On the ambiguity of solutions of the system of the Maxwell equations

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In this paper, we analyze methods of solving the Maxwell equations. First, we investigate a method based on direct differentiation of the Maxwell equations and on reduction of them to the wave equation for the electric field. Unfortunately, this method cannot obtain the closed form solution containing no integrals. It is caused by the physical limitation on our knowledge of the structure of the electron. Analysis of the solution methods using the introduction of potentials and the gauge conditions show that the calculated expressions for the EM fields will be different for each gauge. Thus, we conclude that the Maxwell system of equations can have several solutions. For a unique choice of a single solution, we must fix the gauge. That is, it can be concluded that the gauge condition is a physical condition and different gauge conditions can be realized in different electrodynamic systems.

Keywords: The Maxwell equations, the wave equation, gauge conditions.

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1 Introduction

In this paper, we will analyze methods for solving the Maxwell equations. Although the equations themselves are the most well-known and most used equations in physics, perhaps due to obvious reliability of these equations, some of their aspects remain outside consideration.

For example, the physicists write the Maxwell equations to describe electrodynamic systems but these equations are not solved to find the E and H fields. Instead, as O'Rahilly notes (p. 184 of [1]), everyone uses the retarded potentials introduced by L. Lorenz [2] to obtain the solution. Lorenz has shown too that the wave equations, introduced by B. Riemann [3], and the condition on potentials, now called the gauge condition, are equivalent to the Maxwell equations. However, no one has shown the opposite - if the Maxwell equations are equivalent to the Riemann-Lorenz wave equations? Meanwhile, if the wave equations give a unique solution, the problem: whether the Maxwell equations have a unique solution, remains out of the question. Despite the seeming absurdity of raising such a question because there is no reason to doubt the validity of the Maxwell equations, it can be stated.

First, the question concerns not the validity of the Maxwell equations but possible existence of several solutions of this system. Secondly, when solving this system of equations by means of the potentials, an additional condition linking these potentials is introduced (the gauge condition). In fact, this condition is introduced arbitrarily since in the modern interpretation of electrodynamics, the potentials are treated as a mathematical tool. But if the gauge condition is introduced arbitrarily, and therefore the potentials are also determined with a certain degree of arbitrariness, it is reasonable to ask whether any sets of potentials defined by this condition lead to the same expressions for the fields. The latter is necessary because the fields are solutions of the Maxwell equations. And if the expressions for the fields are different when choosing different gauge conditions, we can conclude that the system of the Maxwell equations has several solutions for the EM fields.

In this paper, we investigate methods for solving the Maxwell equations and show that indeed the system of these equations has more than one solution. To do it we consider a canonical electrodynamic system, namely, the classical (point)charges moving in a vacuum in an arbitrary way and creating the EM fields. We show that a procedure of solving the Maxwell equations to find the EM fields of this system without introducing potentials, has a number of fundamental difficulties that do not allow obtaining the solution itself. This will be done in the next section.

2 Derivation of the wave equation for the electric field

The undoubted merit of Maxwell is the physical justification of the basic equations of electrodynamics. His opponents, the physicists of continental Europe (Kirchhoff, Weber, Riemann, Lorenz) paid more attention to mathematical aspects of the equations they introduced. For example, Riemann had proposed the solution of the wave equation for a scalar potential still in 1858 but withdrew his paper because he found errors in the calculation of the integral in the case of no single but some sources. Although, as it had been later shown, Riemanns method was correct.

We will pay more attention to the mathematical aspect of the Maxwell equations too since we are interested in methods of their solution. The physical interpretation of our analysis will be given in the last section. We will not be interested in the properties and not in the derivation of the Maxwell equations, but only in *methods of solving* these equations.

The original set of equations presented by Maxwell in his work of 1865 [4] and then in his Treatise are equations not only for the electromagnetic fields, but also for potentials, currents and forces acting on these currents. Meanwhile, these equations were presented in so called physical form, that is, rather as relations, but not in the classical form used in the theory of partial differential equations (PDE) when differential operators are being in the *lhs* and the sources in the *rhs* of the equations. Moreover, Maxwell did not distinguish between "the cause and the effect." Obviously, it was almost impossible to obtain solutions of such equations. The followers of Maxwell, so called *Maxwellians*, i.e. Heaviside, Fitzgerald, Lodge and Hertz significantly reduced the number of the initial equations. First, they excluded the forces from the original set of Maxwells equations, and second, they excluded the potentials. After their simplification of Maxwells system of the equations, the potentials are used as auxiliary quantities to determine the EM fields (we use the Gauss units; $\varepsilon = \mu = 1$ for the vacuum)

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\,; \quad \mathbf{H} = \nabla \times \mathbf{A}\,. \tag{1}$$

Despite the number of the basic equation had been essentially reduced to four ones, the general method of solving this system of PDE has not been known.

In order to solve the system of the Maxwell equations, it is necessary to reduce these equations to an equation whose solution is known. The PDEs with known solutions are the Poisson equation and the wave equation. But the Poisson equation contains only spatial variables. Meanwhile two of the Maxwell equations contain the partial time derivatives. Therefore, it is reasonable to reduce the Maxwell equations to the wave equation since the latter contains the time derivative. It can be done either by introducing the potentials (a method dating back to Lorenz), or by direct differentiation of the initial equations for the EM fields.

Let us consider the derivation of the wave equation for the electric field without the introduction of the potentials. First we have two Maxwell equations:

$$\frac{\partial \mathbf{H}}{c\partial t} = -\nabla \times \mathbf{E}\,; \tag{2}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{E}}{c\partial t} + \frac{4\pi \mathbf{j}}{c}\,. \tag{3}$$

Taking the curl of Eq. (2) and the time derivative of Eq. (3), we have

$$\nabla \times \frac{\partial \mathbf{H}}{c\partial t} = -\left[\nabla \times [\nabla \times \mathbf{E}]\right]\,; \tag{4}$$

$$\nabla \times \frac{\partial \mathbf{H}}{c\partial t} = \frac{\partial^2 \mathbf{E}}{c^2\partial t^2} + \frac{4\pi\partial \mathbf{j}}{c^2\partial t}\,. \tag{5}$$

Since the *lhs*'s of the equations are equal, their *rhs*'s should be equal too. As a result, we have

$$-\left[\nabla \times [\nabla \times \mathbf{E}]\right] = \frac{\partial^2 \mathbf{E}}{c^2\partial t^2} + \frac{4\pi\partial \mathbf{j}}{c^2\partial t}\,,$$

or presenting in a form more acceptable to mathematicians when the source is in the *rhs* of the equation, and the differential operators are in its *lhs*

$$-\left[\nabla \times [\nabla \times \mathbf{E}]\right] - \frac{\partial^2 \mathbf{E}}{c^2\partial t^2} = \frac{4\pi\partial \mathbf{j}}{c^2\partial t}\,. \tag{6}$$

We have obtained some differential equation for the E field. Usually this equation is treated as a differential equation that does not contain the longitudinal component of the radiated E field since the operator $\nabla\times$ 'cuts' it from the equation. However, Eq. (6) has no form of the wave equation yet. To reduce this equation to the wave equation, it is necessary to use the vector identity

$$-\left[\nabla \times [\nabla \times \mathbf{E}]\right] = \nabla^2 \mathbf{E} - \nabla\left(\nabla \cdot \mathbf{E}\right)\,,$$

and then remove the last term of that identity. Removal of this term is possible using one more Maxwell equation

$$\nabla\left(\nabla \cdot \mathbf{E}\right) = 4\pi\nabla\rho\,. \tag{7}$$

Indeed if we make term-by-term summation of Eqs. (6) and (7), we have

$$\begin{aligned} -\left[\nabla\times[\nabla\times\mathbf{E}]\right]-\frac{\partial^2\mathbf{E}}{c^2\partial t^2} &= \frac{4\pi\partial\mathbf{j}}{c^2\partial t}\,; \\ + \qquad \nabla\left(\nabla\cdot\mathbf{E}\right) &= 4\pi\nabla\rho\,; \\ \Longrightarrow \quad \nabla^2\mathbf{E}-\frac{\partial^2\mathbf{E}}{c^2\partial t^2} &= 4\pi\left(\nabla\rho+\frac{\partial\mathbf{j}}{c^2\partial t}\right). \end{aligned} \tag{8}$$

Thus, we obtain the wave equation for the electric field in Jefimenko's form [5].

However, there are two problems with this derivation. The first one is related to the procedure of derivation of Eq. (8). To obtain the latter equation, we make summation of two PDEs' of different types – Eq. (6) is of the hyperbolic type and Eq. (7) is of the elliptic type. Strictly speaking, there is no theorem in the theory of partial differential equations that allows doing such a term by term addition. The second problem is related to physical limitations. The solution of Eq. (8) is

$$\mathbf{E}(\mathbf{R};t)=\int\theta(\tau)\delta\left[c^2(t-\tau)^2-(\mathbf{R}-\mathbf{r})^2\right]\left\{\nabla\rho+\frac{\partial\mathbf{j}}{c^2\partial\tau}\right\}\mathtt{d}\mathbf{r}\mathtt{d}\tau\,. \tag{9}$$

where $\theta(t)$ is the step–function.

At the end of the XIX century, when Lorentz derived the wave equation for the E field [6], charges and currents were considered as objects having small but finite dimensions. In modern classical electrodynamics, it is assumed that the classical charge has a finite radius, but we shall ascribe physical significance only to those properties which are independent of the magnitude of the radius (Ch. 19.1 of [7]). However if the integral contains not a charge density but a gradient of this density, it is necessary to know the structure of the charge or the charge distribution over the radius of the electron. Meanwhile, this distribution is unknown even on a quantum level. So we can derive the wave equation for the E field (even ignoring a certain incorrectness of the derivation) but this equation cannot be solved, since its *rhs* is unknown.

If we would use the delta function to describe the charge density, it does not solve problem too because integral (9) will contain three singular functions. Moreover, one singular function should be differentiated. According to the theory of distributions, the integral containing some singular functions can be calculated, but if it is also necessary to calculate the derivative of the distribution (delta-function), this derivative must be transferred to a regular function. However, there are no such functions in the above integral. Therefore, the delta-function also cannot be used as the charge density in the formula of the solution of the wave equation for E.

If we consider the solution of the wave equation for the E fields given in [5] we find that the final result of Jefimenko (Eq. (2-2.12)) is still the retarded integral, and the algorithm how to compute this integral in the general case is absent.

Therefore, we have the following result: in the procedure of solution of the Maxwell equations, the wave equation for the fields cannot be used. In the other words, solving the Maxwell equations is possible only *by introducing the potentials.*

3 Solving the Maxwell equations by introducing the potentials

The second method to solve the Maxwell equations is to rewrite the EM fields via the potentials $\mathbf{A}$, φ, and then to find the solution of the equations for $\mathbf{A}$ and φ .

In the middle of the XIX century Riemann and Lorenz introduced equations that should explain the propagation of electromagnetic potentials with a finite velocity, the wave equations. They presented the solutions of these equations too. Maxwell derived equations for components of the vector potential in a form of homogeneous wave equation (Sec. 784 of [4]) but without solution of these equations. Starting from the form of these equations, Maxwell concluded that light should propagate as waves of a magnetic field. It has been established still in XIX century that propagation of light is a wave process. Hertz experimentally confirmed that propagation of the EM fields is the wave process too. So it was quite reasonable that to solve the Maxwell equations one should reduce them to the wave equations.

As far as the authors know, it was Lorentz who first reduced the Maxwell equations to the wave equations for the potentials [6]. Meanwhile Lorenz treated only the scalar potential as a physical quantity. Analysis of the equipotential surfaces of the uniformly moving single charge allowed to Lorentz to make a conclusion that the size of such a charge should contract in a direction of motion because the surface of the charge should be equipotential too.

But instead of the vector potential, Lorentz introduced a certain auxiliary quantity without associating any physical properties to it. It was an important point in the further interpretation of potentials as abstract values or mathematical tool to solve the Maxwell equations.

Still in 1867 Lorenz showed that the solutions of their wave equations presented in the form of retarded integrals

$$\varphi = \frac{1}{4\pi}\int \frac{[\rho]}{r}\mathbf{dr}\,; \quad \mathbf{A} = \frac{1}{4\pi c}\int \frac{[\mathbf{j}]}{r}\mathbf{dr}\,,$$

and if the potentials φ and $\mathbf{A}$ are connected by the relation

$$\frac{1}{c}\frac{\partial\varphi}{\partial t} + \nabla\cdot\mathbf{A} = \frac{1}{4\pi c}\int \frac{[\partial\rho/\partial t + \nabla\cdot\mathbf{j}]}{r}\mathbf{dr} = 0\,, \tag{10}$$

these potentials are solutions of the Maxwell 's equations too (p. 182 of [1]). So it is obvious that solutions of these equations should be sought using potentials. To do it, one needs:
1) to express the EM fields via the potentials;
2) to separate the potentials in the obtained equations.
Realizing p. 1 for the Maxwell equations, we have for $\nabla\cdot\mathbf{E} = 4\pi\rho$

$$-\nabla\left(\nabla\varphi + \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right) = 4\pi\rho \;\rightarrow\; -\nabla^2\varphi - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = 4\pi\rho\,, \tag{11}$$

and for the equation

$$\nabla\times\mathbf{H} = \frac{\partial\mathbf{E}}{c\partial t} + \frac{4\pi\mathbf{j}}{c} \;\rightarrow$$
$$[\nabla\times[\nabla\times\mathbf{A}]] = -\frac{\partial^2\mathbf{A}}{c^2\partial t^2} - \nabla\frac{\partial\varphi}{c\partial t} + \frac{4\pi\mathbf{j}}{c}\,. \tag{12}$$

Separation of the potentials in Eqs. (11) and (12) should be made by eliminating the term $\nabla\,\partial\varphi/\partial t$ from Eq. (12) or the term $\nabla\cdot\partial\mathbf{A}/\partial t$ from Eq. (11). Using the Lorenz condition (the Lorenz gauge, Eq. (10)), we have for Eq. (11)

$$-\nabla^2\varphi - \frac{1}{c}\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = 4\pi\rho \;\rightarrow\; -\nabla^2\varphi + \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t} = 4\pi\rho\,.$$

Eliminating the scalar potential from Eq. (12) by means of the same relation (10), we obtain a similar wave equation for the vector potential

$$-\nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t} = \frac{4\pi\mathbf{j}}{c}\,.$$

However, the potentials in Eqs. (11) and (12) can be separated in another way, namely by means of Maxwell?s original condition $\nabla \cdot \mathbf{A} = 0$ (the Coulomb gauge). Then the second term in the *lhs* of Eq. (11) vanishes and this equation transforms to the Poisson equation

$$-\nabla\left(\nabla\varphi + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) = 4\pi\rho \;\rightarrow\; -\nabla^2\varphi = 4\pi\rho\,.$$

The solution of this equation is known, it is the scalar potential that propagates instantaneously. Having known the solution for the scalar potential, we are able to treat the term $\nabla\left(\partial\varphi/\partial t\right)$ in Eq. (12) as a source along with the term containing the current density. In this case it is easy to obtain the following wave equation for the vector potential

$$-\nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t} = \frac{4\pi\mathbf{j}}{c} - \nabla\frac{\partial\varphi}{c\partial t}\,.$$

Thus we have two extreme cases of types of propagating the scalar potentials. If the vector potential obeys the wave equation in both cases, the scalar potential propagates either at a speed of light (the Lorenz gauge) or at infinite speed (the Coulomb gauge). Moreover, by accepting the relationship between potentials

$$\nabla\cdot\mathbf{A} + \frac{c}{u^2}\frac{\partial\varphi}{\partial t} = 0\,, \tag{13}$$

we are able to separate the potentials in the Maxwell equations. So we can have a separate (wave) equation for the scalar potential and a separate wave equation for the vector potential, which, however, includes the scalar potential as a source. Since the parameter u can vary continuously in the range from $u = c$ to $u = \infty$, we have infinite number of the equations for the potentials.

The question arises: does the presence of an infinite number of systems of equations for potentials mean the presence of an infinite number of solutions for EM fields, or do all these systems of equations give the same expressions for the EM fields? As we have shown in Sec. 1, the Maxwell equations can be solved only by introducing potentials so the question can be stated in another form: do the Maxwell equations have a unique solution for the EM fields, or they have infinite number of solutions (in accordance to infinite number of possible gauge conditions)?

We note that Eq. (13) does not contain any physical justification it is purely mathematical constrain, and its introduction became to be possible after work of Heaviside when the potentials began to be treated as mathematical symbols. Meanwhile, both Lorenz and Coulomb gauge conditions were initially considered as a link between the physical quantities. The Lorenz gauge can be considered as a consequence of the charge conservation law [2]. Maxwell used certain physical arguments when introducing the Coulomb gauge. So he did not introduce div$\mathbf{A} = 0$, but the condition div$\mathbf{A} = \mathbf{J}$ (here, $\mathbf{J}$ is not the current density; we use the original Maxwells notations, Sec. 99 of [4]) and then explained why the value J should be equal to zero.

In the next section, we study the question if the potentials found in different gauges give the same expressions for EM fields. We will not investigate an infinite number of systems of equations, but consider only "physically justified" gauges their use should give expressions for "physically justified" fields. If we finally obtain that the expressions for the fields are different, we will have the right to conclude that the Maxwell equations cannot be solved unambiguously.

4 On the inequality of the E fields in different gauges

The simplest way to verify if the potentials evaluated in different gauges give the same expressions for the EM fields is to find the relationship between the potentials calculated in the Lorenz and Coulomb gauges. Assuming the electric fields in these gauges are equal, we rewrite the difference between these fields in terms of potentials

$$\begin{aligned} \mathbf{E}_{\mathrm{L}} - \mathbf{E}_{\mathrm{C}} = 0 \quad \Longrightarrow \\ -\nabla\varphi_{\mathrm{L}} + \nabla\varphi_{\mathrm{C}} - \frac{1}{c}\frac{\partial \mathbf{A}_{\mathrm{L}}}{\partial t} + \frac{1}{c}\frac{\partial \mathbf{A}_{\mathrm{C}}}{\partial t} = 0 \quad . \end{aligned} \tag{14}$$

Because the vector potential in two gauges obeys the following equations,

$$\nabla^2 \mathbf{A}_C - \frac{\partial^2 \mathbf{A}_C}{c^2 \partial t^2} = -\frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\nabla\frac{\partial \varphi_C}{\partial t}\,, \tag{15}$$

$$\nabla^2 \mathbf{A}_L - \frac{\partial^2 \mathbf{A}_L}{c^2 \partial t^2} = -\frac{4\pi}{c}\mathbf{J}\,, \tag{16}$$

we have for the difference of the vector potential in two gauges

$$\mathbf{A}_L - \mathbf{A}_C = -\frac{1}{4\pi c}\int G(\mathbf{R}-\mathbf{r};t-\tau)\nabla_r \frac{\partial \varphi_C(\mathbf{r};\tau)}{\partial \tau}\,\mathtt{d}\mathbf{r}\mathtt{d}\tau\,,$$

where $G(\mathbf{R}-\mathbf{r};t-\tau)$ is the Green function of the wave equation. After integration over the τ variable, we have

$$\mathbf{A}_L - \mathbf{A}_C = -\frac{1}{4\pi c}\int \frac{1}{|\mathbf{R}-\mathbf{r}|}\left[\nabla_r \frac{\partial \varphi_C(\mathbf{r};t)}{\partial t}\right] \mathbf{dr}\,, \tag{17}$$

where the square brackets [...] are used (the notations of Jefimenko [5]) for the quantities depending on the retarded time $t_{ret} = t - |\mathbf{R}-\mathbf{r}|/c$.

If we compute the partial time derivative of Eq. (17) and insert the result into Eq. (14), the latter takes a form

$$\nabla\varphi_{\rm C} - \nabla\varphi_{\rm L} + \frac{1}{4\pi c^2}\frac{\partial}{\partial t}\int \frac{1}{|\mathbf{R}-\mathbf{r}|}\left[\nabla_r \frac{\partial \varphi_C(\mathbf{r};t)}{\partial t}\right] \mathbf{dr} = 0\,. \tag{18}$$

One can show that the operator ∇_r that acts on the internal variable in the Coulomb scalar potential can be moved out of the integral

$$\int \frac{1}{|\mathbf{R}-\mathbf{r}|}\left[\nabla_r \frac{\partial \varphi_C(\mathbf{r};t)}{\partial t}\right] \mathbf{dr} = \nabla_R\left(\int \frac{1}{|\mathbf{R}-\mathbf{r}|}\left[\frac{\partial \varphi_C(\mathbf{r};t)}{\partial t}\right] \mathbf{dr}\right)\,.$$

All details of such a procedure are given in the Appendix. So (18) takes a form

$$\nabla_R\,\varphi_{\rm C}(\mathbf{R};t) - \nabla_R\,\varphi_{\rm L}(\mathbf{R};t) + \frac{1}{4\pi c^2}\nabla_R\left(\int \frac{1}{|\mathbf{R}-\mathbf{r}|}\left[\frac{\partial^2 \varphi_C(\mathbf{r};t)}{\partial t^2}\right] \mathbf{dr}\right) = 0\,.$$

We are able to omit the operator ∇_R that acts on all terms in the above expression, that leads to

$$\varphi_L(\mathbf{R},t) - \varphi_C(\mathbf{R},t) = \frac{1}{4\pi c^2}\int \frac{1}{|\mathbf{R}-\mathbf{r}|}\left[\frac{\partial^2 \varphi_C(\mathbf{r},t)}{\partial t^2}\right] \mathbf{dr}\,. \tag{19}$$

We emphase that (19) (as well as (18)) is not the equation [1] but some equality between the expressions for the potentials. This equality – if it is fulfilled – can be treated as a proof of equivalence of the electric field computed in different gauges. We note that a similar criterion is used by Jackson [8]. Therefore, our task is to verify the fulfillment of (19). To do it, we consider two equations for the scalar potential in the Lorenz and Coulomb gauges created by a single classical charge being in arbitrary motion

$$\nabla^2\varphi_L - \frac{1}{c^2}\frac{\partial^2 \varphi_L}{\partial t^2} = -4\pi\rho\,; \tag{20}$$

$$\nabla^2\varphi_C = -4\pi\rho\,. \tag{21}$$

[1] Because they cannot be used to find the difference of the potentials.

where ρ is its charge density. Now we subtract Eq. (21) from Eq. (20), which gives

$$\nabla^2\varphi_L - \nabla^2\varphi_C - \frac{1}{c^2}\frac{\partial^2\varphi_L}{\partial t^2} = 0\,. \tag{22}$$

One can state a question if the functions ρ are identical in both Eqs. (20) and (21). But because we consider a point classical charge, ρ is described by the delta-function in both cases as $\rho(\mathbf{r};t) = q\delta\left(\mathbf{r} - \mathbf{r}_0(t)\right)$, where $\mathbf{r}_0(t)$ is the coordinate of the "center" of the charge determined by its law of motion.

The following steps can be considered as formal (but correct!) operations. Let us move the term with the second time derivative in the *lhs* of Eq. (22) to its *rhs* and act on all terms by the operator

$$\nabla^2\varphi_L - \nabla^2\varphi_C = \frac{1}{c^2}\frac{\partial^2\varphi_L}{\partial t^2}\,. \tag{23}$$

Application of the integral identity

$$\int \frac{\nabla^2 f(\mathbf{r},t)}{|\mathbf{R}-\mathbf{r}|}\,\mathtt{d}\mathbf{r} = 4\pi f(\mathbf{R};t)$$

to Eq. (23) gives

$$\varphi_L(\mathbf{R},t) - \varphi_C(\mathbf{R},t) = \frac{1}{4\pi c^2}\int \frac{1}{|\mathbf{R}-\mathbf{r}|}\frac{\partial^2\varphi_L(\mathbf{r},t)}{\partial t^2}\,\mathtt{d}\mathbf{r}\,. \tag{24}$$

We note that we repeat the procedure of evaluating the difference between the potentials given in [9], although the physical meaning of our procedure is different to the interpretation of Engelhardt.

The key point of our proof is the comparison of Eqs. (19) and (24). Because the *lhs*s of the equations are equal, we conclude that their *rhs* s must be equal too, or

$$\int \frac{1}{|\mathbf{R}-\mathbf{r}|}\left[\frac{\partial^2\varphi_C(\mathbf{r},t)}{\partial t^2}\right]\mathtt{d}\mathbf{r} = \int \frac{1}{|\mathbf{R}-\mathbf{r}|}\frac{\partial^2\varphi_L(\mathbf{r},t)}{\partial t^2}\,\mathtt{d}\mathbf{r}\,. \tag{25}$$

where we omit the factor $1/4\pi c^2$ from both sides of the above equation. This expression can be simplified by removing the second time derivative out of the integral (it is valid for the retarded quantities [5]), and then omit the time derivatives from both sides of the equation too. It gives

$$\int \frac{\varphi_L(\mathbf{r};t)}{|\mathbf{R}-\mathbf{r}|}\,\mathtt{d}\mathbf{r} = \int \frac{[\varphi_C(\mathbf{r};t)]}{|\mathbf{R}-\mathbf{r}|}\,\mathtt{d}\mathbf{r}\,. \tag{26}$$

So, we have obtained the integral relation between the scalar potential calculated in the Coulomb gauge, which depends, however, on the retarded time, and the scalar potential in the Lorenz gauge. Although in most formulas this scalar potential is given as dependent on the retarded coordinates, in Eq. (26) it should be expressed in the current coordinates.

If the relation (26) is fulfilled, then the difference between the expressions for the electric field, calculated in different gauges, will be equal to zero. It should mean too that the expressions for the E field, calculated in any gauge, must be identical.

Eq. (26) allows the further simplification. Considering that the scalar potential in both cases is created by a point charge and writing the expressions for the scalar potential as a solution of the corresponding equation with the source the δ–function, we obtain

$$\int \frac{1}{|\mathbf{R}-\mathbf{r}|}\left(\int \frac{\delta\left\{\mathbf{r}'-\mathbf{r}_0(t-|\mathbf{r}-\mathbf{r}'|/c)\right\}}{|\mathbf{r}-\mathbf{r}'|}\,\mathtt{d}\mathbf{r}'\right)\mathtt{d}\mathbf{r} = \\ = \int \frac{1}{|\mathbf{R}-\mathbf{r}|}\left(\int \frac{\delta\left\{\mathbf{r}'-\mathbf{r}_0(t-|\mathbf{R}-\mathbf{r}|/c)\right\}}{|\mathbf{r}-\mathbf{r}'|}\,\mathtt{d}\mathbf{r}'\right)\mathtt{d}\mathbf{r}\,. \qquad (27)$$

Because the δ-functions in Eq. (27) have different dependences on the variables, it cannot be satisfied in the general case. This result proves that the expressions for the fields, calculated in different gauges, should be different.

5 Conclusions

In this paper, we analyze the methods of solving the Maxwell equations. In fact, only three such complete solutions exist, namely, the method of direct solving equations solely for the EM fields, i.e. without introducing potentials, and two methods for solving equations by introducing potentials. The method of direct solving has been developed by Jefimenko in a number of his articles. However, Jefimenko did not give the closed form expressions for the EM fields. His final expression is written in a form of the retarded integral and therefore is not explicitly calculated. This is explained by the fact that the sources that Jefimenko considered are extended charged bodies. Meanwhile, for the electrodynamic system, mentioned in Sec. 2, Jefimenkos expression cannot be calculated, since it contains singular functions, and the integral contains not only a singular function, but its derivative.

The most used method to solve the Maxwell equations is based on their reduction to two wave equations for potentials written in the Lorentz gauge. Another commonly used method to solve these equations is based on separation of the equations for the potentials in the Coulomb gauge. Meanwhile it is commonly accepted opinion that the expressions for EM fields calculated in any gauge should be identical.

However, our analysis shows that this is not the case. Both Lorenz and Maxwell introduced the conditions, subsequently called gauge conditions, as physical link between the potentials. Only after works of *Maxwellians* (Heaviside, Lodge, Fitzgerald, Hertz), both the potentials and the conditions on the potentials began to be interpreted as purely mathematical subjects. This assumption, however, requires proof of the equivalence of the EM fields under different choice of the gauge condition. This proof has not been made that time. In the next years, several papers containing proof of the equivalence of the EM fields in the Coulomb and Lorentz gauges appear. But the main lack of all these works is in the absence of an example of calculating the equality of such fields in the closed form.

Indeed, the expression for the **E** field in the Coulomb gauge contains a retarded integral of a non-local source. Such an expression cannot be calculated in general form in elementary functions or reduced to special functions. The only calculated example is the EM fields of a single charge, when the latter moves uniformly in a straight line from $x = -\infty$ to $x = +\infty$. As shown by Hnizdo [10], in this case the expressions for the fields are identical in both gauges. This is quite understandable in our consideration. If in Eq. (26) the time variable can be treated as a parameter and coordinate transformations allow eliminating this parameter from the further consideration, the second derivative of the potentials will give zero, which provide equality of the previous relation (25). So the fields $\mathbf{E}_C$ and $\mathbf{E}_L$ are equal in this case.

However, if a time variable cannot be treated as a parameter that is true for any non-stationary process, then the fields will not be equal. This can be confirmed by calculations in a system in which the unit charge is initially at rest and at time $t = 0$ begins to move at a constant speed v along the x–axis. In this case, the difference between the E_x fields can be calculated in the closed form. We note that it is the only possible case when the calculations of the difference between the E fields can be led to the expressions containing no uncomputable integrals.

Meanwhile, our proof shows that the expressions for the fields calculated in different gauges should be different. As a result, we can make the following conclusions:

1. For a system of point classical charges, a direct solution of the Maxwell equations cannot be obtained;

2. Solving the Maxwell equations is possible only via introduction of the potentials;

3. Expressions for EM fields should depend on the choice of a gauge.

Finally we suggest that our results can open certain perspective in the study of solutions of the Maxwell equations. To our point of view, the gauge conditions are physical conditions and therefore they can be determined by the specific type of initial and boundary conditions generated by corresponding distribution of the sources. In particular, of great interest is the study of the properties of the component of the field E created by the scalar potential in the Coulomb gauge. This scalar potential propagates instantaneously and the E component created by this potential must propagate instantaneously too. It is not fully studied how does the latter impact on various charged objects.

6 Appendix

Let us consider transformation of the *rhs* of Eq. (18). Here, we use the notations of Jefimenko that the expression in the squared brackets $[f]$ is a function of the retarded time. Then the notations $[\nabla_r f]$ means that the operator ∇_r acts only on the coordinates and after calculation of the action of this operator, the time dependence associates with the retarded time. Then

$$\nabla_r[f] = [\nabla_r f] + \left[\frac{\partial f}{\partial t}\right] \cdot \nabla_r \left(t - \frac{|\mathbf{R}-\mathbf{r}|}{c}\right) = [\nabla_r f] + \left[\frac{\partial f}{\partial t}\right] \cdot \frac{\mathbf{R}-\mathbf{r}}{c|\mathbf{R}-\mathbf{r}|}$$

Let us calculate the partial derivative of the integrand with respect to the inner variable r.

$$\int \nabla_r \left(\frac{1}{|\mathbf{R}-\mathbf{r}|}\left[\frac{\partial \varphi_C(\mathbf{r};t)}{\partial t}\right]\right) \mathtt{d}\mathbf{r} = \int \left(\nabla_r \frac{1}{|\mathbf{R}-\mathbf{r}|}\right)\left[\frac{\partial \varphi_C(\mathbf{r};t)}{\partial t}\right] \mathtt{d}\mathbf{r} +$$

$$+ \int \frac{1}{|\mathbf{R}-\mathbf{r}|}\left[\nabla_r \frac{\partial \varphi_C(\mathbf{r};t)}{\partial t}\right] \mathtt{d}\mathbf{r} + \int \frac{1}{|\mathbf{R}-\mathbf{r}|}\frac{\partial}{\partial t}\left[\frac{\partial \varphi_C(\mathbf{r};t)}{\partial t}\right] \cdot \frac{\mathbf{R}-\mathbf{r}}{c|\mathbf{R}-\mathbf{r}|} \mathtt{d}\mathbf{r}$$

But the volume integral

$$\int \nabla_r \left(\frac{1}{|\mathbf{R}-\mathbf{r}|} \left[\frac{\partial \varphi_C(\mathbf{r};t)}{\partial t} \right] \right) \mathtt{d}\mathbf{r}$$

is transformed into the surface integral where the surface of integration is expanding to infinity so the value of this integral tends to zero.

Also one can find that

$$\int \left(\nabla_r \frac{1}{|\mathbf{R}-\mathbf{r}|} \right) \left[\frac{\partial \varphi_C(\mathbf{r};t)}{\partial t} \right] \mathtt{d}\mathbf{r} + \int \frac{1}{|\mathbf{R}-\mathbf{r}|} \frac{\partial}{\partial t} \left[\frac{\partial \varphi_C(\mathbf{r};t)}{\partial t} \right] \cdot \frac{\mathbf{R}-\mathbf{r}}{c|\mathbf{R}-\mathbf{r}|} \mathtt{d}\mathbf{r} =$$

$$= -\nabla_R \left(\frac{1}{|\mathbf{R}-\mathbf{r}|} \left[\frac{\partial \varphi_C(\mathbf{r};t)}{\partial t} \right] \right) \mathtt{d}\mathbf{r}$$

So we finally have

$$\int \frac{1}{|\mathbf{R}-\mathbf{r}|} \left[\nabla_r \frac{\partial \varphi_C(\mathbf{r};t)}{\partial t} \right] \mathtt{d}\mathbf{r} = \nabla_R \left(\int \frac{1}{|\mathbf{R}-\mathbf{r}|} \left[\frac{\partial \varphi_C(\mathbf{r};t)}{\partial t} \right] \mathtt{d}\mathbf{r} \right)$$

which means that the operator ∇_r can be removed from the integral with the change $\nabla_r \to \nabla_R$

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