de Broglie's double solution: limitations of the self-gravity approach

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Louis de Broglie presented, at the Solvay Conference of 1927 in Brussels [1], the so-called double solution program aimed at replacing the wave-particle duality by a wave monistic approach in which the particle gets represented by a peaked (soliton-like) wave-packet which in turn is guided by the pilot wave, solution of the linear Schrödinger equation. This approach requires to add a non-linear coupling term to the (otherwise linear) Schrödinger equation, in order to explain the stability of the soliton/particle. Here we explore the properties and limitations of a model aiming at realizing de Broglie's double solution program in terms of self-gravity.

1 Introduction

The irreducibly linear nature of Schrödinger equation is still an open question today. As far as we know, no fundamental non-linearity has been detected yet at the quantum level and the superposition principle is usually accepted to be a universal priciple with the status of a law of Nature. There even exist no-go theorems [2, 3] aimed at proving that linearity is the price to pay to preserve Einsteinian causality, which is another pillar of modern physics. At the other side, non-linear generalisations of the (otherwise linear) Schrödinger equation pave the way to realistic solutions of the measurement problem, in full accordance with de Broglie's double solution program [4, 5, 6]. The basic idea underlying this approach is that non-linearity would be at the source of the so-called collapse process, ultimately explaining the corpuscular properties exhibited by quantum systems. Following this line of thought, self-gravity (in its commonly accepted formulation [7, 8, 9, 10, 11, 12, 13, 14, 15] that we shall from now on denote OSG for orthodox self-gravity) is particularly promising because it enables to predict that the micro-macro (quantum-classical) transition occurs for objects having a mass above say 10^9 a.m.u. [9]. For lighter objects (atoms, molecules, small aggregates) self-gravity is however predicted to be so weak that no measurable violation of the superposition principle is possible, in agreement with all experimental data collected so far. For sufficiently massive objects (having a mass above 10^9 a.m.u.), the non-linearity activated by self-gravity is in principle sufficiently strong for localizing the wave function of the quantum object in a region small compared to its physical size, in which case the object behaves as a localized particle, which is one of the goals of de Broglie's double solution program [16, 6]. Of course, in this approach, the double solution program is, contrary to de Broglie's original formulation, not realized for ALL objects: it would work only if the object is massive enough. Another problem in O.S.G., regarding the realization of de Broglie's program, is that there is no double solution: either the wave function is self-collapsed and then it is no longer a solution of the linear Schrödinger equation, or it fulfills the linear equation which means that self-gravity is so weak that it can consistently be neglected. de Broglie actually faced a similar dilemna in 1927 which brought him to formulate the guidance condition [5] according to which corpuscules/solitons follow the de Broglie-Bohm guidance equation, but there is no indication that this principle is valid in the framework of OSG[1].

In order to explore possible realization of de Broglie's double solution program in the absence of the guidance condition, one of us developed in the past another approach in which the wave function associated to the quantum system is no longer the sum of a solution of the linear Schrödinger equation (the so-called pilot wave) and of a soliton, as originally conceived by de Broglie, but it is a product of these two (this is the so-called factorisability ansatz [17, 6]). One of the reasons for exploring this possibility is the recognition that the superposition principle is no longer valid whenever non-linearities are present. The other reason is the aforementioned difficulty (impossibility?) to derive the guidance

[1]D. Fargue has shown in the past [4] that in the case of free propagation, there exist certain non-linear equations admitting solitonic solutions moving along straight lines, respecting thereby the guidance condition derived from a plane wave type solution of the free linear Schrödinger equation, but in our view this property is merely a consequence of the Galilei invariance of the non-linear equation considered by D. Fargue in his study [6]. As far as we know no confirmation of the validity of the guidance equation has been obtained outside from this particular situation (free evolution plus Galilei invariance)

condition from the non-linear dynamics.

It has been shown in the past that a generalized guidance condition [6] results from the factorisability ansatz, in which, in first approximation, the velocity of the localized soliton is the sum of the de Broglie-Bohm velocity and of an internal velocity. It has also been shown by then that the approximation is good provided the soliton is peaked enough. In order to verify the validity of this result, we shall consider here a particular regime of self-gravity, the so-called quadratic regime [9, 15, 7], valid when the size of the object is large compared to the soliton. In this regime, the evolution is endowed with a remarkable property: it is gaussian; in other words, gaussian states remain gaussian throughout the evolution [8]. There exists however no analytical solution for this (highly non-linear) gaussian evolution (contrary to the free linear Schrödinger equation) but, due to the fact that a gaussian state is characterized by only four real parameters, it is possible to numerically integrate the non-linear dynamics with high accuracy. These preliminary remarks allow us to clearly define the main scope of our paper which consists of an accurate study of the evolution of a self-gravitating quantum system in the quadratic regime, in order to better understand the physical implications of the generalized guidance equation resulting from the factorizability ansatz. Moreover, in order to escape to the aforementioned case, already studied by D. Fargue [4] in the past (free case plus galilei invariance), we considered the situation in which the system is trapped inside an external harmonic potential (which has the merit to preserve the gaussian character of the evolution).

Our main result is twofold. In a first time (section 2), we numerically confirmed the generalized guidance equation, inside its domain of validity, that is to say whenever the soliton is peaked enough. In a second time, we remarked that the internal velocity is in general not small, which implies that the de Broglie guidance condition is violated. Actually, this violation is elucidated (section 3) in terms of a generalisation of Ehrenfest's theorem, valid in the presence of self-gravity, which establishes that peaked solutions of the non-linear Schrödinger equation obey classical dynamics.

This finally brings us to invoke, as a last resort aimed at realizing de Broglie's program, the presence of some stochastic component of the dynamics, aimed at neutralizing the internal velocity. Actually, de Broglie [18] , Bohm and Vigier [19] were also brought to a similar conclusion in the past even though their motivation was different. Nelson [20] proposed an elegant way to add stochasticity to the de Broglie-Bohm dynamics, relaxing thereby the guidance condition while preserving equivariance and respecting the Born rule. In the last section we illustrate Nelson's ideas in the case of the one-dimensional harmonic oscillator. The possible connections with so-called walkers (bouncing oil droplets) [21] and zitterbewegung [22] are sketched in the last section.

2 Factorization ansatz and self-gravity.

2.1 Self-gravity: basics

It is common in the literature to represent quantum effects due to self-gravitation through the Schrödinger-Newton equation [9, 11]

$$i\hbar\frac{\partial\Psi(t,\mathbf{x})}{\partial t} = -\hbar^2\frac{\Delta\Psi(t,\mathbf{x})}{2M} + \int d^3x'|\Psi(t,\mathbf{x}')|^2V(|\mathbf{x}-\mathbf{x}'|)\Psi(t,\mathbf{x}), \quad (1)$$

where $V(d) = -GM^2/d$. This equation can be shown to result from the mean field coupling proposed by Møller [12] and Rosenfeld [14] in the non-relativistic limit [9, 10]. It is valid in principle when the object is an elementary particle. If the object possesses an internal structure, it is necessary to integrate the self-gravitational potential over the internal degrees of freedom of the object [7]. In the case of a rigid homogeneous nanosphere one finds that, at short distance, instead of the Newton potential V, the effective self-interaction can be expressed [7, 23] in terms of $d = |\mathbf{x}_{CM} - \mathbf{x}'_{CM}|$, with $\mathbf{x}_{CM}$ the center of mass of the nanosphere as follows:

$$V^{\text{eff}}(d) = \frac{GM^2}{R}\left(-\frac{6}{5} + \frac{1}{2}\left(\frac{d}{R}\right)^2 - \frac{3}{16}\left(\frac{d}{R}\right)^3 + \frac{1}{160}\left(\frac{d}{R}\right)^5\right) \quad (d \le 2R), \quad (2)$$

where R is the radius of the nanosphere. This expression is valid when d is smaller than twice the radius of the sphere. For larger distances, that is to say whenever d is larger than twice the size of the object, the integration of the internal contributions can be realized easily, making use of Gauss's theorem. Then, we recover the usual Coulomb-like shape, also valid in the case of a non-composite object (1):

$$V^{\text{eff}}(d) = -\frac{GM^2}{d} \quad (d \ge 2R). \quad (3)$$

The resulting integro-differential evolution law of the CMWF now reads

$$i\hbar\frac{\partial\Psi(t,\mathbf{x_{CM}})}{\partial t} = -\hbar^2\frac{\Delta\Psi(t,\mathbf{x_{CM}})}{2M} + \int d^3x'_{CM}|\Psi(t,\mathbf{x'_{CM}})|^2 V^{\text{eff}}(|\mathbf{x}_{CM}-\mathbf{x}'_{CM}|)\Psi(t,\mathbf{x_{CM}}), \tag{4}$$

where $V^{\text{eff}}(|\mathbf{x}_{CM}-\mathbf{x}'_{CM}|)$ is fully defined through equations (2,3). In the limit where the wave function of the center of mass (CMWF) is peaked over a region small in comparison to R, we find that the effective potential is quadratic: $V^{\text{eff}}(d) = \frac{GM^2}{R}\left(-\frac{6}{5}+\frac{1}{2}\left(\frac{d}{R}\right)^2\right)$. This is a very general result, not only valid for spherical objects, as has been shown in Ref.[15]: whenever the extent of the CMWF is small enough, the effective self-gravitational potential is quadratic (see also Refs.[24, 25]). Assuming in first approximation that the nuclei contribution to the mass distribution is spherical (the result has been shown to be very similar in the case of a gaussian distribution [15, 24]) and making use of equation (2), one finds that, in the limit where the extent of the CMWF gets smaller than the size of a nucleus, the contribution of nuclei to self-gravity dominates other contributions [15, 7] so that

$V^{\text{eff}}(d) \approx \frac{GNm^2}{r}\left(\frac{1}{2}\left(\frac{d}{r}\right)^2\right) = \frac{1}{2}M\cdot G\cdot(4\pi/3)\rho_{nucleus}\cdot d^2$, up to an additive constant, with m the mass of a nucleus, r their size, N their number ($N = M/m$), and $(4\pi/3)\rho_{nucleus} = \frac{m}{r^3}$. Taking a nucleus size r of 10^{-2} (10^{-4}) angstroem $G\cdot(4\pi/3)\rho_{nucleus}$ is of the order of 1 (10^6) hertz^2.

Our scope is to study the influence of such a potential, when the object is trapped in a harmonic potential (then the full potential is the sum of an external harmonic potential and of self-gravity).

For instance, in the case of a membrane vibrating along the x direction (or in the case of a linear trap directed along X), the full potential obeys

$$V\Psi(t,\mathbf{x}) = 1/2(kx^2 + k_\infty(y^2+z^2)) + M\cdot G\cdot(4\pi/3)\rho_{nucleus}\cdot\int d^3x'|\Psi(t,\mathbf{x}')|^2(|\mathbf{x}-\mathbf{x}'|)^2)\Psi(t,\mathbf{x}) \tag{5}$$

where k_∞ is supposedly huge, due to the fact that the deformation of the membrane in the (y,z) plane is too small to be noticeable, while k is the spring constant of the membrane along the X direction. Schrödinger equation then separates in Cartesian coordinates and from now on we shall study its reduction along the X direction. The effective reduced Schrödinger-Newton now reads

$$i\,\hbar\frac{\partial\Psi(t,x)}{\partial t} = (-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial x^2} + \frac{k^{ext.}x^2}{2})\Psi(t,x) + \quad (6)$$

$$+(\frac{M\cdot G\cdot(4\pi/3)\rho_{nucleus}}{2}\int dx'|\Psi(t,x)|^2\cdot(x-x')^2)\Psi(t,x)$$

$$=(-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial x^2} + (\frac{k^{ext.}x^2}{2} + \frac{k^{self}(x-x_0)^2 + k^{self}\sigma_x^2}{2}))\Psi(t,x)$$

where $k^{ext.}$ represents the spring along the X direction; we also introduced the parameters k^{self}, x_0 and σ_x^2 defined through

$$k^{self} = \frac{M\cdot G\cdot(4\pi/3)\rho_{nucleus}}{2}\int dx|\Psi(t,x)|^2\ ,\ x_0 = \frac{\int dx|\Psi(t,x)|^2 x}{\int dx|\Psi(t,x)|^2}\ ,$$

and $\sigma_x^2 = \frac{\int dx|\Psi(t,x)|^2(x-x_0)^2}{\int dx|\Psi(t,x)|^2}$.

We are not the first to study this dynamical system. In the framework of OSG a similar study was already performed in the past [15, 24], motivated by the ambition to experimentally observe manifestations of self-gravity through their influence on the dynamical properties of a mesoscopic object trapped in a harmonic potential. In those studies however, the self-gravitational interaction always appeared to be very small and was treated as a perturbation. Actually, self-gravity is so weak that up to now it was impossible to experimentally observe its potential manifestations. In our approach however, we do not impose that the norm of the wave function is equal to unity, as is the case in OSG. The reason therefore is that we consider that the wave function is attached to a solitonic self-collapsed solution of the non-linear dynamics which behaves as a corpuscle [6], and ought not to be confused with the pilot wave, solution of the linear Schrödinger equation for which normalisation to unity ($\int dx|\Psi(t,x)|^2 = 1$) is traditionally required, in agreement with Born's probabilistic interpretation. This constraint is generally overlooked in the framework of usual (linear) quantum mechanics because the normalization of a solution of a linear equation does not affect the properties of the solution, such a solution being always defined up to an arbitrary multiplicative factor. On the contrary this is no longer so as far as we consider non-linear equations. In our approach, we consider that the normalisation factor of the solitonic wave representing the corpuscle is huge compared to unity, and it is rather the external potential that is treated as a perturbation. This is why we explore the solutions of the non-linear dynamics in a regime which has been unexplored before, regime in which the effective coupling constant characterizing self-gravity is quite larger than the spring constant attached to the external (trapping) potential. In the practice, we considered a situation for which the ratio between the non-linear and linear spring constants $\frac{k^{self}}{k^{ext.}}$ is equal to 10^3.

2.2 Factorisation ansatz, double solution and self-gravity.

In this section we shall impose the factorisation ansatz according to which the wave function $\Psi(t,x)$ solution of equation (6) can be written as the product of a pilot wave $\Psi^L(t,x)$ and of a peaked soliton $\Psi^{NL}(t,x)$:

$$\Psi(t,x) = \Psi^L(t,x) \cdot \Psi^{NL}(t,x), \tag{7}$$

where $\Psi^L(t,x)$ is solution of the linear Schrödinger equation:

$$i\hbar \frac{\partial \Psi^L(t,x)}{\partial t} = \left(-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial x^2} + \frac{k^{ext.}x^2}{2}\right)\Psi^L(t,x) \tag{8}$$

We further impose that $\Psi(t,x)$, $\Psi^L(t,x)$ (and thus $\Psi^{NL}(t,x)$) are gaussian: $\Psi(t,x) = e^{-Ax^2+Bx+C}$, $\Psi^L(t,x) = e^{-A^Lx^2+B^Lx+C^L}$, $\Psi^{NL}(t,x) = e^{-A^{NL}x^2+B^{NL}x+C^{NL}}$, where the factors $A, A^L, A^{NL}, B, B^L, B^{NL}, C, C^L$, C^{NL} are complex, time-dependent, numbers. The real part of C, C^L and C^{NL} is constrained by the normalisation of the wave function (which remains constant throughout time), and their imaginary part can be seen as a global, purely time-dependent, phase, irrelavant from the physical point of view. Having this in mind, one sees that the full wave function $\Psi(t,x)$, the pilot wave $\Psi^L(t,x)$ and the soliton $\Psi^{NL}(t,x)$ can be consistently parameterized by 4 real parameters; the full wave function for instance is entirely defined (up to an irrelevant global phase) by the 4 real parameters a_1, a_2, b_1, b_2: $a_1 = Re(A), a_2 = Im(A), b_1 = Re(B), b_2 = Im(B)$. A similar parameterization holds for the pilot wave and the soliton. If now we impose the constraints (6,7,8), the dynamics reads (making use of the fact that $x_0 = \frac{b_1}{2a_1}$):

$$\begin{aligned}\hbar\dot{a_1} &= 4\frac{\hbar^2}{m}a_1 \cdot a_2\\ \hbar\dot{a_2} &= -2\frac{\hbar^2}{m}(a_1^2 - a_2^2) + \frac{k^{ext.} + k^{self}}{2}\\ \hbar\dot{b_1} &= 2\frac{\hbar^2}{m}(a_1 b_2 + a_2 b_1)\\ \hbar\dot{b_2} &= -2\frac{\hbar^2}{m}(a_1 b_1 - a_2 b_2) + \frac{k^{self} b_1}{2a_1},\end{aligned} \tag{9}$$

$$\begin{aligned}a_1^{NL} = a_1 - a_1^L, a_2^{NL} = a_2 - a_2^L,\\ b_1^{NL} = b_1 - b_1^L, b_2^{NL} = b_2 - b_2^L\end{aligned} \tag{10}$$

$$\begin{aligned}\hbar\dot{a_1^L} &= 4\frac{\hbar^2}{m}a_1^L \cdot a_2^L\\ \hbar\dot{a_2^L} &= -2\frac{\hbar^2}{m}((a_1^L)^2 - (a_2^L)^2) + \frac{k^{ext.}}{2}\\ \hbar\dot{b_1^L} &= -2\frac{\hbar^2}{m}(a_1^L b_2^L + a_2^L b_1^L)\\ \hbar\dot{b_2^L} &= -2\frac{\hbar^2}{m}(a_1^L b_1^L - a_2^L b_2^L)\end{aligned} \tag{11}$$

2.3 Generalized guidance equation.

In a previous paper [6], one of us (T.D.) showed the following property (from now on denoted the generalized guidance condition):

-whenever ϕ_{NL} remains peaked throughout time in a sufficiently small region, the velocity of its barycentre $\mathbf{v}_{drift} = \frac{d\mathbf{x_0}(t)}{dt}$ obeys, in good approximation, the generalized guidance equation

$$\begin{aligned}\mathbf{v}_{drift} = \frac{\hbar}{m}\nabla\varphi_L(\mathbf{x_0}(t), t) + \frac{< \Psi_{NL}|\frac{\hbar}{im}\nabla|\Psi_{NL} >}{< \Psi_{NL}|\Psi_{NL} >}\\ = \mathbf{v}_{dB-B} + \mathbf{v}_{int.},\end{aligned} \tag{12}$$

which contains the well-known Madelung-de Broglie-Bohm [26, 27, 28] contribution ($\mathbf{v}_{dB-B} = \frac{\hbar}{m}\nabla\varphi_L(\mathbf{x_0}(t), t)$, where φ_L represents the phase of the pilot wave Ψ^L ($\frac{\hbar}{m}\bigtriangledown\varphi_L(t,x) = \frac{\hbar}{m|\Psi^L(t,x)|^2} Im.((\Psi^L(t,x))^*\bigtriangledown\Psi^L(t,x))$ (with $\varphi_L(t,x)$ the phase of the pilot wave) evaluated at the barycentre $\mathbf{x_0}(t)$ of Ψ^L, plus a new contribution, an "internal" velocity reflecting the contributions of the internal structure of the soliton ($\mathbf{v}_{int.} = \frac{<\Psi_{NL}|\frac{\hbar}{im}\bigtriangledown|\Psi_{NL}>}{<\Psi_{NL}|\Psi_{NL}>}$).

In order to test the range of validity of the generalized guidance condition through a concrete example, we numerically solved the dynamical equations (9,10,11). By doing so, we were able to obtain plots of the parameters x_0, p_0, σ_x^2 and σ_p^2 in function of time. These four parameters respectively represent the average values of the position and the velocity and their variances; in the case of a gaussian wave packet $\Psi(t,x) = e^{-Ax^2+Bx+C}$ they are in one to one correspondence with the four complex parameters a_1, a_2, b_1, b_2 introduced in the previous section:

$$x_0 = b_1/2a_1, p_0 = (b_2 a_1 - a_2 b_1)/a_1, \sigma_x^2 = 1/a_1, \sigma_p^2 = 4\hbar^2|A|^2/a_1 \quad (13)$$

Similar relations hold in the case of the gaussian wave packets Ψ^L and Ψ^{NL}. In order to check that the generalized guidance condition is well satisfied, we had to evaluate, in function of time,

i) v_{drift} the time derivative of the position of the barycentre of the solitonic wave packet $x_0^{NL} = \int d^3x'|\phi^{NL}(t,x')|^2 x' / \int d^3x'|\phi^{NL}(t,x')|^2 = b_1^{NL}/2a_1^{NL}$,

ii) the de Broglie-Bohm velocity

$$\begin{aligned} v_{dBB} &= (\frac{\hbar}{m|\Psi^L(t,x))|^2} Im.(\Psi^L(t,x))^* \bigtriangledown_x \Psi^L(t,x)))|_{x_0^{NL}} \\ &= \frac{1}{m} \cdot (-2a_2^L x_0^{NL} + b_2^L) \ , \end{aligned}$$

iii) the internal velocity

$$v_{int} = \frac{p_0}{m} = (b_2^{NL} a_1^{NL} - a_2^{NL} b_1^{NL})/a_1^{NL} \ .$$

As can be seen from figure 1, $v_{drift} = \frac{dx_0^{NL}}{dt}$ and $v_{dBB} + v_{int.}$ are not distinguishable with naked eyes, which establishes the validity of the generalized guidance condition, at least for the choice of parameters made in this case. Actually, we may also retrieve the generalized guidance condition by an analytic argument, remarking that, combining (ii)

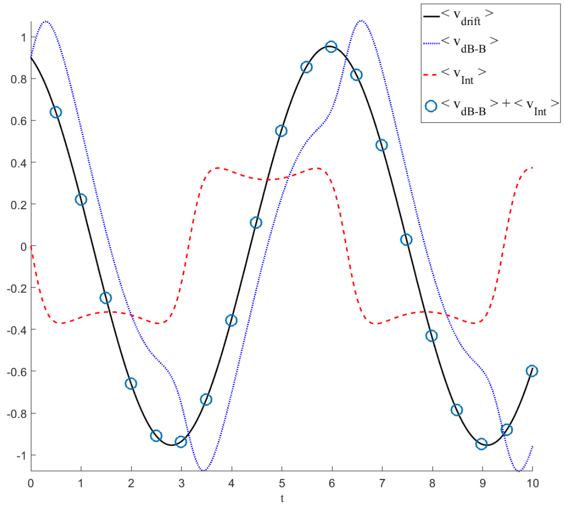

Figure 1: Plot of $v_{dBB}, v_{int.}, v_{dBB} + v_{int.}$ and $v_{drift} = \frac{d}{dt}(x_0^{NL})$ in function of time; space and time were rescaled and are of the order of unity.

and (iii) above we obtain $v_{dBB} + v_{int} = \frac{1}{m} \cdot (b_2^L + b_2^{NL} - \frac{b_1^{NL}}{a_1^{NL}} \cdot (a_2^L + a_2^{NL}))$ $= \frac{1}{m} \cdot (b_2 - 2 \cdot x_0^{NL} \cdot a_2)$. Now, the average velocity of the "full" wave packet Ψ is nothing else than $\frac{1}{m} \cdot (b_2 - \frac{b_1}{a_1} \cdot a_2) \approx \frac{1}{m} \cdot (b_2 - 2 \cdot x_0^{NL} \cdot a_2^{NL})$ because in the regime considered by us $x_0 = \frac{b_1}{2a_1} \approx x_0^{NL}$, due to the fact that the weight (size) of the pilot wave is quite smaller (larger) than the weight (size) of the soliton ($a_1^L <<< a_1^{NL}$). Then the velocity of the full packet also coincides with the drift velocity of the soliton.

This being said, it is worth noting that the drift velocity looks classical, as is obvious from a phase space representation of the average trajectory $(x_0^{NL}, \frac{dx_0^{NL}}{dt})$ which looks perfectly circular provided we rescale one of the variables appropriately as can be seen from figure 2.

This property was already noted before in ref.[15], whom authors remarked that when the dynamics obeys equations (9,10,11), the first moments of the distribution of positions and velocities are not affected at all by the non-linear coupling. One can prove this result rigorously on the basis of the dynamics (9,10,11), but this property is not a coinci-

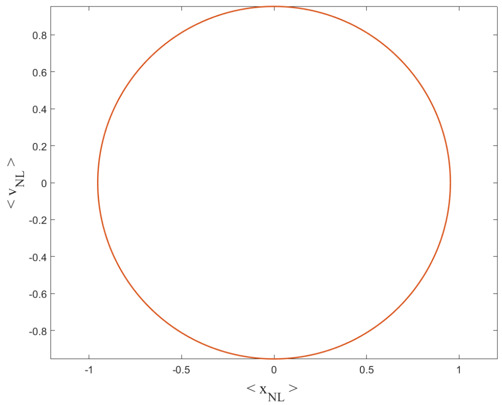

Figure 2: Plot of $(x_0^{NL}, \frac{dx_0^{NL}}{dt})$ in appropriate units, illustrating the validity of a generalised Ehrenfest's theorem.

dence; as was noted in [24, 6], one can explain the emergence of classical trajectories in terms of a generalized Ehrenfest theorem as we shall discuss in the next section. The second moments σ_x^2 and σ_p^2 are nevertheless modified by the evolution. As is very explicitly shown in figure 3, in our case, self-gravity results into a breathing of the wave packet [29, 8].

Actually there exists a stable radius for the gaussian packets for which self-gravity and diffusion exactly compensate each other. In full analogy with coherent oscillator states the size of these generalized coherent states is of the order of $\sqrt{\frac{\hbar}{\sqrt{k^{self}m}}}$ as was confirmed by our simulations (this is so at least when we impose that the ratio $k^{self}/k^{ext.}$ is equal to 10^3, in which case the small fluctuations of the shape of the generalized coherent states induced by the external harmonic potential are negligible). This can be seen from figure 3 which shows that when $\sigma_x^2 \approx \frac{\hbar}{\sqrt{k^{self}m}}$ the shape of the wave packet remains stable, whereas if it is disturbed, the size of the wave packet oscillates (breathes) around this equilibrium value.

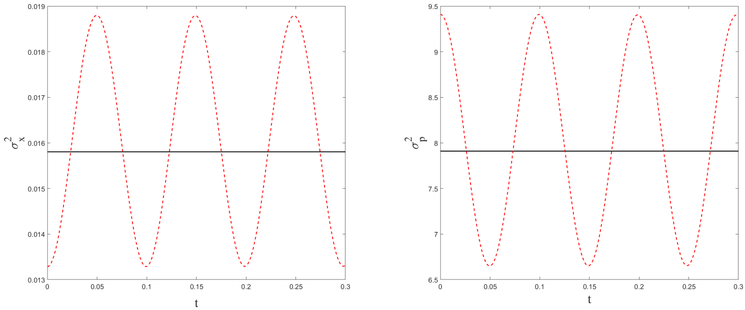

Figure 3: Time evolution of the second moments σ_x^2 and σ_p^2 with self-gravity for a "static" state (black) such that $\sigma_x^2 \approx \frac{\hbar}{\sqrt{k^{self} m}}$ and for an arbitrary state (red). For more details go to Appendix: Numerical simulations.

3 Generalised Ehrenfest theorem.

Equations (1) and (5) are invariant under Galilean transformations, which expresses that self-gravity does not result into self-acceleration, in accordance with Noether's theorem. This implies that even if a harmonic oscillator is self-gravitating, the average position of its center of mass rigorously obeys Newton equation and indeed we find that

$$\frac{d^2 < x >}{dt^2} = -\frac{k}{M} < x > \tag{14}$$
$$+M \cdot G \cdot (4\pi/3)\rho_{nucleus} \cdot \int d^3x \int d^3x' |\Psi(t, \mathbf{x})|^2 |\Psi(t, \mathbf{x'})|^2 \cdot 2(x - x'),$$

where $\int dx \int dx' |\Psi(t, \mathbf{x})|^2 |\Psi(t, \mathbf{x'})|^2 \cdot 2(x - x') = 0$ by symmetry. $< x >$ will thus oscillate at the classical frequency $\omega_c/2\pi$.

This constraint imposes severe limitations to our original program which was to realize de Broglie's double solution program, and thus to derive de Broglie-Bohm dynamics as a consequence of the factorization ansatz. This goal is reached only whenever de Broglie-Bohm trajectories coincide with classical trajectories which is not at all the result that we were looking for [6]. In other words, we aimed at derive de Broglie-Bohm dynamics and instead we found classical dynamics, a rather disappointing result. It is worth noting at this level that OSG exhibits the same features: in the classical limit (for macroscopic objects counting at ;east of the order of 10^{23} nucleons), in situations where self-gravitational collapse results in an extremely peaked localization of the CMWF of the object, we may neglect quantum fluctuations around the barycentre of the CMWF so that in virtue of the generalized Ehrenfest's theorem dynamics is FAPP classical. From this point of view, in both approaches (ours and OSG), we predict that localized objects obey classical dynamics in good approximation wich is however satisfactory if we wish to deal with the classical limit. However the classical limit is only one side of the measurement problem. Another side of the measurement problem is related to the probabilistic nature of the quantum predictions. This brings us to the last section.

4 Nelson dynamics

4.1 Stochastic trajectories and Born rule.

In OSG it is taken for granted from the beginning that the Born rule is obeyed that is to say that the modulus squared of the wave function assigned to a quantum system represents the stochastic distribution of its possible positions. The double solution program is more ambitious: in de Broglie's view, the dynamics (or guidance condition) must be complemented with the condition of equivariance: in order to mimic the Born rule at all times, it is sufficient, whenever the guidance condition is fulfilled, to impose that at a certain time the distribution of positions obeys the Born rule.

The problem to solve is now the following: how can we justify that at a certain time the distribution of positions obeys the Born rule. There are different ways to tackle this problem (see [30] for a review). Some arguments are based on the fact that the Born rule would represent an equilibrium condition for the dynamics which is asymptotically reached after suficiently long times. This approach is very close in mind to the approach followed by Boltzmann when he attempted to derive his famous H-theorem (see [31] for a review). In the latter approach, the probabilistic nature of the quantum predictions ought to be imputed, ultimately, to the sensitivity of the dynamics to initial conditions. As in classical chaos theory [31], sensitivity to the initial conditions provides the tool necessary for deriving a stochastic dynamics from a deterministic one. We shall no longer make reference to the aforementioned approach, among others because in all cases it is taken for granted that the guidance condition is satisfied to begin with, which is not true in our approach. There exists another approach however which could rescue ours, and bring a solution to our main problem which is that we derive classical dynamics instead of de Broglie-Bohm dynamics from the factorization ansatz and self-gravity. A possible solution consists in replacing, in "our" guidance equation, the internal velocity by an appropriate combination of deterministic and stochastic velocity. This alternative was already discussed in a previous paper [6], and its possible connections with the so-called zitterbewegung were underlined by then. Here we shall briefly present a model, due to Nelson, in which the velocity field is the sum of the de Broglie-Bohm field with a stochastic velocity field and with a third field, the osmotic velocity field, aimed at guaranteeing that equivariance is satisfied.

4.2 Nelson model

Nelson [20] conceived a model in which the drift velocity $\mathbf{b}(\mathbf{x}, t)$ is supplemented by a stochastic velocity[2]:

$$d\mathbf{x} = \mathbf{b}(\mathbf{x}, t)dt + d\mathbf{w}(t), \tag{15}$$

where $E_t[dw_i(t), dw_j(t)] = 2\nu\delta_{ij}dt$ $(i, j = 1, 2, 3)$, $\nu > 0$ being the diffusion coefficient and E_t representing the expectation value at time t. In our simulations we choose each dw_i to obey an independent gaussian distribution centered around zero; the variance of the distribution is fixed by the requirement that $E_t[dw_i(t), dw_i(t)] = 2\nu dt$ (these constraints characterize a white noise process *à la* Wiener). The conservation equation then takes the form:

$$\frac{\partial\rho}{\partial t} = -div(\mathbf{b}\rho) + \nu \triangle \rho \tag{16}$$

Nelson rewrites the conservation in the form

$$\frac{\partial\rho}{\partial t} = -div((\mathbf{b} - \nu\frac{\nabla\rho}{\rho})\rho), \tag{17}$$

and imposes that $\mathbf{b} = \nu\frac{\nabla\rho_Q}{\rho_Q} + \mathbf{v_{dBB}}$ where $\mathbf{v_{dBB}}$ represents the de Broglie-Bohm velocity

[2] Actually, our formulation of the stochastic dynamics slightly differs from Nelson's original proposal. It can be traced back to Bohm and Hiley [32]; see Ref. [33] for a detailed presentation and a comparison of both approaches.

$\mathbf{v_{dBB}} = (\frac{\hbar}{m|\Psi^L(t,\mathbf{x})|^2} Im.(\Psi^L(t,\mathbf{x}))^* \nabla \Psi^L(t,\mathbf{x})))$ and $\rho_Q(t,\mathbf{x}) = |\Psi^L(t,\mathbf{x})|^2$.

Nelson attributed to the parameter ν the value $\hbar/2m$. In the literature $\nu \frac{\nabla \rho_Q}{\rho_Q} \rho$ is called the osmotic current and $\nu \frac{\nabla \rho_Q}{\rho_Q}$ the osmotic velocity. It is easy to check that, resulting from the presence of the osmotic velocity, the quantum distribution $\rho_Q(t,\mathbf{x}) = |\Psi^L(t,\mathbf{x})|^2$ is an equilibrium distribution in the sense that it is equivariant: if at a certain time t, $\rho(t,\mathbf{x}) = \rho_Q(t,\mathbf{x}) = |\Psi^L(t,\mathbf{x})|^2$, the equality will be respected for all times in the future.

As was noted by Bohm [32], Nelson's model presents another advantage: it also explicitly provides a mechanism directly responsible for the irreversible onset of quantum equilibrium. Our numerical simulations confirm that indeed $\rho(t,\mathbf{x})$ irreversibly converges to $\rho_Q(t,\mathbf{x})$ throughout time, a property that we studied in depth in another paper [34].

4.3 1-D harmonic oscillator and Nelson's dynamics

In order to illustrate the stochastic process previously introduced (see [34] and references therein for a detailed treatment), we simulated Nelson's dynamics in the case of a 1-dimensional harmonic oscillator.

Consider a wave function at time $t = 0$ in the coherent state:

$$\Psi(x,0) = \left(\frac{\beta}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\beta}{2}(x-\bar{x}_t)^2} \tag{18}$$

where $\beta = \frac{m\omega}{\hbar}$ and $\bar{x}_t$ the mean position in which the wave function is initially centered.

This state evolves in time according to the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m}{2} \omega^2 x^2 \right] \Psi \tag{19}$$

which possesses particular solutions in the form of coherent states:

$$\Psi(x,t) = \left(\frac{\beta}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\beta}{2}(x-\bar{x}_t)^2 + i\frac{\bar{p}_t x}{\hbar} + i\varphi(t)} \tag{20}$$

where φ is a phase containing the energy, $\bar{x}_t$ and $\bar{p}_t$ are the mean position and the mean momentum at time t:

$$\bar{x}_t = \bar{x}_0 \, cos\,(\omega t) \qquad \text{and} \qquad \bar{p}_t = -m\omega \bar{x}_0 \, sin\,(\omega t) \tag{21}$$

Imposing that the wave function is such a coherent state, we numerically solved (15) for a bunch of initial conditions.

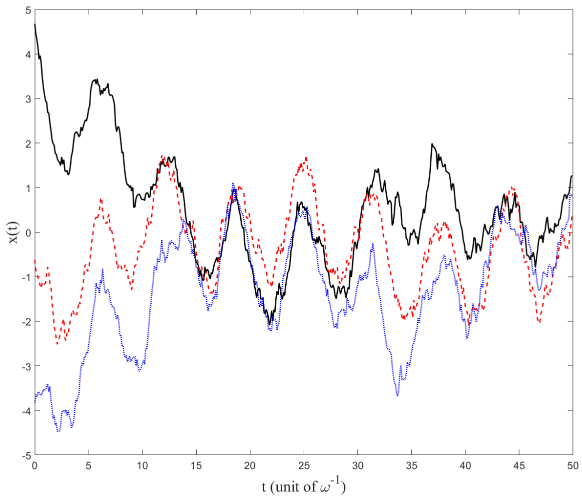

Figure 4: Numerical solutions of the Ito stochastic differential equation (15) for three different initial conditions. We took $\bar{x}_0 = 1$ and expressed the results in natural units.

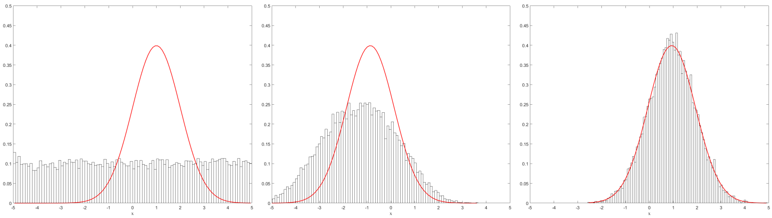

Figure 5: The evolution of the non-equilibrium distribution is illustrated with histograms of the resulting positions at three different times. The curve in red corresponds to $|\Psi|^2$. We started from a uniform distribution of initial conditions.

As we can see on figure (4), the trajectories are affected by the stochastic evolution but keep oscillating at the same period because

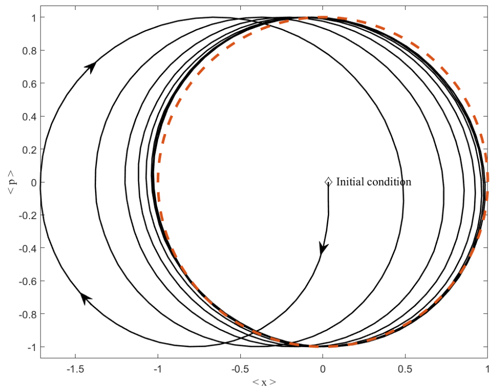

Figure 6: Phase space diagram of the expectation position and momentum in appropriate units. The asymptotic behavior, characteristic of the coherent state (20) is represented by the red dotted curve (equivalent to the classical trajectory plotted in figure 2).

of the deterministic part of the Ito process. Due to the properties of the Fokker-Planck equation (16) the current averaged over many realizations of the Ito process converges in time to the de Broglie-Bohm current. Following [34], one can illustrate step-by-step the relaxation process caracterising the evolution towards the Born distribution associated to the quantum state equilibrium state (20) (see figure 5). At equilibrium it means that the brownian motion is balanced by the osmotic velocity and the de Broglie-Bohm velocity is recovered on average (as well as the Born distribution of positions-in agreement with equivariance). This can also be seen on a phase-space diagram by plotting the averaged momentum in function of the averaged position (see figure 6). Note that when the equilibrium is attained, averaged trajectories obey classical dynamics resulting from the (usual) Ehrenfest theorem.

5 Conclusions and open questions

5.1 Open question: possible connection with bouncing oil droplets

As was stressed in previous papers, [16, 6], recently, de Broglie's point of view has been revived, be it indirectly, by experimental observations in hydrodynamics, which show that so-called walkers exhibit many of the features of the de Broglie-Bohm dynamics [35, 36, 37, 21].

Walkers consist of oil droplets bouncing at the surface of a vibrating bath of oil, excited in a Faraday resonance regime; the walkers are prevented from coalescing into the bath because the vibration creates an air film at the interface between the surface of the bath and the droplet. As was noted in [16], *...Walkers exhibit rich and intriguing properties.... For instance, when the walker passes through one slit of a two-slit device, it undergoes the influence of its "pilot-wave" passing through the other slit, in such a way that, after averaging over many trajectories, the interference pattern typical of a double-slit experiment is restored despite the fact that each walker passes through only one slit. The average trajectories of the drops exhibit several other quantum features such as orbit quantization [38], quantum tunneling, single-slit diffraction, the Zeeman effect and so on. Another surprising features is a pseudo-gravitational interaction that has also been observed between two droplets [39]. ... These observations suggest that a 'fluidic', hydrodynamical formulation of wave mechanics is possible, in which the droplet would play the role of de Broglie's soliton, while the properties of the environmental bath are assigned to the pilot wave of de Broglie....*

In view of these observations, de Broglie's original ideas have regained a certain prominence recently, since these walkers/bouncers were realized in the laboratory with artificial macroscopic systems. These unexpected developments not only show that de Broglie's ideas encompass a large class of systems, but they might in the future also allow us to build a bridge between quantum and classical mechanics, where ingredients such as nonlinearity, solitary waves and wave monism play a prominent role (see [16] for a review, and [40, 41] for an alternative approach).

Originally, the ansatz (7) has been introduced by us [17] in order to describe the phenomenology of walkers. In the case of droplets, our basic motivation for imposing the factorization ansatz is that walkers prepared at different positions and represented by Ψ^{NL} always see the same bath (environment) represented by Ψ^{L}. In the same paper [17], we extended this idea to arbitrary quantum systems, for instance to elementary particles and/or atoms molecules and so on. If, following the guidelines of de Broglie's double solution program, we retain the lessons of the present paper and apply them in order to simulate the phenomenology of droplets, we are led to conclude that noise could play the role of a monitoring parameter in the quantum-classical transition. It is worth noting that in the case of bouncing oil droplets a stochastic disturbance of velocities is often present, due to the periodical forcing imposed to the bath which unavoidably generates noise. This could explain why dB-B trajectories are never observed directly at the level of droplets and why interferences result from the averaging of a large number of trajectories. The ergodic nature of Nelson trajectories as well as the convergence of the distribution of positions to the Born distribution has been studied elsewhere [34]. In the same paper, we proposed experimental tests aimed at validating the relevance of Nelson's formalism in order to mimic the dynamics of droplets.

In the present paper we moreover suggest that quantum dynamics *à la* Nelson (section 4) emerges when noise is strong, while in the noiseless limit the mechanics is classical (section 3). This idea is illustrated in figures (7,8,9) where we show trajectories obtained in the case of the double slit experiment, ranging from a quantum behaviour (with noisy dB-B trajectories à la Nelson, fig. 9, see [42] for a similar work) to a purely classical, noiseless behaviour (with straight line [43, 44, 45, 46]) , fig.7) passing through an in-between region where superpositions are still present (fig.8). In order to compute the trajectories, we imposed that the velocity along Y is constant while the velocity along X is characterized by a real parameter ϵ chosen in the interval $[0, 1]$, and obeys the constraint $V_X = (1 - \epsilon) V_{Nelson} + \epsilon V_{classical}$, (see appendix and equation (25) for a more rigorous treatment). Figures (7,8,9) respectively correspond to the choices ($\epsilon = 1, \epsilon = 1/2, \epsilon = 0$). This model could maybe explain qualitatively why interference effects are observed [39] in certain double slit experiments performed with droplets and absent in others [47, 48] (see [49] for a review).

5.2 Conclusions

In this paper, we have studied the predictions of a model in which self-gravity is treated in the quadratic regime, valid when the size of the quantum system is larger than the size (mean square root deviation in position) of its center of mass wave function. Compared to more conventional approaches to self-gravity (OSG), we also imposed two unusual

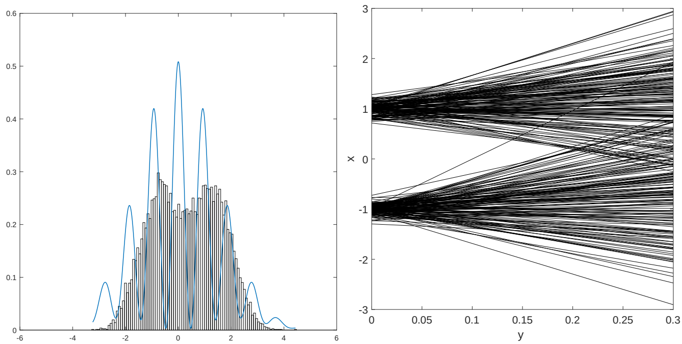

Figure 7: Numerical simulations of the double-slit experiment: classical trajectories (right), and distribution of arrivals on a screen (left). The curve (blue) corresponds to the quantum probability $|\Psi|^2$.

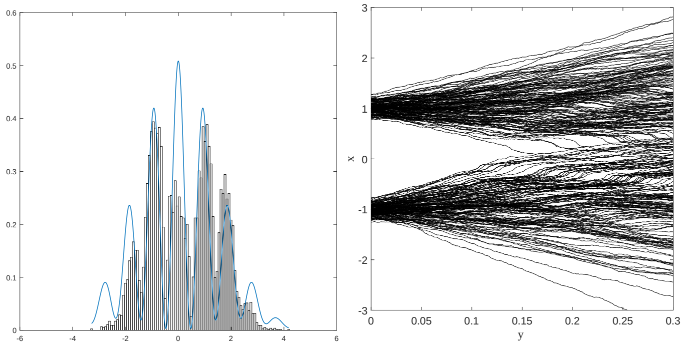

Figure 8: Numerical simulations of the double-slit experiment: hybrid (intermediate between quantum and classical) trajectories (right), and distribution of arrivals on a screen (left). The curve (blue) corresponds to the quantum probability $|\Psi|^2$.

conditions:

i) the L_2 norm of the center of mass wave function is huge (which ipso facto blows the effective coupling strength between the object and self-gravity);

ii) according to the factorization ansatz (7), the wave function factorizes into the product of a pilot wave Ψ^L solution of the linear Schrödinger

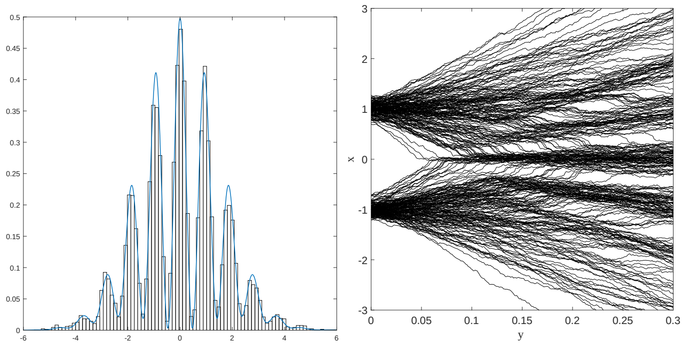

Figure 9: Numerical simulations of the double-slit experiment: *à la* Nelson trajectories (right), and distribution of arrivals on a screen (left). The curve (blue) corresponds to the quantum probability $|\Psi|^2$.

equation and of a (peaked) solitonic wave packet that represents the localized corpuscle, in accordance with de Broglie's double solution program.

We have numerically integrated the highly non-linear dynamics associated to the evolution of a self-gravitating gaussian wave packet trapped in an external harmonic potential. By doing so we have confirmed that the generalized guidance equation is well-satisfied, but also that the trajectories of the soliton are classical. Contrary to our primary hope which was to retrieve de Broglie-Bohm dynamics we noted that the internal velocity may not be neglected and even more, that it conspires, in agreement with the generalized Ehrenfest theorem presented in section 3, to restore classical dynamics.

If we wish to fulfill de Broglie's double solution program, a possibility consists of replacing the internal velocity by a stochastic velocity field, while preserving equivariance. This program has already been achieved in the past by Nelson (see section 4) whose dynamics provides useful guidelines if we wish to combine non-linearity and stochasticity in order to mimic the predictions of standard (linear) quantum mechanics. If, for instance, we wish to incorporate zitter bewegung [22] in a model of self-gravity [6], we learn from Nelson's model that we shall most probably have to derive and to justify the presence in the drift velocity of an osmotic velocity à la Nelson.

As has been noted by Gisin [50], the combination of non-linearity and stochasticity is also an essential tool in order to respect causality (expressed through the no signalling condition [51, 16, 52]). This combination is present in Nelson's model but also in spontaneous localisation models *à la* GRW [53], and even in all "shut up and compute" simulations of the Monte-Carlo type where an effective collapse is implemented. Our analysis suggests that, probably, all approaches inspired by the double solution program of de Broglie [1, 5, 54] will have to include stochasticity as an essential ingredient in order to tackle the hard task of mimicking quantum mechanics with realistic models.

It is worth noting that in the past Bohm and Vigier [19] as well as de Broglie [18] arrived to a similar conclusion; de Broglie attributed the origin of the stochastic quantum "zero-point field fluctuations" to a quantum thermostat or subquantum field, the study of which constitutes still today one of the main open questions in the framework of realistic hidden variables theories.

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Appendix: Numerical simulations

Firstly, the coupled equations (9) and (11) are rewritten in dimensionless form by introducing a characteristic time $\tau = \sqrt{\frac{m}{k^{ext.}}}$ and a characterstic length $L = \sqrt{\frac{\hbar\tau}{m}}$. Then we made the substitution $A \to \alpha/L^2$, $B \to \beta/L$, $x \to x'\,L$ and $t \to t'\,\tau$. These equations reach the following dimensionless form:

$$\begin{aligned}
\dot{\alpha_1} &= 4\alpha_1 \cdot \alpha_2 \\
\dot{\alpha_2} &= -2({\alpha_1}^2 - {\alpha_2}^2) + \frac{1+K}{2} \\
\dot{\beta_1} &= 2(\alpha_1\beta_2 + \alpha_2\beta_1) \\
\dot{\beta_2} &= -2(\alpha_1\beta_1 - \alpha_2\beta_2) + K\frac{\beta_1}{2\alpha_1},
\end{aligned} \tag{22}$$

$$\begin{aligned}
\dot{\alpha}_1^L &= 4\frac{\hbar^2}{m}\alpha_1^L \cdot \alpha_2^L \\
\dot{\alpha}_2^L &= -2((\alpha_1^L)^2 - (\alpha_2^L)^2) + \frac{1}{2} \\
\dot{\beta}_1^L &= 2(\alpha_1^L \beta_2^L + \alpha_2^L \beta_1^L) \\
\dot{\beta}_2^L &= -2(\alpha_1^L \beta_1^L - \alpha_2^L \beta_2^L)
\end{aligned} \tag{23}$$

which are easily solved with the well known Runge-Kutta 4th order method (RK4)[55]. Moreover, the problem is then parametrized by a unique dimensionless parameter $K = \frac{k^{self}}{k^{ext.}}$ which expresses the strength of the self-gravitation compared to the spring constant of the harmonic oscillator

• In the case of the figures 1 to 3 we plot the static case corresponding to $a_1(0) = \frac{\sqrt{1+K}}{2}$, $a_2(0) = 0$, $b_1(0) = 10$, $b_2(0) = 0.9$ and $K = 1000$. These values are imposed by the static conditions $\dot{a_1}(0) = \dot{a_2}(0) = 0$ in (22). For the pilot wave we chose $a_1^L(0) = 1$, $a_2^L(0) = 0$, $b_1^L(0) = 0$, $b_2^L(0) = 0.9$.

In order to be sure of the validity of the simulations we performed two realizations of the RK4 algorithm with different choices of the so-called tolerance parameter γ [34]. The accuracy of the computation depends on that parameter γ. We then compare the distance between the two trajectories; if it is less than some value, the result of the previous iteration of the RK algorithm is trusted. Otherwise, we perform another iteration with a smaller γ and we compare it to the last realization of the RK4. This procedure is repeated until the value of the distance between the two realizations of the RK algorithm is a minimal value of γ.

This method is very useful but is limited to a deterministic evolution and does not apply to a stochastic evolution. However, in the case of Nelson dynamics (section 4.2) we used the Euler-Maruyama method for stochastic processes to approximate the solution of the Ito equation (15). Following the same procedure as in Euler's method, the evolution time T is divided into N small discrete time steps Δt. For each time steps a random variable normally distributed $\Delta W_i = \sqrt{\Delta t}\mathcal{N}(0,1)$ is generated. The scheme of the integration has the following form:

$$x_{i+1} = x_i + b(x_i, i\,\Delta t)\Delta t + \sqrt{2\nu}\,\Delta W_i. \tag{24}$$

The question is then: how to choose the time step Δt in order to trust the result of the numerical simulations ? One way to answer this question is to consider an invariant of the dynamics. As previously said, it is known that the equilibrium condition $P = |\Psi|^2$ remains invariant under Nelson's dynamics. Consequently, we can test the validity of the simulation by starting directly from the equilibrium condition for different value of Δt. The resulting numerical simulation is trusted if the ensemble remains distributed according to $|\Psi|^2$.

• In the case of the figure 4 we plot three realizations of (15) with arbitrary initial conditions, for a coherent state (20). For the figures 5 and 6 the simulation is performed for 5 000 uniformly distributed initial conditions. For the figures 4 to 6 we chose $\nu = 0.5$, $\beta = 1$ and $\Delta t = 0.001$.

• Finally, in figures (7,8,9) we mix Nelson's dynamics (15) and classical dynamics according to the following rule:

$$dx = (1-\epsilon)\ [b(x,t)dt + \ dw(t)] + \epsilon\, V_{classical} dt, \tag{25}$$

where ϵ is a constant parameter quantifying the degree of classicality [56, 57] and $V_{classical}$ is the classical velocity associated to straight-line trajectories. The numerical simulation is then performed for this evolution and in the context of the double-slit experiment. To do so, we used the following wave function expressed in the coordinates (x,t):

$$\psi(x,t) = \frac{\exp\left(-\frac{\left(x-x_s+\frac{\hbar k}{m}t\right)^2}{2\sigma^2\left(1+\frac{it}{\sigma^2}\right)} - ik\left(x-x_s+\frac{\hbar k}{2m}t\right)\right)}{\sqrt{\sigma\left(1+\frac{it}{\sigma^2}\right)}} + \frac{\exp\left(-\frac{\left(x+x_s-\frac{\hbar k}{m}t\right)^2}{2\sigma^2\left(1+\frac{it}{\sigma^2}\right)} + ik\left(x+x_s-\frac{\hbar k}{2m}t\right)\right)}{\sqrt{\sigma\left(1+\frac{it}{\sigma^2}\right)}}, \tag{26}$$

where $\pm x_s$ and σ are the position and the width of each slit. We assumed that the wave is described by a plane wave in the y-direction so that the total wavefunction can be put in the product form $\Psi(x,y,t) = \psi(x,t)\phi(y,t)$. In order to avoid any numerical complexity it is also assumed that $\phi(y,t)$ is unaffected by the effect of the double-slit. We invite the reader interested in the details to consult the textbook of Peter R. Holland [28]. The initial positions for each slit are normally distributed around $\pm x_s$.

In order to compute classical trajectories, we associated for each position a normally distributed random velocity. We established the histograms of the distribution of positions on the arrival after a time 0.3 (in dimensionless units).The average value as well as the standard deviation of the distribution of classical velocities have been chosen in such a way that the spread of the resulting classical probability has approximately the same order of magnitude as the quantum (Nelson) one. We chose $x_s = 1$, $\sigma = 0.15$, $\Delta t = 0.001$ and $\nu = 0.5$.

References

[1] G. Bacciagaluppi and A. Valentini. Quantum Theory at the Crossroads: Reconsidering the 1927 Solvay Conference. Cambridge University Press (2010), and quant-ph/0609184.

[2] N. Gisin. Weinberg's non-linear quantum mechanics and superluminal communications. *Phys. Lett. A*, 143(1,2):1–2, 1990.

[3] J. Polchinski. Weinberg's nonlinear quantum mechanics and the Einstein-Podolsky-Rosen paradox. *Phys. Rev. Lett.*, 66(4):397–400, Jan. 1991.

[4] D. Fargue. Permanence of the corpuscular appearance and non linearity of the wave equation. In S. Diner et al., editor, *The wave-particle dualism*, pages 149–172. Reidel, 1984.

[5] D. Fargue. Louis de Broglie's double solution: a promising but unfinished theory. *Annales de la Fondation Louis de Broglie*, **42**, 19, 2017.; available on the website http://aflb.ensmp.fr/AFLB-421/table421.htm

[6] T. Durt. L. de Broglie's double solution and self-gravitation. *Annales de la Fondation Louis de Broglie*, **42**, 73, 2017.; available on the website http://aflb.ensmp.fr/AFLB-421/table421.htm

[7] S. Colin, T. Durt, and R. Willox. Can quantum systems succumb to their own (gravitational) attraction? *Class. Quantum Grav.*, 31:245003, 2014.

[8] S. Colin, T. Durt, and R. Willox. Crucial tests of macrorealist and semi-classical gravity models with freely falling mesoscopic nanospheres. *Phys. Rev. A*, 93, 0612, 2016.

[9] L. Diósi. Gravitation and quantum-mechanical localization of macro-objects. *Phys. Lett. A*, 105:199–202, 1984.

[10] Giulini,D. and Großardt, A. The Schrödinger-Newton equation as non-relativistic limit of self-gravitating Klein-Gordon and Dirac fields. *Classical and Quantum Gravity*, 29:215010, 2012.

[11] K. R. W. Jones. Newtonian Quantum Gravity. *Aust. J. Phys.*, 48:1055–1081, 1995.

[12] C. Møller. The energy-momentum complex in general relativity and related problems. In A. Lichnerowicz and M.-A. Tonnelat, editor, *Les*

Théories Relativistes de la Gravitation - Colloques Internationaux CNRS 91. CNRS, 1962.

[13] R. Penrose. On Gravity's Role in Quantum State Reduction. *General Relativity and Gravitation*, 28(5):581–600, 1996.

[14] L. Rosenfeld. On quantization of fields. *Nucl. Phys.*, 40:353–356, 1963.

[15] H. Yang, H. Miao, D-S. Lee, B. Helou, and Y. Chen. Macroscopic quantum mechanics in a classical spacetime. *Phys. Rev. Lett.*, 110:170401, 2003.

[16] S. Colin, T. Durt, and R. Willox. L. de Broglie's double solution program: 90 years later. *Annales de la Fondation Louis de Broglie*, **42**, 19, 2017.; available on the website http://aflb.ensmp.fr/AFLB-421/table421.htm

[17] T. Durt Generalized guidance equation for peaked quantum solitons and effective gravity. *Euro. Phys. Lett.*, 114, n° 1, 2016.

[18] L. de Broglie. Interpretation of quantum mechanics by the double solution theory *Annales de la Fondation Louis de Broglie*, **12**, 4, 1987, english translation from a paper originally published in the book Foundations of Quantum Mechanics- Rendiconti della Scuola Internazionale di Fisica Enrico Fermi, IL Corso, B. d' Espagnat ed. Academic Press N.Y., 1972.

[19] D. Bohm and J.P. Vigier. Model of the Causal Interpretation of Quantum Theory in Terms of a Fluid with Irregular Fluctuations
Phys. Rev., 96, 208, 1954.

[20] Nelson (Derivation of the Schroedinger equation from Newtonian Mechanics, Phys. Rev. 150, 4, 1079-1085, 1966.

[21] J. Bush. *Pilot-wave hydrodynamics. Annu. Rev. Fluid Mech.*, 47:269–292, 2015.

[22] D. Hestenes. The Zitterbewegung Interpretation of Quantum Mechanics *Founds. of Phys.*, **20**, 10, 1990.

[23] R. Penrose. On the Gravitization of Quantum Mechanics 1: Quantum State Reduction. *Foundations of Physics*, Vol. 44, Issue 5, 2014.

[24] A. Grossardt, J. Bateman, H. Ulbricht, and A. Bassi. Optomechanical test of the Schrödinger-Newton equation. *Phys. Rev. D*, 93:096003, 2016.

[25] A. Grossardt, Approximations for the free evolution of self-gravitating quantum particles *Phys. Rev. A*, 94: 022101, 2016.

[26] D. Bohm. A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. I. *Phys. Rev.*, 85(2):166–179, 1952.

[27] D. Bohm. A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables. II. *Phys. Rev.*, 85(2):180–193, 1952.

[28] P.R. Holland. The Quantum Theory of Motion Cambridge University Press, 1993.

[29] E.R. Arriola and J. Soler. A Variational Approach to the Schrödinger-Poisson System: Asymptotic Behaviour, Breathers, and Stability. *J. Stat. Phys.*, 103(5-6):1069–1105, 2001.

[30] A. Drezet. How to justify Born's rule using the pilot wave theory of de Broglie? *Annales de la Fondation Louis de Broglie*, **42**, 103, 2017.; available on the website http://aflb.ensmp.fr/AFLB-421/table421.htm

[31] C. Efthymiopoulos, G. Contopoulos and A.C. Tzemos . Chaos in de Broglie - Bohm quantum mechanics and the dynamics of quantum relaxation. *Annales de la Fondation Louis de Broglie*, **42**, 73, 2017.; available on the website http://aflb.ensmp.fr/AFLB-421/table421.htm

[32] D. Bohm and B.J Hiley. Non-locality and locality in the stochastic interpretation of the Quantum theory. *Annales de l'I.H.P. Physique théorique*, Tome 49, no. 3 , p. 287-296, 1988.

[33] A. Kyprianidis. The principles of a stochastic formulation of quantum theory. *Foundations of Physics*, Vol. 22, Issue 12, 1992.

[34] M. Hatifi, S. Colin, R. Willox and T. Durt. Bouncing oil droplets, de Broglie's quantum thermostat and convergence to equilibrium, submitted to *Chaos*, April 2018.

[35] Y. Couder and E. Fort. Single-Particle Diffraction and Interference at a Macroscopic Scale. *Phys. Rev. Lett.*, 97:15410, 2006.

[36] Y. Couder, A. Boudaoud, S. Protière and E. Fort. Walking droplets, a form of wave-particle duality at macroscopic scale? *Europhysics News*, Vol. 41, No. 1, 14-18, 2010.

[37] Y. Couder, S. Protière, E. Fort and A. Boudaoud. Dynamical phenomena: walking and orbiting droplets. *Nature* 8, 437 (7056), 208, 2005.

[38] M. Labousse, A. U. Oza, S. Perrard, and J. W. Bush. Pilot-wave dynamics in a harmonic potential: Quantization and stability of circular orbits, Physical Review E 93, 033122, 2016.

[39] Y. Couder and E. Fort. Single-Particle Diffraction and Interference at a Macroscopic Scale. *Phys. Rev. Lett.*, 97:154101 2006.

[40] C. Borghesi. Dualité onde-corpuscule formée par une masselotte oscillante dans un milieu élastique : étude théorique et similitudes quantiques *Annales de la Fondation Louis de Broglie*, **42**, 161, 2017.; available on the website http://aflb.ensmp.fr/AFLB-421/table421.htm

[41] C. Borghesi. Equivalent quantum equations with an external potential in a system inspired by bouncing droplets experiments. *Found. Phys.*, 47, vol.8, 2017.

[42] G. Grossing. Sub-quantum thermodynamics as a basis of emergent quantum mechanics, Entropy 12, 1975-2044, 2010.

[43] M. Labousse, Etude d'une dynamique a memoire de chemin: une experimentation theorique, Ph.D. thesis, Universite Pierre et Marie Curie UPMC Paris VI, 2014.

[44] E. Fort, A. Eddi, A. Boudaoud, J. Moukhtar, and Y. Couder, Path-memory induced quantization of classical orbits, Proceedings of the National Academy of Sciences 107, 17515-17520, 2010.

[45] R. Dubertrand, M. Hubert, P. Schlagheck, N. Vandewalle, T. Bastin, T. and J. Martin. Scattering theory of walking droplets in the presence of obstacles. *New J. Phys.*, **18**, (11), 113037, 2016.

[46] L. Tadrist, J.-B. Shim, T. Gilet, and P. Schlagheck, Faraday instability and subthreshold faraday waves: surface waves emitted by walkers, 2017.

[47] A. Andersen, J. Madsen, C. Reichelt, S.R. Ahl, B. Lautruo, C. Ellegaard, M.T. Levinsen and T.Bohr. Double-slit experiment with single wave-driven particles and its relation to quantum mechanics. *Phys. Rev. E* , **92** (1), 013006, 2015.

[48] T. Bohr, A. Andersen, B. Lautrup. Bouncing droplets, pilot-waves, and quantum mechanics. in Recent Advances in Fluid Dynamics with Environmental Applications (J. Klapp et al. Editors), 335, Springer (2016).

[49] G. Pucci, D.M. Harris, L.M. Faria and J.W. Bush. Walking droplets interacting with single and double slits. *J. Fluid. Mech.*, **835**, 1136, 2018.

[50] N. Gisin. Stochastic Quantum Dynamics and Relativity. *Helv. Hys. Acta*, 62(4):363–371, 1989.

[51] J. S. Bell. On the EPR paradox. *Physics*, 1: 165, 1964.

[52] T. Durt. About the possibility of supraluminal transmission of information in the Bohm- Bub theory. *Helv. Phys. Acta*, 72:356376, 1999.

[53] G. C. Ghirardi, A. Rimini, and T. Weber. Unified dynamics for microscopic and macroscopic systems. *Phys. Rev. D*, 34:470–491, Jul. 1986.

[54] L. de Broglie. Une tentative d'interprétation causale et non linéaire de la mécanique ondulatoire: la théorie de la double solution. Paris: Gauthier-Villars, (1956). English translation: Nonlinear wave mechanics: A causal interpretation. Elsevier, Amsterdam 1960.

[55] Press. W. H. Numerical recipes 3rd edition: The art of scientific computing. Cambridge university press, 2007.

[56] H. Nitta and T. Kudo, Time of arrival of electrons in the double-slit experiment, Physical Review A 77, 14102, 2008.

[57] P. Ghose, A continuous transition between quantum and classical mechanics. I, Found.Phys. 32, 871-892, 2002.

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