A note on the Reasonable Induction of Quantum Physics

Norman Joseph Goldstein

Clear System Solutions RD Inc.
27-11751 King Road, Richmond, BC
Canada V7A 3B5

ABSTRACT. Lucien Hardy has laid out a program for deducing quantum physics from reasonable axioms. One step of the process is the inductive construction of level N quantum physics (QP) from level 2 QP (the Bloch sphere). In this note, we point out a detail to be filled in in Hardy's program, and supply the proof. By this stage of Hardy's derivation, we know that level 2 state space is QP, and that level N state space is identified as a subset of the Hermitian NxN matrices. The key technical result of this article is that an embedding of level 3 state space into the Hermitians is rather rigid – to convert it to standard QP requires only a conjugation and/or scaling of select entries of the 3x3 Hermitians. The corresponding result, for level N at least 4, follows by induction.

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1 Introduction

Lucien Hardy [1] has laid out a program for deducing quantum physics from reasonable axioms. One step of the process is the inductive construction of level N quantum physics (QP) from level 2 QP (the Bloch sphere). In this note, we point out a detail to be filled in in Hardy's program, and supply the proof. By this stage of Hardy's derivation, we know that level 2 state space is QP, and that level N state space is identified as a subset of the Hermitian NxN matrices. The key technical result of this article is that an embedding of level 3 state space into the Hermitians is rather rigid – to convert it to standard QP requires only a conjugation and/or scaling of select entries of the 3x3 Hermitians. The corresponding result, for level N at least 4, follows by induction. We denote by PU(N), the projective unitary group, the isometry group for level N QP.

The detail in question occurs in §8.7 of [1], after Equation 90. Taking the case $N = 3$ for concreteness, the discussion begins with the three basis states

$$S_1 = |1\rangle\langle 1|, \quad S_2 = |2\rangle\langle 2|, \quad S_3 = |3\rangle\langle 3|$$

The point of the discussion is to conclude that any rank 1 state may be reached via a succession of group actions on the above states, ending with Equation 91. The above states are next rotated to a new basis, using a transformation, U_{12}, in the 12 subspace :

$$S_1' = |1'\rangle\langle 1'| = U_{12}\,|1\rangle\langle 1|\,U_{12}^{\dagger} \quad S_2' = |2'\rangle\langle 2'| = U_{12}\,|2\rangle\langle 2|\,U_{12}^{\dagger} \quad S_3 = |3\rangle\langle 3|$$

The next step is to apply a similar rotation, $U_{1'3}$, in the $1'3$ subspace. However, the $1'3$ subspace is not known to consist of rank 1 matrices, and the argument breaks down at this point.

The current article takes a different approach to proving that level N state space may be embedded into the rank 1 Hermitian matrices. Consider the two matrices

$$A = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad B = \frac{1}{3}\begin{pmatrix} 1 & 1 & i \\ 1 & 1 & 1 \\ -i & 1 & 1 \end{pmatrix}$$

Both A and B restrict to (unnormalized) states in the three fundamental level 2 subsystems, but matrix B is not rank 1. The key technical result of this paper, Proposition 1 [p58], is that any level 3 embedding will almost consist of rank 1 matrices : A change of fiducial basis, consisting of scaling and conjugating the off-diagonal entries of the Hermitian matrices, as in the above example, is all that is needed to convert a general level 3 embedding into a level 3 embedding consisting of rank 1 matrices. The scaling amounts to rotating the fiducial basis in state space, while conjugation amounts to changing their orientation. This is the full orthogonal group on the 2-dimensional fiducial spaces defined by Hardy [1, §8.7].

In §2 we analyze Hermitian matrices, and give a convenient characterization of being rank 1. In §3, we review Hardy's approach to embedding level N state space into the Hermitian matrices, and refine this for the case $N = 3$, to embed state space into the rank 1 Hermitian matrices. Section 4 extends the $N = 3$ results to show how to embed state space, for arbitrary N, onto standard QP.

Throughout, $S_1, \ldots, S_N$ denote a state basis for a level N state space, $\mathfrak{S}$, as defined by Hardy [1]. A *standard* level M subspace (relative to the $\{S_i\}$) has basis $S_{i_1}, \ldots, S_{i_M}$ for some M of the $\{S_i\}$. If A is a subset of a vector space, V, we denote by, $< A >$, the linear span of A in V. For a complex matrix, M, its conjugate transpose is denoted, M^*.

2 Hermitian Matrices

A matrix is *decomposable* if it can be written as the product of a row vector with a column vector i.e. if the matrix has rank at most 1. Every non-negative decomposable Hermitian $N \times N$ matrix, A, may be written as

$$A \;=\; c\,c*, \;\; c \in \mathbb{C}^N, \;\; \text{a column vector}$$

A *principal* submatrix of an $N \times N$ matrix, A, is obtained by deleting a set of rows, and the corresponding set of columns, from A. The i^{th} *principal* submatrix is the $(N-1)\times(N-1)$ submatrix of A obtained by deleting the i^{th} row and column from A.

Definition 1. A matrix is *principally 2-decomposable* if its principal 2×2 submatrices are decomposable.

Hardy [1] shows that, when embedding state space into the Hermitian matrices, the state matrices have the following properties :

Principally 2-decomposable This is because a norm 1 trace 1 Hermitian 2×2 matrix is necessarily singular.

Non-negative diagonal The vanishing of the principal 2×2 determinants shows that all the diagonal elements are of the same sign (or 0). The trace being 1 means the diagonal elements must all be non-negative.

We call such matrices, *positive principally 2-decomposable*, and it is convenient to characterize them :

Lemma 1. *A Hermitian matrix, A, is positive principally 2-decomposable, if and only if it can be written in the following form :*

$$\begin{aligned} A_{ii} &= x_i{}^2, \; x_i \geq 0 \\ A_{ij} &= x_i x_j \omega_{ij}, \; \omega_{ij} \in \mathbb{C}, \; \left|\omega_{ij}\right| = 1 \\ \omega_{ij} &= \overline{\omega_{ji}}, \; \omega_{ii} = 1 \end{aligned} \tag{1}$$

The x_i are uniquely determined. If $x_i x_j > 0$, then w_{ij} is uniquely determined.

Démonstration. The x_i are uniquely determined as the non-negative square roots of the diagonal entries of A. The principal 2×2 submatrix for $i < j$

is

$$\begin{pmatrix} {x_i}^2 & A_{ij} \\ \overline{A_{ij}} & {x_j}^2 \end{pmatrix}$$

The determinant is 0, so $\left|A_{ij}\right|^2 = (x_i x_j)^2$. The lemma follows from this. □

Using the notation of Equation 1, let X be the diagonal matrix with elements $x_1, x_2, \ldots, x_N$, and define the matrix $W = \{\omega_{ij}\}$. We evidently have,

$$A = XWX$$

We call W, the *modulus* matrix for A.

As we would like to embed the states into the rank 1 matrices, we use the following characterization :

Lemma 2. *Using Equation 1 to describe an $N \times N$ positive principally 2-decomposable matrix, A, let $\alpha_i = \omega_{i,i+1}$, $i < N$, be the elements of the super-diagonal of the modulus matrix, W. Then A is decomposable if and only if the ω_{ij} may be chosen so that*

$$\omega_{ij} = \alpha_i \alpha_{i+1} \ldots \alpha_{j-1},\; i \le j \tag{2}$$

Démonstration. Suppose that $A = c\, c^*$ is decomposable. So, $x_i = |c_i|$, we can set $c_i = x_i \eta_i$, and $\omega_{ij} = \eta_i \overline{\eta_j}$. The decomposition Equation 2 follows.

Conversely, suppose that decomposition Equation 2 holds for A. We will show that the modulus matrix for A has rank 1, by showing that the $(j+1)^{th}$ column of W is α_j times the j^{th} column of W. It is clear that $\omega_{ij}\alpha_j = \omega_{i,j+1}$ for $i \le j$. For $i > j$, we have

$$\omega_{ij} = \overline{\omega_{ji}} = \overline{\alpha_j}\, \overline{\alpha_{j+1}} \ldots \overline{\alpha_{i-1}}$$

and we see that multiplying by α_j again yields $\omega_{i,j+1}$.

□

For 3×3 principally 2-decomposable matrices, we use the following notation for the modulus matrix :

$$W = \begin{pmatrix} 1 & \bar{\gamma} & \beta \\ \gamma & 1 & \bar{\alpha} \\ \bar{\beta} & \alpha & 1 \end{pmatrix} \tag{3}$$

and we note that

Remark 1. *W is rank 1 if and only if $\beta = \overline{\alpha}\,\overline{\gamma}$ i.e. $\alpha\beta\gamma = 1$.*

We end this section with a well-known result :

Lemma 3. *Let H be the $N \times N$ Hermitian matrices with the usual inner product $\langle A, B\rangle = \mathrm{Tr}(A^*B)$. Suppose that T belongs to a connected group of orthogonal operators on H that preserve the decomposable Hermitian matrices. Then $T \in$ PU(N).*

Démonstration. According to [2] Theorem 6 Corollary 1, in our situation there are two possibilities for T :

1. $T(A) = \epsilon S^*AS$ for some $n \times n$ matrix S, $\epsilon = \pm 1$, or
2. $T(A) = \epsilon S^*A^tS$ for some $n \times n$ matrix S, $\epsilon = \pm 1$.

The identity of the group is of type #1, with $S = I$ and $\epsilon = 1$, so by continuity, all the group elements satisfy $T(A) = S^*AS$ for some matrix S. Orthogonality implies

$$\langle A, B\rangle = \langle T(A), T(B)\rangle = \mathrm{Tr}(S^*ASS^*BS) = \mathrm{Tr}(SS^*ASS^*B)$$

so that $A = SS^*ASS^*$. Taking $A = I$, we see that SS^* is a positive square root of I, so that $SS^* = I$, i.e $S \in U(n)$ and $T \in PU(n)$. □

3 Basic Embeddings

Hardy [1] Appendix 3.4 decomposes N-level state space as

$$<\mathfrak{S}>= \bigoplus_{i=1\ldots N} <S_i> \oplus \bigoplus_{i<j} V_{ij}^{2D}$$

where the S_i are a state basis, and the V_{ij}^{2D} are the fiducial subspaces. The induced embedding to Hermitian matrices, H_N, is

Definition 2 (Basic Embedding)**.**

- $S_i \mapsto E_i$, where E_i is the 0 matrix except for a 1 at the i^{th} slot of the diagonal.
- V_{ij}^{2D} is mapped to the corresponding off-diagonal entries of the 2×2 block containing E_i and E_j. Explicitly, given an orthonormal basis, $u_{ij}, u_{ij}^{\perp}$, for V_{ij}^{2D},
  - u_{ij} is mapped to $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the ij sub-block.

— $u_{ij}^{\perp}$ is mapped to $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ in the ij sub-block.

As it stands, this is an isometric linear embedding of state space into the Hermitian matrices. The embedding needs to be further specialized, in order for the image of the (pure) states to be standard QP i.e. to consist of the rank 1 matrices. We note that a level N basic embedding restricts to a basic embedding of any standard level M subspace.

Definition 3 (Hermitian Scaling)**.** Let $\kappa \in \mathbb{C}$ be a unit length complex number. The linear isometry *Hermitian scaling* by κ

$$L_{ij}^{\kappa} : H_N \longrightarrow H_N$$

multiplies the ij entry of a Hermitian matrix by κ and the ji entry by $\overline{\kappa}$.

Definition 4 (Hermitian Conjugation)**.** The linear isometry *Hermitian conjugation*

$$C_{ij} : H_N \longrightarrow H_N$$

conjugates both the ij and ji entries of a Hermitian matrix.

The two operators, Hermitian scaling and conjugation, also preserve the positive principally 2-decomposable matrices, Definition 1 [p55].

Once we have a basic embedding of state space into the Hermitians, Hermitian conjugation and Hermitian scaling are equivalent to changing the choice of basis in the relevant V_{ij}^{2D} – Hermitian scaling rotates u_{ij} and $u_{ij}^{\perp}$, and Hermitian conjugation changes the sign of $u_{ij}^{\perp}$. At times, it will be convenient to work with explicit bases of state space, and at other times more convenient to work with Hermitian matrices via a given basic embedding.

The Hermitian scaling, L_{13}^{κ} transforms $\{\alpha\beta\gamma = 1\}$ to $\{\alpha\beta\gamma = \kappa\}$. Similarly, the Hermitian conjugation, C_{13}, transforms $\{\alpha\beta\gamma = 1\}$ to $\{\alpha\overline{\beta}\gamma = 1\}$. These two examples make it clear that not any level 3 embedding, as constructed in Definition 2, will have its image contained in $\{\alpha\beta\gamma = 1\}$. It turns out, that we only need to account for Hermitian scaling and Hermitian conjugation, in choosing bases for the V_{ij}^{2D}, in order to ensure that a basic embedding of N-level state space maps into the rank 1 matrices.

The following proposition is the main technical result of this paper.

Proposition 1. *The image of a level 3 basic embedding, as constructed in Definition 2, satisfies one of the constraints listed in Table 1.*

Table 1. *Level 3 embedding constraints*

$\alpha\beta\gamma = \tau$	$\overline{\alpha}\beta\gamma = \tau$	$\alpha\overline{\beta}\gamma = \tau$	$\alpha\beta\overline{\gamma} = \tau$

α, β, γ are as in Equation 3 [p56]. For a level 3 embedding, one of these four constraints is valid for every matrix in the image of the embedding. The modulus, τ, is a constant unit length complex number.

The lengthy proof of Proposition 1 is written out in Appendix A. Table 1 is distilled from Table 2 [p66]. This result captures the rigidity of a level 3 basic embedding.

We point out that state space cannot be contained in a vector subspace of the Hermitians, since state space linearly spans the Hermitians ($K = N^2$, as expressed by Hardy [1]§8.2). As well, state space is the homogeneous space of a connected Lie group, so state space is a connected real analytic manifold. For example

Remark 2. *A level 3 embedding satisfies* exactly *one constraint of Table 1.*

The reason is, that if two constraints were satisfied, one of α, β, γ would be constant, imposing a linear constraint onto state space.

Definition 5 (Modulus 1)**.** A basic embedding of level N state space is *modulus 1* if, on each standard level 3 state subspace, the modulus, τ, from Table 1 is 1.

Proposition 2. *A level N state space has a modulus 1 embedding.*

Démonstration. Choose a generic matrix, of a basic embedding, where each coefficient is non-zero. Scale all the off-diagonal entries to be positive. For any 3×3 principal submatrix, we have $\alpha = \beta = \gamma = 1$, so the modulus is 1 for the embedding of this level 3 subspace, regardless of where the conjugation occurs in the constraint from Table 1. □

Lemma 4. *A modulus 1 embedding of level 3 state space into the Hermitians, may be altered to embed into the rank 1 matrices, in both the following ways :*

1. *Conjugate none, one, or both elements in the top row (and so, also, in the leftmost column).*
2. *Conjugate none, one or both elements in the rightmost column (and so, also, in the bottom row).*

Démonstration. The embedding satisfies one of the constraints from Table 1, with $\tau = 1$. Either of the two proposed schemes can be used to ensure a resulting $\alpha\beta\gamma = 1$. By Lemma 1 [p57], the image is then contained in the rank 1 matrices. □

4 Induction

Let $\mathfrak{S}_N$ denote level N state space. This section contains a proof of

Proposition 3. *For all $N \geq 2$,*

1. *Any basic embedding of $\mathfrak{S}_N$ may be altered by conjugation and scaling to be contained in the rank 1 Hermitian matrices.*
2. *Any basic embedding of $\mathfrak{S}_N$, contained in the rank 1 matrices, is onto standard QP.*

The proof is by induction on N.

4.1 N=2

Hardy [1] shows that any level 2 basic embedding is necessarily contained in the rank 1 Hermitian matrices, and is onto standard QP.

4.2 N=3

Proposition 2 and Lemma 4 confirm the first claim.

As for the second claim, we note that the $3x3$ rank 1 Hermitian matrices are the complex projective space, $P^2_{\mathbb{C}}$. Let $\mathfrak{S}_{2A}$ and $\mathfrak{S}_{2B}$ be two principal level 2 subspaces of $\mathfrak{S} = \mathfrak{S}_3$. By the induction hypotheses, each of these is a linear $P^1_{\mathbb{C}} \subset P^2_{\mathbb{C}}$. Since these two subspaces intersect transversely in $P^2_{\mathbb{C}}$, we must have that $\dim \mathfrak{S} = \dim P^2_{\mathbb{C}}$ i.e. $\mathfrak{S}$ is an open subset of $P^2_{\mathbb{C}}$.

Viewing G as a subgroup of the orthogonal group for the positive definite probability transition pairing, the closure, $\overline{G}$, of G is compact and acts transitively on $\overline{\mathfrak{S}}$, so that $\overline{\mathfrak{S}}$ is also a manifold, and compact, so must be equal to all of $P^2_{\mathbb{C}}$. Since G leaves $\mathfrak{S}$ invariant, G must leave invariant all the rank 1 Hermitian matrices ($\mathfrak{S}$ being dense in $P^2_{\mathbb{C}}$). By Lemma 3, $G \subset PU(3)$. But, according to our assumptions, Hardy [1]§A3.3, G contains all the $PU(2)$ for the principal subspaces, so that $G = PU(3)$. In particular, G is compact and so is $\mathfrak{S}$, so that $\mathfrak{S} = P^2_{\mathbb{C}}$.

4.3 $N \geq 4$

Lemma 5. *Let A be a generic matrix in the image of a modulus 1 embedding of level N state space. Assume that the 1^{st} and N^{th} principle submatrices are both decomposable. Then, either A is decomposable as is, or A becomes decomposable after conjugating A_{1N} (and A_{N1}). Note that A_{1N} and A_{N1} do not belong to either of the two principal submatrices.*

Démonstration. We illustrate the proof with the modulus matrix for A :

$$W = \begin{pmatrix} 1 & \alpha_1 & \dots & d & f = A_{1N} \\ & 1 & \dots & c & e \\ & & \ddots & \vdots & \vdots \\ & & & 1 & \alpha_{N-1} \\ & & & & 1 \end{pmatrix} \tag{4}$$

Because of the hypotheses, and recalling Lemma 2 [p56], we need only show that

$$\tilde{f} = \alpha_1 \alpha_2 \dots \alpha_{N-1} \tag{5}$$

where $\tilde{f}$ denotes either f or $\overline{f}$. In the former case, there is nothing to do, and in the latter case, conjugate f. So, assume, then, that Equation 5 does not hold.

Since $N \geq 4$, we have the two distinct standard level 3 subspaces corresponding to the following two principal 3×3 submatrices of W :

$$W_1 = \begin{pmatrix} 1 & d & f \\ & 1 & \alpha_{N-1} \\ & & 1 \end{pmatrix} \qquad W_2 = \begin{pmatrix} 1 & \alpha_1 & f \\ & 1 & e \\ & & 1 \end{pmatrix}$$

where, by Lemma 2 [p56] and Equation 4

$$\begin{aligned} d &= \alpha_1 \alpha_2 \dots \alpha_{N-2} \\ e &= \alpha_2 \alpha_3 \dots \alpha_{N-1} \end{aligned}$$

Proposition 1 implies that W_1 and W_2 impose constraints on f, and, by assumption, f does not satisfy Equation 5. Consequently the constraints on f are

$$\left(f = \overline{d}\, \alpha_{N-1} \vee f = d\, \overline{\alpha_{N-1}}\right) \wedge (f = \overline{e}\, \alpha_1 \vee f = e\, \overline{\alpha_1})$$

We consider the case $f = \overline{d}\alpha_{N-1} = \overline{e}\, \alpha_1$:

$$\begin{aligned} \overline{\alpha_1}\, \alpha_{N-1} &= \overline{\alpha_{N-1}}\, \alpha_1 \\ {\alpha_{N-1}}^2 &= {\alpha_1}^2 \end{aligned}$$

However, the α's can not be constrained to a lower dimensional space, for the following reason : The two principal subspaces of the lemma imply that state space contains two standard $P_{\mathbb{C}}^{N-2}$'s that intersect transversely in standard $P_{\mathbb{C}}^{N-1}$, so state space must have (real) dimension at least $2(N-1)$. In our current situation, the following $2(N-1)$ variables map onto state space, so they must be analytically independent :

— The magnitudes $S^{N-1} = \{x \in \mathbb{R}^N : {x_1}^2 + {x_2}^2 \ldots + {x_N}^2 = 1\}$

— ${S^1}^{N-1}$ = The superdiagonal elements of the modulus matrix : $\alpha_1, \ldots, \alpha_{N-1}$

We next consider the case $f = \overline{d}\, \alpha_{N-1} = e\, \overline{\alpha_1}$:

$$\begin{array}{rcl} \overline{\alpha_2} \ldots \overline{\alpha_{N-2}} & = & \alpha_2 \ldots \alpha_{N-2} \\ 1 & = & {\alpha_2}^2 \ldots {\alpha_{N-2}}^2 \end{array}$$

which, again, puts a constraint on the α's, as $N \geq 4$, and so is inadmissible.

The remaining two cases are the same as the first two, so our assumption was incorrect, and f must, indeed, satisfy Equation 5.

$\square$

With the above lemma, we proceed to prove the first claim of the proposition :

By Proposition 2, we may assume a modulus 1 embedding. By the inductive assumption, we can conjugate the elements of the 1^{st} principal submatrix (the lower right $(N-1) \times (N-1)$ submatrix) to lie in the rank 1 Hermitian matrices.

Now, consider the upper left 3×3 principal submatrix, M_3, of the full embedding. We can conjugate within its top row (and left column) to convert M_3 into a level 3 rank 1 embedding, according to Lemma 4 [p59]. This leaves unchanged the 1^{st} principal submatrix.

We, next, step along the top row of the embedding, addressing the upper left principal submatrices $M_k \in \{M_4, M_5, \ldots, M_N\}$, invoking Lemma 5 [p61] to convert M_k into a rank 1 embedding : M_{k-1} is decomposable, and so is the upper left $(k-1) \times (k-1)$ submatrix of the 1^{st} principal submatrix. This leaves unchanged the 1^{st} principal submatrix. Since M_N is the full matrix, this completes the proof of the first claim of the proposition.

The proof of the second claim of the proposition is mutatis mutandis the same as the case, $N = 3$.

5 Conclusions

Hardy[1] gives a novel approach to the foundations of quantum physics. In the current article, we have supplied the mathematics that fills in a step in Hardy's program – the induction of level N QP from level 2 QP.

Below, we discuss some of the results obtained in the current article.

QP state space is closed We have shown, Proposition 3 [p60], that state space is a topologically closed subset of Euclidean space. From the measurement perspective, this has the following meaning :

A sequence of states, (S_k), is *weakly convergent* if for every measurement, f, the real sequence $(f(S_k))$ is convergent. The sequences of states that we consider are not restricted to being ensembles, as all the states in an ensemble are identical. State space is *closed* when every weakly convergent sequence of states, converges to some state i.e.

$$\exists S \in \mathfrak{S} \;\; f(S_k) \longmapsto f(S) \quad \text{for every measurement, } \; f$$

For a non-QP scenario, when state space is embedded into Euclidean space, as we have done, every weakly convergent sequence of states does converge to a point of $\overline{\mathfrak{S}}$, and it is tempting to just replace $\mathfrak{S}$ with $\overline{\mathfrak{S}}$ and G with $\overline{G}$. But what physical meaning would there be to the points of $\overline{\mathfrak{S}} \setminus \mathfrak{S}$? Would the axioms extend to $\overline{\mathfrak{S}}$? We are spared these question in the QP scenario.

Level 3 embeddings The rigidity of level 3 embeddings, Proposition 1 [p58], is derived from the fundamental constraint Equation 12 [p67], which relates the three moduli of the Hermitian matrix state representation, to the matrix elements of the group acting on this state space. Some relevant observations are :

— The 4-dimensional level 3 state space is contained in the 5-dimensional variety of trace 1 principally 2-decomposable matrices, X^{5D}, as per the discussion after Definition 1 [p55]. A priori, the level 3 state space would simply be contained in the 9-dimensional Hermitian matrices, or the 8-dimensional affine subspace of trace 1, or the 7-dimensional sphere within this affine space.

— The isotropy group of a state has the rich structure of PU(2). Moreover, the isotropy group representation on the 9-dimensional vector space of Hermitian matrices decomposes, with the various pieces being interrelated due to the constraint of the states lying in X^{5D}.

— The axioms further mandate a tight connection between level 2 state space and level 3 state space : The subspace axiom describes how a level 3 state may be restricted to a level 2 subspace, and every isometry of a level 2 subspace is the restriction of a level 3 isometry, Hardy [1].

The above highlights that the level 2 inductive assumption both constrains the level 3 state space, while also bringing a rich structure of its own. This combination is conducive to reducing degrees of freedom, as in the fundamental constraint.

6 Acknowledgements

Although this paper is several pages long, it is but a detail in the broad program laid out and developed by Lucien Hardy [1]. To him is owed the thanks for this conceptualization of quantum state spaces, and for many of the results supporting the current article. I would also like to thank James Carrell for his support during the writing of this paper.

References

[1] Lucien Hardy. *Quantum Theory from Five Reasonable Axioms*, arXiv :quant-ph/0101012v4 (Feb 1, 2008)

[2] Ehud Moshe Baruch and Raphael Loewy, *Linear Preservers on Spaces of Hermitian or Real Symmetric Matrices*, Linear Algebra and its Applications 183 ; 89-102 (1993), Elsevier Science Publishing Co., Inc, New York.

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A Level 3 Constraints

This section contains the proof of Proposition 1 [p58]. We have a basic embedding of level 3 state space, $\mathfrak{S}$, into the Hermitian matrices, according to Definition 2 [p57]. The goal is to construct Table 2 for this embedding.

Definition 6 (Trace 1 Principally 2-Decomposable)**.** Referring to Equation 3 [p56], let

$$X^{5D} = \left\{ \begin{pmatrix} x^2 & xy\bar{\gamma} & xz\beta \\ xy\gamma & y^2 & yz\bar{\alpha} \\ xz\bar{\beta} & yz\alpha & z^2 \end{pmatrix} : x^2 + y^2 + z^2 = 1, |\alpha| = |\beta| = |\gamma| = 1 \right\}$$

be the 5 (real) dimensional variety of principally 2-decomposable trace 1 Hermitian 3×3 matrices.

X^{5D} is roughly parameterized by $S^2 \times S^1 \times S^1 \times S^1$, and, according to Remark 1 [p57], contains the rank 1 matrices

$$P^2_{\mathbb{C}} = \{A \in X^{5D} : \alpha\beta\gamma = 1\}$$

The level 3 embedding has its image contained in X^{5D}, but not necessarily in the $P^2_{\mathbb{C}}$ subvariety.

We follow Hardy, and decompose the Hermitian matrices, H, as

$$H = V^{4D}_{12} \oplus <S_3> \oplus V^{2D}_{23} \oplus V^{2D}_{13}$$

where

— V^{4D}_{12} is the 4-dimensional vector space of the level 2 state subsystem $\mathfrak{S}_{12}$.

— V^{2D}_{23} and V^{2D}_{13} are the two fiducial subspaces for the other two fundamental level 2 subsystems.

Let G be the automorphism group of the level 3 state space, and G_{12} the connected component of the identity of the subgroup leaving invariant $\mathfrak{S}_{12}$ (or, equivalently, fixing S_3). Let $g \in G_{12}$. The action of g on H can be described as

— $g_0 = g|_{V^{4D}_{12}}$ acts as an element of PU(2). According to the discussion in [1] §A3.3, the restriction map, $G_{12} \longrightarrow \mathrm{PU}(2)$, is onto.

— $g|_{<S_3>}$ is the identity.

— $g_1 = g|_W \in \mathrm{Aut}\,(W)$, since g leaves $W = V^{2D}_{23} \oplus V^{2D}_{13}$ invariant.

The derivation of the table proceeds through several steps, and exploits the above state space decomposition and group action, to derive constraints on the group elements and states. In the end, the group dependence is explicitly eliminated, except for the parameter, σ, as shown in Table 2.

Table 2. *Level 3 embedding constraints : Derivation*

ϵ	$\beta' = \beta$	$\beta' = \overline{\beta}$
1	$\gamma = \sigma\overline{\beta}\,\overline{\alpha}$	$\gamma = \sigma\beta\,\overline{\alpha}$
-1	$\gamma = \overline{\sigma}\beta\,\alpha$	$\gamma = \overline{\sigma}\overline{\beta}\,\alpha$

α, β, γ are as in Definition 6. For a level 3 embedding, one of these four constraints is valid for every matrix in the image of the embedding. The parameter, σ, is a constant unit length complex number. The parameters ϵ, β' and σ arise during the derivation of the table.

A.1 Coordinates

Keeping in mind Definition 6 for the format of a member of X^{5D}, the four real components of $xz\beta$ and $yz\overline{\alpha}$ are the components of an orthonormal basis for W (actually, uniformly scaled by $\sqrt{2}$). Correspondingly, denote the 4×4 matrix of g_1 as the block matrix

$$g_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D are 2×2 real matrices.

At this point we change notation, slightly, and view α and β as the 2-vectors

$$a = \begin{pmatrix} \operatorname{Re}\alpha \\ \operatorname{Im}\alpha \end{pmatrix} \qquad b = \begin{pmatrix} \operatorname{Re}\beta \\ \operatorname{Im}\beta \end{pmatrix}$$

With this in mind, the action of g_1 is

$$g_1 \begin{pmatrix} xb \\ y\overline{a} \end{pmatrix} = \begin{pmatrix} xAb + yB\overline{a} \\ xCb + yD\overline{a} \end{pmatrix} = \begin{pmatrix} x'b' \\ y'\overline{a'} \end{pmatrix} \tag{6}$$

We factor out the z's, since z is fixed by these transformations. The second equality expresses that the group action takes states to states.

As for g_0, it is a member of PU(2) for $\mathfrak{S}_{12}$:

$$g_0 = \begin{pmatrix} u & v \end{pmatrix} = \begin{pmatrix} \cos t & \omega \sin t \\ \overline{\eta} \sin t & -\overline{\eta}\omega \cos t \end{pmatrix}$$

where u, v are an orthonormal pair of vectors in $\mathbb{C}^2$, defined in terms of t, ω, η, where t is real, and ω, η are unit length complex numbers. The action of g_0 is

$$g_0 \begin{pmatrix} x \\ \gamma y \end{pmatrix} = xu + \gamma y v = \begin{pmatrix} x' \\ \gamma' y' \end{pmatrix} \kappa \tag{7}$$

where κ is a unit length complex number.

A.2 Fundamental Constraint

The first components of Equations 6 and 7 have the same magnitude (x') :

$$|xAb + yB\overline{a}|^2 = \left|xu^1 + \gamma y v^1\right|^2$$

Collect in monomials of x and y :

$$0 = x^2\left(|Ab|^2 - \left|u^1\right|^2\right) + y^2\left(|B\overline{a}|^2 - \left|v^1\right|^2\right) + 2xy\left(b^t A^t B\overline{a} - \mathrm{Re}(\overline{u^1}\gamma v^1\right) \qquad (8)$$

The states where $y = 0$ comprise the $\mathfrak{S}_{13}$ subspace, where xz is generically non-zero, so the first coefficient in Equation 8 vanishes on $\mathfrak{S}_{13}$:

$$|Ab|^2 = \left|u^1\right|^2 = \cos^2 t$$

But, b is arbitrary on $\mathfrak{S}_{13}$, so

$$A = \cos t\, E \quad \text{, for some } \; E \in O(2) \qquad (9)$$

Similarly, working with the second coefficient of Equation 8, we have

$$B = \sin t\, F \quad \text{, for some } \; F \in O(2) \qquad (10)$$

and the first two coefficients of Equation 8 vanish identically on *the entire level 3 state space*. All that remains of Equation 8 is the third term. But, xyz is generically non-zero, so that the third term vanishes, too :

$$b^t A^t B\, \overline{a} = \mathrm{Re}(\overline{u^1}\gamma v^1) = \cos t \sin t\, \mathrm{Re}(\omega\gamma) \qquad (11)$$

Substitute Equations 9 and 10 into Equation 11 :

$$\mathrm{Re}(\omega\gamma) = b^t E^t F\, \overline{a} \qquad (12)$$

A.3 The α, β, γ constraints

Referring to Equation 12, let

$$L = E^t F$$

so that

$$Re(\omega\gamma) = b^t L\, \overline{a}$$

Choose a transformation $g \in G_{12}$ with $\omega = 1$ and another transformation with $\omega = i$, to get the two equations

$$\operatorname{Re}\gamma = b^t L_1 \overline{a} \tag{13a}$$
$$\operatorname{Im}\gamma = -b^t L_i \overline{a} \tag{13b}$$

where $L_1, L_i \in O(2)$ are constant. From this we calculate

$$\begin{array}{rcl} \operatorname{Re}(\omega\gamma) & = & \omega_x b^t L_1 \overline{a} + \omega_y b^t L_i \overline{a} \\ & = & b^t(\omega_x L_1 + \omega_y L_i)\overline{a} \\ L = L_\omega & = & \omega_x L_1 + \omega_y L_i \end{array}$$

Since $L \in O(2)$, we must have $I = L^t L$, so that

$$0 = {L_1}^t L_i + {L_i}^t L_1$$

i.e. ${L_1}^t L_i$ is a skew symmetric element of $O(2)$, so that

$$\begin{array}{rcl} {L_1}^t L_i & = & \epsilon J, \ \ \epsilon = \pm 1 \\ J & = & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SO}(2) \end{array}$$

Rewrite Eq 13b as

$$\operatorname{Im}\gamma = -\epsilon\, b^t L_1 J\, \overline{a} \tag{14}$$

For simplicity, we set $\epsilon = 1$, but keep in mind, whatever solution we get for γ, that $\overline{\gamma}$ is also a solution, corresponding to $\epsilon = -1$.

With regards to L_1, there are two cases to consider :

case $L_1 \in$ **SO(2)** Set $L_1 = R_\theta$, rotation by some θ, and define $b' = b$.

case $L_1 \notin$ **SO(2)** Set $L_1 = KR_\theta$ where

$$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \text{and define } b' = Kb = \overline{b}$$

Equations 13a and 14, can now be rewritten as

$$\begin{array}{rcl} \operatorname{Re}\gamma & = & b'^t R_\theta\, \overline{a} \\ \operatorname{Im}\gamma & = & -b'^t R_\theta J\, \overline{a} \end{array}$$

and these two equations can be combined to express

$$\gamma = \operatorname{Re}\gamma + i \operatorname{Im}\gamma = b'^t R_\theta\,(I - Ji)\,\overline{a}$$

where

$$I - Ji = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \begin{pmatrix} 1 & i \end{pmatrix}$$

so that

$$\gamma = b'^t \begin{pmatrix} \sigma \\ -i\sigma \end{pmatrix} \overline{\alpha}$$

where $\sigma = \cos\theta + i \sin\theta$. We conclude that

$$\gamma = \begin{cases} \sigma\overline{\beta'}\overline{\alpha}, & \epsilon = 1 \\ \overline{\sigma}\beta'\alpha, & \epsilon = -1 \end{cases}$$

where $\beta' = \beta$ or $\overline{\beta}$. The four possible constraints are listed in Table 2. □