wave programme instead because the double solution theory was plagued with difficulties (this is recounted by de Broglie in Ref. [9]). The main difference with the Pilot wave model is that no particle is postulated, but the discrete aspect inherent to quantum phenomena is assumed to be due to the singularity of a physical wave, different from the pilot wave obeying the Schrödinger equation. Such a physical wave would solve the first issue mentioned in the preceding paragraph, concerning pilot waves living in a multiconfiguration space. But what about the second, dynamical aspect? This is the question we will examine in this paper. We note at the outset that the Double solution theory has not up to now become a full fledged research programme that would allow to recover, at least in principle, the results of standard quantum mechanics. So our remarks in the present paper are rather intended to foresee the consequences of any potential development of the theory with regard to the topic of understanding the dynamics of semiclassical systems.

In Sec. 2 we will give a brief presentation of the Double solution theory. The main characteristics of semiclassical systems will be exposed in Sec. 3, and the idea of the "*dynamical mismatch*" between the pilot wave dynamics and the classical trajectories will be recalled. Sec. 4 will be devoted to introduce a modification of the double solution theory in order to solve the dynamical mismatch problem affecting the Bohmian model. We will give our conclusions in Sec. 5.

2 Theory of the Double solution

In the Pilot wave model, the wavefunction ψ in the position representation is decomposed as [2, 3]

$$\psi(\mathbf{x},t) = R_\psi(\mathbf{x},t)\exp(iS_\psi(\mathbf{x},t)/\hbar) \tag{1}$$

where $R_\psi(\mathbf{x},t)$ is a real positive function. Since ψ obeys the Schrödinger equation, R_ψ and S_ψ obey the coupled equations

$$\frac{\partial R_\psi^2(\mathbf{x},t)}{\partial t} + \frac{1}{m}\nabla\cdot\left(R_\psi^2(\mathbf{x},t)\nabla S_\psi(\mathbf{x},t)\right) = 0 \tag{2}$$

and

$$\frac{\partial S_\psi(\mathbf{x},t)}{\partial t} + \frac{(\nabla S_\psi(\mathbf{x},t))^2}{2m} + V(\mathbf{x},t) + Q_\psi(\mathbf{x},t) = 0, \tag{3}$$

where $V(\mathbf{x},t)$ is the usual potential and $Q_\psi(\mathbf{x},t)$ is a term known as the quantum potential given by

$$Q_\psi(\mathbf{x},t) \equiv -\frac{\hbar^2}{2m}\frac{\nabla^2 R_\psi}{R_\psi}. \tag{4}$$

The momentum and the velocity of the particle are introduced via a configuration space field defined from the polar phase function through the "guiding equation"

$$\mathbf{p}_\psi(\mathbf{x},t) = m\mathbf{v}_\psi(\mathbf{r},t) = \nabla S_\psi(\mathbf{x},t). \tag{5}$$

$\mathbf{v}_\psi(\mathbf{r},t)$ is proportional to the standard quantum mechanical current density associated with the Schrödinger equation, so that the particle is guided along the probability flow.

In order to introduce the Double solution theory, de Broglie argues [10] that $\psi(\mathbf{x},t)$ is a statistical wave, not a physical wave, and that a particle can hardly be guided by a statistical quantity. He introduces a wave

$$u(\mathbf{x},t) = a(\mathbf{x},t)\exp(iS_\psi(\mathbf{x},t)/\hbar) \tag{6}$$

having the same phase as $\psi(\mathbf{x},t)$ but an amplitude $a(\mathbf{x},t)$ proportional to $R_\psi(\mathbf{x},t)$ everywhere but in a small singular region. This singular region accounts for the discrete, particle-like aspect of quantum mechanics. Whether $u(\mathbf{x},t)$ should be a soliton-like solution of a non-linear equation, or if it can taken to be a singular solution of the linear Schrödinger equation has remained an open question [11]. The important point for de Broglie is that the guiding equation (5) still holds. This is formalized, in a nonlinear context, by writing [10]

$$u(\mathbf{x},t) = u_0(\mathbf{x},t) + w(\mathbf{x},t) \tag{7}$$

where $u_0(\mathbf{x},t)$ is the solitonic "bump" (a solution of a nonlinear equation having negligible amplitude except in a compactly localized and mobile reigon), while $w(\mathbf{x},t)$ is the physical (unnormalized) wave similar to $\psi(\mathbf{x},t)$:

$$w(\mathbf{x},t) = c\psi(\mathbf{x},t) \tag{8}$$

where c is a constant. Hence according to the Theory of the Double solution, the solitonic bump is guided according to Eqs. (5) and (8) by a linear wave, the physical wave $w(\mathbf{x},t)$.

3 Classical dynamics in quantum systems and the Dynamical Mismatch

The investigations of the quantum-classical correspondence, which has its origins in the early days of quantum mechanics were revived in the 1980's and 1990's in the context of quantum chaos [6]. It is today well-established that several types of quantum systems – known generically as semiclassical systems – display the manifestations of properties belonging to the classical analog of these systems. This is due to the fact that the wavefunction propagates essentially along the trajectories of the corresponding classical system; indeed in these cases the semiclassical approximation to the path integral propagator, given by [12]

$$K(\mathbf{x}_0,\mathbf{x},t) = \sum_k \frac{1}{2i\pi\hbar}\left|\det\frac{\partial^2\mathcal{S}_k}{\partial\mathbf{x}\partial\mathbf{x}_0}\right|^{1/2}\exp\left(i\mathcal{S}_k(\mathbf{x}_0,\mathbf{x},t)/\hbar + i\phi_k\right), \tag{9}$$

becomes excellent up to certain time scales. Here the sum runs on all the classical trajectories k connecting $\mathbf{x}_0$ to $\mathbf{x}$ in the time t. $\mathcal{S}_k$ is the classical action for the kth trajectory and the determinant is the inverse of the Jacobi field familiar from the classical calculus of variations, reflecting the local density of the paths; ϕ_k is a phase accounting for reflections and conjugate points encountered along the kth trajectory.

Eq. (9) has observable consequences, like the recurrence of the wavefunction along classical periodic orbits that has been seen experimentally for example in atomic spectra [13]. The corresponding de Broglie-Bohm trajectories are not classical: the observed recurrences can be explained in terms of hundreds of different types of Bohmian trajectories that return in the assigned time to the starting point so as to produce the observed recurrences [14]. This is hardly surprising since according to Eq. (9) the *waves* propagate along the trajectories of the corresponding classical system, whereas according to Eq. (5) the solitonic singularity propagates along the *current density.* The current density at some given point results from all the waves with non-vanishing amplitude that interfere at that point (in the simplest example discussed by Einstein [15] criticizing the Pilot-wave model, a particle in an infinite well is described by two semiclassical counter-propagating waves accounting for the to and fro motion; their interference results in a static current density). Typically semiclassical systems are excited, and the fine-grained dynamics is incredibly complex. The current density hence displays a high sensitivity relative to the initial wavefunction: two slightly different initial wavefunctions can give rise to very different de Broglie-Bohm trajectories. However the semiclassical propagator (9) depends solely on the system, defined by the classical Hamiltonian whose canonical quantization yields the Hamiltonian of the quantum system.

We have argued elsewhere [5] why this *dynamical mismatch* between de Broglie Bohm trajectories and classical trajectories could be seen as a difficulty for the Pilot wave model in accounting for the emergence of classical dynamics. Indeed, the classical dynamics is already visible in the structure and properties of the semiclassical systems, while the Bohmian model predicts highly non-classical trajectories. The claim that decoherence and interaction with a complex environment will render the non-classical pilot-wave dynamics classical appears as somewhat constrained, since on the one hand classical trajectories are already at play in the closed, non-interacting system, and on the other hand the decoherence mechanism is not a resource specific to the Bohmian model but a standard quantum mechanical effect that is known to provide at best a practical solution to understand the effective average disappearance of interferences, not a fundamental solution that would be applicable to an ontological account [16].

4 Towards a new Double solution theory?

The dynamical mismatch we have just mentioned also holds for the Theory of the Double solution, because it is constructed, by Eqs. (6) and (7) so as to recover the guiding equation (5). Now in the usual Bohmian model involving a point-like particle, it seems that there is no way to have a dynamics defined by something different than the guiding equation (5). The reason is that the quantum waves interfere and that the particle needs to avoid the regions where the wavefunction vanishes, and this is exactly what the current density achieves.

However, since the solitonic bump is a wave, from a conceptual view point it can interfere with background waves, disappear or reappear. Therefore, contrary to the particle of the Pilot-wave model, it is possible to envisage a double solution theory whose starting point would be different from Eq. (6). The bump can then be ascribed to follow a dynamical law different from the guiding equation (5).

As a starting point, let us write the wavefunction $\psi(\mathbf{x},t)$ in a generic semiclassical form as

$$\psi(\mathbf{x},t)=\sum_k \psi^k(\mathbf{x},t) \tag{10}$$

where

$$\psi^k(\mathbf{x},t)=\psi(\mathbf{x}_0^k,t=0)\left|\det\frac{\partial^2\mathcal{S}_k}{\partial\mathbf{x}\partial\mathbf{x}_0^k}\right|^{1/2}\exp\left(i\mathcal{S}_k(\mathbf{x}_0^k,\mathbf{x},t)/\hbar+i\phi_k\right). \tag{11}$$

As in Eq. (9) the sum over k runs on the classical trajectories starting at points $\mathbf{x}_0^k$ within the regions in which the initial wavefunction $\psi(\mathbf{x}_0,t=0)$ has a non-vanishing amplitude.

Let us now introduce field functions

$$w^k(\mathbf{x},t)=c\psi^k(\mathbf{x},t) \tag{12}$$

where c is a global constant. Note that the relative weight of each $w^k(\mathbf{x},t)$ is given by a classical quantity, the amplitude $\det\partial^2\mathcal{S}_k/\partial\mathbf{x}\partial\mathbf{x}_0^k$ along each classical path. Following the steps leading to the double solution, we have $u^k(\mathbf{x},t)\approx w^k(\mathbf{x},t)$ except that for one of the fields $u^k(\mathbf{x},t)$

say $u^{k_b}(\mathbf{x},t)$ we will have a bump representing the discrete quantum. We therefore put

$$u^k(\mathbf{x},t) = u_0^k(\mathbf{x},t) + w^k(\mathbf{x},t) \tag{13}$$

where

$$u_0^k(\mathbf{x},t) = 0 \text{ for } k \neq k_b. \tag{14}$$

The idea sketched here is that of a solitonic bump traveling on a single semiclassical wave $u_0^{k_b}(\mathbf{x},t)$. This requires that

- (i) the initial position of the soliton lies randomly within the region in which the initial field $w(\mathbf{x}_0, t = 0)$ has a non-vanishing amplitude;
- (ii) the nonlinear wave is driven by essentially classical dynamics, since each phase $\mathcal{S}_k(\mathbf{x}_0^k, \mathbf{x}, t)$ is a solution of the Hamilton-Jacobi equation [that is Eq. (3) with $Q_\psi = 0$];
- (iii) the amplitude of the solitonic wave has to be strongly coupled to all the waves $w^k(\mathbf{x},t)$ (and not only to the wave $w^{k_b}(\mathbf{x},t)$ having the same dynamics).

Point (iii) is the most important: in order to recover the correct statistical predictions when a position measurement is made, a mechanism coupling the solitonic wave $u_0^{k_b}(\mathbf{x},t)$ to all the linear waves $w^k(\mathbf{x},t)$ should ensure that the interference effects are properly taken into account. The simplest mechanism one could think of is coupling the amplitudes of the different fields, so that the amplitude of $u_0^{k_b}(\mathbf{x},t)$ is controlled by the amplitude of $w(\mathbf{x},t)$. This would indeed account for the effects observed in semiclassical systems, eg the fact that if two different periodic orbits with amplitudes A_1 and A_2 and actions $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same period, their recurrence strength is given by $|A_1 \exp i\mathcal{S}_1/\hbar + A_2 \exp i\mathcal{S}_2/\hbar|^2$.

Eqs. (10)-(14) were given here for systems in the semiclassical regime (ie, $\hbar/\mathcal{S}_k \to 0$), but we could further speculate what these relations would become deep in the quantum regime ($\mathcal{S}_k \approx \hbar$). In that case the semiclassical propagator (9) should be replaced by the standard expression for the propagator in terms of a path integral. There would not be a discrete number of functions $w^k(\mathbf{x},t)$ anymore but an infinite and continuous number of such fields defined from any arbitrary path, the contribution of each path κ being proportional to $\exp\left(i\mathcal{S}_\kappa(\mathbf{x}_0, \mathbf{x}, t)/\hbar\right)$.

Eq. (14) defining the solitonic wave for a given preparation of the system would then hold for one of these paths κ, yielding typically a random, Brownian like motion.

5 Conclusion

In this paper we have recalled the existence of a dynamical mismatch between trajectories of the Bohmian model and classical motion in semiclassical systems. This dynamical mismatch has serious implications concerning the empirical acceptability of the de Broglie-Bohm theory as describing the *real* behaviour of the quantum world.

In this context, we have discussed whether the theory of the Double Solution, initially (and ultimately) favoured by de Broglie over the Pilot-wave model, could avoid this dynamical mismatch. We have sketched how this could be the case, namely by assuming that the solitonic bump is attached to a single semiclassical wave, rather than to the entire wavefunction. From this perspective, the Double Solution theory appears to be more flexible than the Bohmian model, though it should be kept in mind that this programme faces serious difficulties [17] for multiparticle generalisations if the linear waves are taken to be defined over our 4-dimensional space-time (rather than in configuration space), as advocated by de Broglie [18].

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