Deterministic Electron Spin

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ABSTRACT. We study the Pauli equation for the electron in the context of a localised field solution to the Schrödinger equation. This yields a non-probabilistic theory of electron spin. Our results are relevant, in particular, to Toyoki Koga's work on the Schrödinger equation.

We also explain the statement that the direction of the spin magnetic moment of the electron is given by the Hopf map; this seems to have been assumed or ignored in the literature.

Key words: Schrödinger equation, electron spin, spin vector, magnetic moment, angular momentum, Hopf map, electromagnetic field, Pauli equation, Hamiltonian, Pauli matrices, measurement, localised field.

1 Introduction

After Schrödinger discovered the equation named after him, it was found that this equation did not completely describe the electron. It is necessary to ascribe to the electron an intrinsic angular momentum and magnetic moment. This phenomenon is called *electron spin* since it appears that the electron is spinning.

The experimental results of Stern and Gerlach can be understood if it is assumed that when "measured" in any direction, the component of the spin angular momentum vector $\boldsymbol{s}$ is found to be $\pm\hbar/2$. Following the Copenhagen interpretation of Quantum Mechanics, this is assumed to mean that the component in any direction *is* $\pm\hbar/2$, and that it is only meaningful to speak about one component at a time. However, it is assumed permissible to add the squares of the components in three orthogonal directions, despite the obvious contradiction.

We interpret spin measurement differently. This is explained after our discussion of the Pauli equation.

Also, it must be assumed that the electron's magnetic moment vector $\boldsymbol{\mu}$ satisfies

$$\boldsymbol{\mu} = -\frac{e}{m}\boldsymbol{s} \tag{1}$$

where e is the electron charge and m is its mass.

A non-relativistic theory of the electron, including its magnetic moment, was developed in 1927 by Pauli [1] (see [2] for a modern outline and references to textbooks) who showed how to extend the Schrödinger theory. Here we introduce a theory analogous to Pauli's, but based on the assumption that the electron is a localised field.

Toyoki Koga ([3], [4], Chapter IV of [5]) developed such an approach to the Schrödinger equation. According to Koga, the electron is a localised field described by the ψ function, and is neither a wave nor a particle; it is spherically symmetric in the absence of external forces [3] but its shape gets distorted by external fields [4].

This is a theory without observers, measurements, probabilities and expectation values. Koga showed that his solution of the Schrödinger equation for a free electron yielded a de Broglie wave by averaging over an ensemble. He was of the view that all of quantum theory could be obtained from his results by considering ensembles.

Koga studied the Schrödinger equation and the Dirac equation (Chapter V of [5]) but not the Pauli equation. This may be because he found it of no use in developing his general relativistic theory of the electron, including its internal gravitational field (Chapter VI of [5]). The latter was motivated by his solution to the Dirac equation. But the deterministic theory of the Pauli equation is a good illustration of his ideas.

We also discuss the relation of the Hopf map [6] to the Pauli spin theory. The paper [7] gives details of the Hopf map and some information about how the Pauli matrices arise. It also mentions that the Hopf map gives the spin direction of a spin 1/2 particle. Other introductions to the Hopf map and its relation to quaternions are [8] and [9].

In the well-known book by Penrose [10] it is asserted, without naming Hopf, that the Hopf map gives the direction of a 2-spinor.

We do not address the question of whether electron spin is real or only apparent. It should be noted that Pauli's paper [1] does not mention

spin at all.

The double valued nature of the spin suggests that the electron's wavefunction (we use the traditional term for the electron field) should contain two components:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{2}$$

In other words, $\psi = \psi(x, y, z, t) = \psi(\boldsymbol{x}, t)$ is a map from $\mathbb{R}^3 \times \mathbb{R}$ to $\mathbb{C}^2$.

Since the map ψ represents a localised field, we assume that $|\psi_1| \to 0$ and $|\psi_2| \to 0$ as $|\boldsymbol{x}| \to \infty$.

If $\psi_2 = 0$ at all points of space we say that the electron is in a *spin-up* state; the spin vector $\boldsymbol{s}$ is assumed to have the direction $(0, 0, 1) \in S^2$. If $\psi_1 = 0$ everywhere, the state is *spin-down* with spin direction $(0, 0, -1)$.

We assume that the ratio ψ_2/ψ_1 is independent of position. The reason is that we expect it to always give the spin direction, just as it does when it has the value 0 (up) or ∞ (down).

We assume that we are working in an inertial frame.

Following Koga, we write the Schrödinger equation for an electron as

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + U\psi \tag{3}$$

where ψ is a complex valued function on $\mathbb{R}^3 \times \mathbb{R}$. Here m is the mass of the electron and U is the potential energy.

2 A free electron

By a free electron we mean one with no external forces acting on it. We take $U = 0$ for a free electron. For convenience, we assume that we are studying an electron which is at rest in our frame. In order to study electron spin, we postulate that ψ_1 and ψ_2 are related solutions to the Schrödinger equation for a free electron:

$$\psi_j = a_j \exp(iS/\hbar) \tag{4}$$

where $a_j = |\psi_j|$ are complex valued functions of position and time.

We also assume that the ratio ψ_2/ψ_1 is constant (independent of time as well as position) for a free electron. If this ratio gives the spin

direction, then the assumption above is equivalent to the assertion that in the absence of a torque, an electron does not precess.

Although this paper is motivated by Koga's work on the Schrödinger equation, it applies to any theory of the electron considered as a localised field. It is therefore not necessary, for our purposes, to give details about Koga's solution. See [3] for Koga's treatment of the Schrödinger equation. When an electron is in an external electric field, Koga studies what happens in [4]. These matters are also explained in Chapter IV of his book [5] where he changes his terminology for the wavefunction from "wavelet" to "elementary field".

We can write

$$\psi = \psi_0 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{5}$$

where ψ_0 is a solution to the Schrödinger equation and α and β are constants (for a free electron) with $|\alpha|^2 + |\beta|^2 = 1$. This expression is unique up to multiplication by a complex number of absolute value 1. We can make it unique by (for example) assuming that α is real and positive.

If we identify $\mathbb{R}^4$ with $\mathbb{C}^2$ as in [7], we have $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in S^3$, the unit sphere in $\mathbb{R}^4$. This brings up the question: what is the direction of ψ, if any? In other words, what is the spin axis? There is a possible answer to this. In 1931, Hopf [6] defined a map which we denote

$$f : S^3 \to S^2 \tag{6}$$

as an example of a continuous map between spheres that is not null-homotopic.

We define the Hopf map f here and justify its use later. The following definition and several equivalent ones are given in [7]; this one is the most convenient for us.

A general point $P \in S^3$ can be described as

$$P = e^{i\xi} \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \end{pmatrix} \tag{7}$$

where $0 \leq \theta \leq \pi$. Here θ is unique and ϕ can also be made unique by putting suitable bounds on it; ξ is arbitrary. Let

$$f(P) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \in S^2. \tag{8}$$

We see that

(i) f is continuous, independent of ξ and depends only on the ratio of the components of P, $e^{i\phi}\tan(\theta/2)$,

(ii) every point of S^2 is $f(P)$ for some $P \in S^3$,

(iii) $f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (0,0,1)$ (the north pole of S^2),

(iv) $f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0,0,-1)$ (the south pole of S^2), and

(v) $f(P)$ uniquely determines P (except for the value of ξ).

As a consequence of (i), we can extend the domain of f to all the points of $\mathbb{C}^2$ except the origin. There is also an S^2-valued map $f(\psi)$ where $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. It should be noted that $f(\psi)$ depends on t alone; for a free electron, $f(\psi)$ is constant. Thus, f is a candidate for the direction map. But is f compatible with the Pauli spin theory? In other words, for a free electron, do $f(\psi)$ and the spin angular momentum vector $\boldsymbol{s}$ have the same direction in $\mathbb{R}^3$?

By properties (iii) and (iv), $f(\psi)$ gives the spin direction of a spin-up or spin-down electron.

3 An electron in an electromagnetic field

We accept the Pauli equation as given in the literature. One difference in our approach is that we always consider angular momentum as a vector in $\mathbb{R}^3$.

In the Schrödinger (or Pauli) equation, the potential energy U is a sum of terms due to various forces acting on the electron. The vector potential of the electromagnetic field also modifies the kinetic energy term. These are all scalar operators; they simply multiply ψ. None of them take into account the fact that the electron has an intrinsic magnetic moment.

Suppose an electron is placed in a magnetic field $\boldsymbol{B}$. Then it experiences a torque $\boldsymbol{\mu} \times \boldsymbol{B}$ where $\boldsymbol{\mu}$ is its magnetic moment. The potential energy term due to the magnetic field is

$$V = -\boldsymbol{B} \cdot \boldsymbol{\mu} \tag{9}$$

which leads to

$$V = \frac{e}{m}(B_1 s_1 + B_2 s_2 + B_3 s_3) \tag{10}$$

where B_1, B_2, B_3 are the components of $\boldsymbol{B}$.

Since ψ has two components, the three real components s_1, s_2, s_3 of the spin angular momentum $\boldsymbol{s}$ must be represented in the Pauli equation by 2×2 matrices, say S_1, S_2, S_3. These satisfy well known commutativity relations and consequently we can make the choice

$$S_j = \frac{\hbar}{2}\sigma_j \tag{11}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. These are the so called Pauli spin matrices.

Thus, in the Pauli equation, corresponding to V is the operator

$$\frac{\hbar e}{2m}(B_1\sigma_1 + B_2\sigma_2 + B_3\sigma_3). \tag{12}$$

For us, the spin vector $\boldsymbol{s}$ is a real vector in $\mathbb{R}^3$, not a triple of matrices as stated in most textbooks. A spin measurement, such as passing an electron through a Stern-Gerlach apparatus, is actually a rotation of the spin axis due to the torque exerted by the magnetic field. This agrees with the view of Doran and Lasenby in their book [11] that spin measurement is really spin polarisation.

4 Pauli and Hopf

Let $\boldsymbol{B} = |\boldsymbol{B}|\boldsymbol{n}$ where $\boldsymbol{n}$ is a unit vector with components n_1, n_2, n_3. This means that $B_j = |\boldsymbol{B}|n_j$ for $j = 1, 2, 3$ and $\boldsymbol{n} \in S^2$ is the direction of $\boldsymbol{B}$.

Now there is a unique θ such that $0 \leq \theta \leq \pi$ and $\cos\theta = n_3$. Then, since ${n_1}^2 + {n_2}^2 + {n_3}^2 = 1$, there is ϕ such that $\sin\theta\cos\phi = n_1$ and $\sin\theta\sin\phi = n_2$. Note that for $P \in S^3$ the Hopf map satisfies

$$f(P) = \boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{13}$$

if and only if P is of the form

$$P = e^{i\xi} \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi} \sin(\theta/2) \end{pmatrix} \tag{14}$$

for some ξ.

Similarly, we find $Q \in S^3$ such that $f(Q) = -\boldsymbol{n}$. Since $-\boldsymbol{n}$ is obtained from $\boldsymbol{n}$ by replacing θ with $\pi - \theta$ and ϕ with $\phi + \pi$, we get

$$Q = e^{i\xi} \begin{pmatrix} \sin(\theta/2) \\ -e^{i\phi} cos(\theta/2) \end{pmatrix}. \tag{15}$$

Without loss of generality, we will take $\xi = 0$ or any other convenient value. With the usual inner product, $\{P, Q\}$ forms an orthonormal basis for $\mathbb{C}^2$. The two complex vectors P and Q are eigenvectors of the matrix $n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3$ corresponding to the eigenvalues 1, -1. See the paper [7] for details.

We are now concerned with the question of the relation, if any, between $\boldsymbol{n}$ and the directions of ψ and $\boldsymbol{s}$.

First consider a special case. Suppose $\boldsymbol{B}$ is parallel to the $+z$-axis, i.e., $n_1 = n_2 = 0$, $n_3 = 1$. We can take $P = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In this case, using $\psi = \psi_1 P + \psi_2 Q$, $\sigma_3 P = P$ and $\sigma_3 Q = -Q$, we see that the Pauli equation degenerates into a pair of independent scalar equations, one for each ψ_j. If $\psi_2 = 0$, we have a spin-up electron; by definition, its spin axis has the same direction, $(0, 0, 1)$, as $\boldsymbol{B}$.

In the general case, there are complex-valued maps ψ_P and ψ_Q, uniquely determined by ψ, such that $\psi = \psi_P P + \psi_Q Q$. Again, the Pauli equation degenerates into a pair of independent scalar equations, one for ψ_P and the other for ψ_Q. If $\psi_Q = 0$ then $f(\psi) = f(P) = \boldsymbol{n}$ while if $\psi_P = 0$ then $f(\psi) = f(Q) = -\boldsymbol{n}$.

Conversely, if $f(\psi) = \boldsymbol{n}$ then $\psi_Q = 0$; if $f(\psi) = -\boldsymbol{n}$ then $\psi_P = 0$.

Suppose we rotate axes in $\mathbb{R}^3$, making $\boldsymbol{n}$ the north pole of S^2 (and $-\boldsymbol{n}$ the south pole), and change bases in $\mathbb{C}^2$ from $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ to $\{P, Q\}$. The vectors $\boldsymbol{s}$ and $\boldsymbol{B}$ are unchanged, but now $\boldsymbol{B}$ is parallel to the new $+z$-axis. With the new basis of $\mathbb{C}^2$, P is represented by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and Q by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Similarly, $\boldsymbol{n}$ is now represented by $(0, 0, 1)$. Although

the Hopf map changes, its values at P and Q remain the same: $\boldsymbol{n}$ and $-\boldsymbol{n}$. Thus, as in the special case, the direction of $\psi_P P$ is $\boldsymbol{n}$, which is the direction of $\boldsymbol{B}$. As in the special case, $\psi_P P$ stands for a spin-up electron. So its direction coincides with those of $\boldsymbol{n}$ and $\boldsymbol{s}$. Considering the limit as $\boldsymbol{B} \to 0$, we see that for a free electron ψ and $\boldsymbol{s}$ have the same direction.

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