Lorentz-covariant two-particles quantum relativistic equation in a Lagrangian approach.

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ABSTRACT. We propose a new, multitime, generalisation of the Dirac equation to the N particle case which is invariant under Lorentz transformations. It is derived from a Lorentz covariant Lagrangian density. We focus on the two-particle case ($N = 2$) for which we show that the associated conserved density, in the single time regime, is not definite positive, similar to the $N = 1$ Klein-Gordon equation.

1 Introduction.

This work was motivated by the recent work of one of us [1] who studied a multitime and multiparticle generalization of the single particle Dirac equation which reduces, when all particles are independent, to a system of Dirac equations originally studied by Wentzel [2]. This multitime equation generalizes the single time Bohm-Hiley equation [3] which is itself a multiparticle generalization of the (single particle) Dirac equation. As is shown in [1], Lorentz invariance of the multitime equation is not always guaranteed, making use of a necessary condition for Lorentz invariance outlined by L. de Broglie in the single particle case in 1934 [4]. We explain here the lack of relativistic invariance in terms of Lagrangian densities, and propose a new multitime Dirac equation which derives from an invariant Lagrangian density. In the paper,we focus on a 2 particles formulation. We show that the conserved density is not definite positive, in the case of the single time equation. Finally we consider implications of this equation concerning the gravitational interaction in the Newtonian limit.

2 Previous results: the multitime generalisation of Bohm-Hiley equation.

Wentzel originally considered a multitime system of N single particles Dirac equations in which the particles were all separated by spacelike distances so that they did not interact with each other[1]. Here we shall instead consider the multitime equation as an effective equation, also valid for interacting Dirac fermions (e.g. electrons, protons). In the rest of the paper, we shall restrict ourselves to a 2 particles formulation. When $N = 2$, adding the contributions of electro-magnetic interactions and of an interaction potential V, the multitime generalisation of Bohm and Hiley's multiparticle single time equation reads thus [1]

$$\{(\mathbb{1}^A P_0^A - \overrightarrow{\alpha}^A \cdot \overrightarrow{P}^A - m_A c\alpha_0^A) \otimes \mathbb{1}^B + \mathbb{1}^A \otimes (\mathbb{1}^B P_0^B - \overrightarrow{\alpha^B} \cdot \overrightarrow{P^B} - m_B c\alpha_0^B) \\ +V(\overrightarrow{r}^A, t^A, \overrightarrow{r}^B, t^B)\mathbb{1}^A \otimes \mathbb{1}^B\}\mathbf{\Psi}_{AB} = 0, \quad (1)$$

where the matrices $\alpha_x, \alpha_y, \alpha_z$ and α_0 are the Dirac 4 times 4 matrices, $c \cdot P_0^{A(B)}$ represents the operator $i\hbar\frac{\partial}{\partial t} + eA_0$ acting on the $A(B)$ particle only: $P_0^{A(B)} = i\frac{\hbar}{c}\frac{\partial}{\partial t_{A(B)}} + \frac{e}{c}A_0(\overrightarrow{r}^{A(B)}, t^{A(B)})$ (with A_0 the electric field ϕ, that is to say the time-component of the relativistic quadrivector $\mathbf{A}$); similarly, the operator $\overrightarrow{P}^A$ represents the 3-components operator $\frac{\hbar\overrightarrow{\nabla}}{i} + e\overrightarrow{A}$ acting on the $A(B)$ particle only:
$P_{x_i}^{A(B)} = \frac{\hbar}{i}\frac{\partial}{\partial_{x_i}^{A(B)}} + eA_{x_i}(\overrightarrow{r}^{A(B)}, t^{A(B)}),\ i = 1, 2, 3, x_1 = x, x_2 = y, x_3 = z$ (with $\overrightarrow{A}$ the electro-magnetic potential vector).

The Dirac α matrices are defined as follows:

$$\alpha_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\alpha_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \alpha_z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

[1]However, as was noted by Bohm and Hiley, this does not mean that they are independent, because the quantum theory is a nonlocal theory, a was confirmed by the violation of Bell's inequalities in numerous experiments in the meanwhile.

$V(\overrightarrow{r}^A, t^A, \overrightarrow{r}^B, t^B)$ represents an interaction potential between the two particles (e.g. electrons, protons) .

The equation (1) constitutes a multitime generalisation of the two particles single time Bohm-Hiley equation [3] which we get after imposing $t_A = t_B$ and $A_0 = 0$.

As is shown in [1], equation (1) is not Lorentz covariant when N=2. What is meant hereby is that when the spatio-temporal coordinates of the A and B particles transform under a Lorentz transformation there exists no matricial transformation of the components of the 16 components spinor $\mathbf{\Psi}_{AB}$ that will restore Lorentz covariance[2].

3 Lagrangian approach: Why the Lorentz invariance of the multitime generalisation of B-H equation is not always guaranteed.

3.1 An apparent paradox.

At first sight, the impossibility to find linear spinorial transformations under which the multitime generalisation of the two particles Bohm-Hiley equation would be Lorentz invariant is a counternatural result. In order to show this, let us consider a situation in which N particles (fermions) are prepared in far away regions of space. One is free to assign to each particle an arbitrarily fixed local external electro-magnetic field but without losing generality we shall choose in a first time to cancel the electro-magnetic interaction. As we shall see in what follows, this choice simplifies the expressions of the Lagrangians; our choice to neglect the electro-magnetic interaction is thus essentially dictated by convenience and simplicity. Then the electric and magnetic potentials as well as the potential V in equation (1) can be considered to be equal to zero, in which case we can construct a solution of (1) which consists of the tensor product of N (here N=2) independent of each other solutions of the free single particle Dirac equation. Each equation being Lorentz invariant, it is difficult to explain why when more than one particle is considered Lorentz invariance gets broken.We expect indeed that the tensor product

[2]It is worth noting that the result shown in [1] concerns a restricted class of transformations, not all possible linear transformations. However this class contains the tensor product of the local spinorial transformations associated to the A and B particles separately.As a consequence, the lack of Lorentz invariance put into evidence in [1],also rules out the multitime equation (1) even in the restricted domain of validity considered by Wentzel because the proof is still valid in the case of nointeracting particles separated by spacelike distances.

of the local spinorial transformations associated to the A and B particles will guarantee Lorentz covariance, but this is not true as shown in [1].

This is clearly a counterintuitive results, which contradicts Bohm and Hiley who wrote about the multitime equation ... *"The Lorentz invariance of the formalism is thus evident"*...[3], and Wentzel according to who ... *"the relativistic invariant character of the theory is evident"*... [2]. In order to understand this apparent paradox, let us consider the Lagrangian densities respectively associated to the 1-particle and 2-particles Dirac equations (resp. N=1 and 2).

3.2 Single particle case.

In terms of the quadrispinor $\mathbf{\Psi}$, where

$$\mathbf{\Psi} = \begin{pmatrix} \Psi_1(t,\mathbf{x}) \\ \Psi_2(t,\mathbf{x}) \\ \Psi_3(t,\mathbf{x}) \\ \Psi_4(t,\mathbf{x}) \end{pmatrix}, \tag{2}$$

the 1-particle Lagrangian density reads
$\mathcal{L} = \mathbf{\Psi}^\dagger(i\hbar\partial_t - \frac{\hbar c}{i}\overrightarrow{\alpha}\cdot\overrightarrow{\bigtriangledown} - mc^2\alpha_0)\mathbf{\Psi}$

Lorentz covariance of Dirac single particle equation

For simplicity we shall consider here the restricted case where only the components 1 and 3 differ from zero, and do not depend on x and y. Then the Lagrangian density reads

$$\begin{aligned}\mathcal{L}_{13} &= \mathbf{\Psi}_{13}^\dagger M_{Dirac}\mathbf{\Psi}_{13} \\ &= \big(\Psi_1(t,z),\Psi_3(t,z)\big)\begin{pmatrix} P_0 - mc & -P_3 \\ -P_3 & P_0 + mc\end{pmatrix}\begin{pmatrix}\Psi_1(t,z) \\ \Psi_3(t,z)\end{pmatrix}\end{aligned} \tag{3}$$

where we introduced the Dirac matrix M_{Dirac}, which is in the present case the restriction to the components 1 and 3 of the matrix
$(i\hbar\partial_t - \frac{\hbar c}{i}\overrightarrow{\alpha}\cdot\overrightarrow{\bigtriangledown} - mc^2\alpha_0)$.

Under a Lorentz boost along Z, t and z will transform under the corresponding Lorentz transform and we get

$$\begin{pmatrix} t' \\ z' \end{pmatrix} = \begin{pmatrix} ch\gamma & -sh\gamma \\ -sh\gamma & ch\gamma \end{pmatrix}\begin{pmatrix} t \\ z \end{pmatrix} \tag{4}$$

Then, the Dirac matrix reexpressed in terms of the new spatio-temporal coordinates reads

$$M'_{Dirac} = \begin{pmatrix} P'_0 ch\gamma + P'_3 sh\gamma - mc & -P'_3 ch\gamma - P'_0 sh\gamma \\ -P'_3 ch\gamma - P'_0 sh\gamma & P'_0 ch\gamma + P'_3 sh\gamma + mc \end{pmatrix} \tag{5}$$

As is well-known (see e.g. [5, 6] for updated formulations), Dirac equation is again valid in the boosted, primed, reference frame provided the components 1 and 3 in $\mathbf{\Psi}_{13}$ obey [4] the bispinorial transformation T:

$$\mathbf{\Psi'}_{13} = \begin{pmatrix} \Psi'_1(t,z) \\ \Psi'_3(t,z) \end{pmatrix} = \begin{pmatrix} ch\frac{\gamma}{2} & -sh\frac{\gamma}{2} \\ -sh\frac{\gamma}{2} & ch\frac{\gamma}{2} \end{pmatrix} \begin{pmatrix} \Psi_1(t,z) \\ \Psi_3(t,z) \end{pmatrix} = T\mathbf{\Psi}_{13} \tag{6}$$

If we simultaneously transform $\mathbf{\Psi}_{13}$ under T (6) and t and z under the Lorentz transform (4), the Lagrangian density, reexpressed in terms of the primed quantities reads

$$\mathcal{L}'_{13} = (\mathbf{\Psi}'_{13})^\dagger (T^{-1})^\dagger M'_{Dirac} (T^{-1}) \mathbf{\Psi}'_{13} \tag{7}$$

where

$$T^{-1} = \begin{pmatrix} ch\frac{\gamma}{2} sh\frac{\gamma}{2} \\ sh\frac{\gamma}{2} ch\frac{\gamma}{2} \end{pmatrix} \tag{8}$$

Lorentz invariance under a boost along Z then results from the fact that $\mathcal{L}'_{13}$ has the same form as $\mathcal{L}_{13}$:

$$(T^{-1})^\dagger M'_{Dirac} (T^{-1}) = \begin{pmatrix} P'_0 - mc & -P'_3 \\ -P'_3 & P'_0 + mc \end{pmatrix}, \tag{9}$$

which ensures that the same Dirac equations are satisfied by the primed quantities (primed spinor in primed spatio-temporal coordinates).

Similar results hold for the components 2 and 4 of the Dirac spinor which can be shown in the present case to transform according to

$$\mathbf{\Psi'}_{24} = \begin{pmatrix} \Psi'_2(t,z) \\ \Psi'_4(t,z) \end{pmatrix} = \begin{pmatrix} ch\frac{\gamma}{2} & +sh\frac{\gamma}{2} \\ +sh\frac{\gamma}{2} & ch\frac{\gamma}{2} \end{pmatrix} \begin{pmatrix} \Psi_2(t,z) \\ \Psi_4(t,z) \end{pmatrix} \tag{10}$$

It is worth noting that the spinorial transformation law that renders Dirac equation invariant can be expressed in terms of the Dirac matrices. We get then,

$$T_{4x4} = exp(-\frac{\gamma}{2}\alpha_z) = ch\frac{\gamma}{2}\mathbb{1} - sh\frac{\gamma}{2}\alpha_z, \tag{11}$$

which encapsulates (6) and (10).

This is a very well-known result [6] that ultimately can be understood in terms of the properties of the generators of the Lie group to which these representations of the Lorentz group belong [5].

3.3 The two particles case.

The two particles multitime Dirac equation (1) is associated to the Lagrangian density

$$\begin{aligned} \mathcal{L}^{AB} &= \mathbf{\Psi}^{\dagger}_{AB}\{(i\hbar\mathbb{1}^A\partial_t^A - \frac{\hbar c}{i}\overrightarrow{\alpha^A} \cdot \overrightarrow{\bigtriangledown} - m_A c^2\alpha_0^A) \otimes \mathbb{1}^B \\ &+ \mathbb{1}^A \otimes (i\hbar\mathbb{1}^B\partial_t^B - \frac{\hbar c}{i}\overrightarrow{\alpha^B} \cdot \overrightarrow{\bigtriangledown} - m_B c^2\alpha_0^B)\}\mathbf{\Psi}_{AB} \end{aligned} \tag{12}$$

Here again we assume for simplicity that the 2 and 4 components of the quadrispinor of the A and B systems are equal to 0, and that the state only depends on t^A, t^B, z^A and z^B. The bipartite system is then represented by the quadrispinor $\mathbf{\Psi}_{AB}$, where, in absence of interaction between the A and B subsystems, no entanglement occurs and we can without loss of generality consider factorisable solutions of the type

$$\begin{aligned} \mathbf{\Psi}_{AB} &= \begin{pmatrix} \Psi_{11}(t^A,z^A,t^B,z^B) \\ \Psi_{13}(t^A,z^A,t^B,z^B) \\ \Psi_{31}(t^A,z^A,t^B,z^B) \\ \Psi_{33}(t^A,z^A,t^B,z^B) \end{pmatrix} = \begin{pmatrix} \Psi_1(t^A,z^A)\cdot\Psi_1(t^B,z^B) \\ \Psi_1(t^A,z^A)\cdot\Psi_3(t^B,z^B) \\ \Psi_3(t^A,z^A)\cdot\Psi_1(t^B,z^B) \\ \Psi_3(t^A,z^A)\cdot\Psi_3(t^B,z^B) \end{pmatrix} \\ &= \mathbf{\Psi}_{13}^A \otimes \mathbf{\Psi}_{13}^B. \end{aligned} \tag{13}$$

Under a Lorentz boost, $\mathcal{L}^{AB} = \mathcal{L}^A_{13}(\mathbf{\Psi}^\dagger)^B_{13}\mathbf{\Psi}^B_{13} + (\mathbf{\Psi}^\dagger_{13})^A\mathbf{\Psi}^A_{13}\mathcal{L}^B_{13}$ will transform to $\mathcal{L}'^{AB} = \mathcal{L}'^A_{13}(\mathbf{\Psi}'^\dagger)^B_{13}((T^B)^{-1})^\dagger((T^B)^{-1})\mathbf{\Psi}'^B_{13}$ $+ (\mathbf{\Psi}'^\dagger_{13})^A((T^A)^{-1})^\dagger((T^A)^{-1})\mathbf{\Psi}'^A_{13}\mathcal{L}'^B_{13}$.

As in the single particle case, Lorentz invariance under a boost along Z then imposes that $\mathcal{L}'_{13}$ has the same form as $\mathcal{L}_{13}$ but one should note that the transformation T is not unitary[3], so that

$(\mathbf{\Psi}'^\dagger)^A_{13}\mathbf{\Psi}'^A_{13} \neq (\mathbf{\Psi}'^\dagger)^A_{13}((T^A)^{-1})^\dagger((T^A)^{-1})\mathbf{\Psi}'^A_{13}$

and

$(\mathbf{\Psi}'^\dagger)^B_{13}\mathbf{\Psi}'^B_{13} \neq (\mathbf{\Psi}'^\dagger)^B_{13}((T^B)^{-1})^\dagger((T^B)^{-1})\mathbf{\Psi}'^B_{13}$ which explains why and how Lorentz invariance gets broken in the two particles case.

4 Lorentz invariance restored.

4.1 Lorentz invariant Lagrangian density.

One can however remedy the problem, noting that, as has been shown for instance in Messiah's standard textbook[6], the transformation T obeys the identity $\alpha_0 \cdot T^{-1} \cdot \alpha_0 = T^\dagger$, which implies that

$(\mathbf{\Psi}'^\dagger)^A_{13} \cdot \alpha^A_0 \cdot \mathbf{\Psi}'^A_{13} = (\mathbf{\Psi}'^\dagger)^A_{13}((T^A)^{-1})^\dagger \cdot \alpha^A_0 \cdot ((T^A)^{-1})\mathbf{\Psi}'^A_{13}$

and

$(\mathbf{\Psi}'^\dagger)^B_{13} \cdot \alpha^B_0 \cdot \mathbf{\Psi}'^B_{13} = (\mathbf{\Psi}'^\dagger)^B_{13}((T^B)^{-1})^\dagger \cdot \alpha^B_0 \cdot ((T^B)^{-1})\mathbf{\Psi}'^B_{13}$.

Therefore, in order to restore Lorentz covariance it suffices to choose (in the factorisable case) the new Lagrangian density

$\tilde{\mathcal{L}}^{AB} = \mathcal{L}^A_{13} \cdot (\mathbf{\Psi}^\dagger)^B_{13} \cdot \alpha^B_0 \cdot \mathbf{\Psi}^B_{13} + (\mathbf{\Psi}^\dagger_{13})^A \cdot \alpha^A_0 \cdot \mathbf{\Psi}^A_{13} \cdot \mathcal{L}^B_{13}$.

By construction, it is Lorentz invariant provided space and time transform according to the Lorentz transform while simultaneously the spinorial components of $\mathbf{\Psi}_{AB}$ will transform (in the present situation) according to

[3] Ultimately non-unitarity is related to the fact that $\mathbf{\Psi}^\dagger\mathbf{\Psi}$ is not a Lorentz scalar but is the time-component of a quadrivector [6] as can be checked by direct computation with the help of (11); this being said, the counterintuitive nature of the lack of Lorentz covariance of the equation (1) is due to non-unitarity, which is not common in every day quantum physics; Bohm and Hiley for instance wrote the following [3] about the single time N-particle equation):... *"As is well known, the transformation between frames in field theory is unitary. From this it follows that at least as far as probabilities are concerned, the many-body Dirac equation is Lorentz invariant both in form and in content."*...

$$\mathbf{\Psi'}_{AB} = \begin{pmatrix} \Psi'_{11}(t^A, z^A, t^B, z^B) \\ \Psi'_{13}(t^A, z^A, t^B, z^B) \\ \Psi'_{31}(t^A, z^A, t^B, z^B) \\ \Psi'_{33}(t^A, z^A, t^B, z^B) \end{pmatrix} = \tag{14}$$

$$\mathbf{\Psi'}^A_{13} \otimes \mathbf{\Psi'}^B_{13}. = \begin{pmatrix} ch\frac{\gamma}{2} & -sh\frac{\gamma}{2} \\ -sh\frac{\gamma}{2} & ch\frac{\gamma}{2} \end{pmatrix}^A \otimes \begin{pmatrix} ch\frac{\gamma}{2} & -sh\frac{\gamma}{2} \\ -sh\frac{\gamma}{2} & ch\frac{\gamma}{2} \end{pmatrix}^B \mathbf{\Psi}_{AB}$$

In order to tackle the general case where electro-magnetic fields do not necessarily vanish, while the subsystems A and B might happen to interact (be it by direct electro-magnetic interaction and/or via the potential V) and thus to get entangled, it is natural to choose the Lagrangian density

$$\begin{aligned} \tilde{\mathcal{L}}^{AB} =& \mathbf{\Psi}^\dagger_{AB}\{(\mathbb{1}^A P_0^A - c\overrightarrow{\alpha}^A \cdot \overrightarrow{P}^A - m_A c^2 \alpha_0^A) \otimes \alpha_0^B \\ &+ \alpha_0^A \otimes (\mathbb{1}^B P_0^B - c\overrightarrow{\alpha^B} \cdot \overrightarrow{P^B} - m_B c^2 \alpha_0^B) \\ &+ V(\overrightarrow{r}^A, \overrightarrow{t}^B, \overrightarrow{r}^B, \overrightarrow{t}^A)\alpha_0^A \otimes \alpha_0^B\}\mathbf{\Psi}_{AB}, \end{aligned} \tag{15}$$

where V is a Lorentz scalar[4].

In this more general situation, covariance is ensured provided the spinor $\mathbf{\Psi}_{AB}$ transforms according to the law

$\mathbf{\Psi'}_{AB} = T^A \otimes T^B \mathbf{\Psi}_{AB}$

From the new Lagrangian density, we derive the new multitime equation

$$\begin{aligned} &\{(\mathbb{1}^A P_0^A - c\overrightarrow{\alpha}^A \cdot \overrightarrow{P}^A - m_A c^2 \alpha_0^A) \otimes \alpha_0^B \\ &\quad + \alpha_0^A \otimes (\mathbb{1}^B P_0^B - c\overrightarrow{\alpha^B} \cdot \overrightarrow{P^B} - m_B c^2 \alpha_0^B) \\ &+ V(\overrightarrow{r}^A, t^A, \overrightarrow{r}^B, t^B)\alpha_0^A \otimes \alpha_0^B\}\mathbf{\Psi}_{AB} = 0, \end{aligned} \tag{16}$$

[4]The existence of a manifestly covariant interaction potential between two particles is seriously constrained by the famous no-go theorem of Currie, Jordan and Sudarshan [7, 8]. Soon, we shall nevertheless consider the case where V would represent the instantaneous Newtonian gravitational interaction between the A and B particles, having in mind that the standard expression *à la* Newton of this potential can reasonably be considered to be valid in the classical (nonrelativistic) limit.

The generalisation of the single spinor matricial transformation T_{4x4} (11) to the two particle case reads

$$T^{AB}_{16x16} = T^{A}_{4x4} \otimes T^{B}_{4x4} = exp(-\frac{\gamma}{2}(\alpha^A_z + \alpha^B_z))$$
$$= (ch\frac{\gamma}{2}\mathbb{1}^A - sh\frac{\gamma}{2}\alpha^A_z) \otimes (ch\frac{\gamma}{2}\mathbb{1}^B - sh\frac{\gamma}{2}\alpha^B_z), \tag{17}$$

Lorentz covariance is then guaranteed "by construction", but in appendix we show that covariance can also be established, applying the method outlined in reference [1]. Let us now reconsider the single time version of the $N = 2$ multitime equation (18).

$$\begin{aligned}&\{(\mathbb{1}^A P^A_0 - c\overrightarrow{\alpha}^A \cdot \overrightarrow{P}^A - m_A c^2 \alpha^A_0) \otimes \alpha^B_0\\ &\quad + \alpha^A_0 \otimes (\mathbb{1}^B P^B_0 - c\overrightarrow{\alpha^B} \cdot \overrightarrow{P^B} - m_B c^2 \alpha^B_0)\\ &+V(\overrightarrow{r}^A, t^A, \overrightarrow{r}^B, t^B)\alpha^A_0 \otimes \alpha^B_0\}\mathbf{\Psi}_{AB} = 0,\end{aligned} \tag{18}$$

5 Conserved currents.

5.1 Two times conserved current with a not definite positive densityfor the multitime two-particle equation.

The conservation equation associated to (18) is obtained as usually by multiplying it at the left by $\mathbf{\Psi}^{\dagger}{}_{AB}$, and substracting its adjoint multiplied at the right by $\mathbf{\Psi}_{AB}$. By doing so we find the two times conservation equation

$$\frac{\partial \rho_A}{\partial t_A} + \frac{\partial \rho_B}{\partial t_B} + div_1(\overrightarrow{J}_A) + div_2(\overrightarrow{J}_B) = 0 \tag{19}$$

with

$$\begin{aligned}\rho_A(t_A, t_B, \overrightarrow{x}_1, \overrightarrow{x}_2) &= \mathbf{\Psi}^{\dagger}_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2)(\mathbb{1}^A \otimes \alpha^B_0)\mathbf{\Psi}_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2) =\\ &(\mathbf{\Psi}^{AB}_{11})^*\mathbf{\Psi}^{AB}_{11} + (\mathbf{\Psi}^{AB}_{21})^*\mathbf{\Psi}^{AB}_{21} + (\mathbf{\Psi}^{AB}_{31})^*\mathbf{\Psi}^{AB}_{31} + (\mathbf{\Psi}^{AB}_{41})^*\mathbf{\Psi}^{AB}_{41}\\ &(\mathbf{\Psi}^{AB}_{12})^*\mathbf{\Psi}^{AB}_{12} + (\mathbf{\Psi}^{AB}_{22})^*\mathbf{\Psi}^{AB}_{22} + (\mathbf{\Psi}^{AB}_{32})^*\mathbf{\Psi}^{AB}_{32} + (\mathbf{\Psi}^{AB}_{42})^*\mathbf{\Psi}^{AB}_{42}\\ &-(\mathbf{\Psi}^{AB}_{13})^*\mathbf{\Psi}^{AB}_{13} - (\mathbf{\Psi}^{AB}_{23})^*\mathbf{\Psi}^{AB}_{23} - (\mathbf{\Psi}^{AB}_{33})^*\mathbf{\Psi}^{AB}_{33} - (\mathbf{\Psi}^{AB}_{43})^*\mathbf{\Psi}^{AB}_{43}\\ &-(\mathbf{\Psi}^{AB}_{14})^*\mathbf{\Psi}^{AB}_{14} - (\mathbf{\Psi}^{AB}_{24})^*\mathbf{\Psi}^{AB}_{24} - (\mathbf{\Psi}^{AB}_{34})^*\mathbf{\Psi}^{AB}_{34} - (\mathbf{\Psi}^{AB}_{44})^*\mathbf{\Psi}^{AB}_{44},\end{aligned} \tag{20}$$

$$\rho_B(t,_A, t_B, \overrightarrow{x}_1, \overrightarrow{x}_2) = \mathbf{\Psi}^\dagger_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2)(\alpha_0^A \otimes 1\!\!1^B)\mathbf{\Psi}_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2) =$$
$$(\mathbf{\Psi}_{11}^{AB})^*\mathbf{\Psi}_{11}^{AB} + (\mathbf{\Psi}_{12}^{AB})^*\mathbf{\Psi}_{12}^{AB} + (\mathbf{\Psi}_{13}^{AB})^*\mathbf{\Psi}_{13}^{AB} + (\mathbf{\Psi}_{14}^{AB})^*\mathbf{\Psi}_{14}^{AB}$$
$$(\mathbf{\Psi}_{21}^{AB})^*\mathbf{\Psi}_{21}^{AB} + (\mathbf{\Psi}_{22}^{AB})^*\mathbf{\Psi}_{22}^{AB} + (\mathbf{\Psi}_{23}^{AB})^*\mathbf{\Psi}_{23}^{AB} + (\mathbf{\Psi}_{24}^{AB})^*\mathbf{\Psi}_{24}^{AB}$$
$$-(\mathbf{\Psi}_{31}^{AB})^*\mathbf{\Psi}_{31}^{AB} - (\mathbf{\Psi}_{32}^{AB})^*\mathbf{\Psi}_{32}^{AB} - (\mathbf{\Psi}_{33}^{AB})^*\mathbf{\Psi}_{33}^{AB} - (\mathbf{\Psi}_{34}^{AB})^*\mathbf{\Psi}_{34}^{AB}$$
$$-(\mathbf{\Psi}_{41}^{AB})^*\mathbf{\Psi}_{41}^{AB} - (\mathbf{\Psi}_{42}^{AB})^*\mathbf{\Psi}_{42}^{AB} - (\mathbf{\Psi}_{43}^{AB})^*\mathbf{\Psi}_{43}^{AB} - (\mathbf{\Psi}_{44}^{AB})^*\mathbf{\Psi}_{44}^{AB} \quad (21)$$

$$\overrightarrow{J}_A = \mathbf{\Psi}^\dagger_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2)(\overrightarrow{\alpha}^A \otimes \alpha_0^B)\mathbf{\Psi}_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2)$$
$$\overrightarrow{J}_B = \mathbf{\Psi}^\dagger_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2)(\alpha_0^A \otimes \overrightarrow{\alpha}^B)\mathbf{\Psi}_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2).$$

In 1927, L. de Broglie interpreted the single time conservation associated to the multiparticle non-relativistic Schroedinger equation in relation with velocities in configuration space. The double time conservation equation (19) ought to be interpreted in configuration space-time, the difference of positive terms in eqns.(20,21) showing clearly that the density is not definite positive.

5.2 Non-definite positive conserved current with a not definite positive densityfor the single time two-particle equation.

If we consider the non-relativistic limit where all velocities are small relatively to the speed of light, it is natural to consider a single inertial frame, for instance the frame at rest relatively to the center of mass of the two particles. The conservation equation associated to the single time version of (18), written in this frame in terms of the "Newtonian" space-time coordinates $(t, x_A, y_A, z_A, x_B, y_B, z_B)$ reads

$$\frac{\partial \rho_{AB}}{\partial t} + div_1(\overrightarrow{J}_A) + div_2(\overrightarrow{J}_B) = 0 \text{ with}$$
$$\overrightarrow{J}_A = \mathbf{\Psi}^\dagger_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2)(\overrightarrow{\alpha}^A \otimes \alpha_0^B)\mathbf{\Psi}_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2)$$
$$\overrightarrow{J}_B = \mathbf{\Psi}^\dagger_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2)(\alpha_0^A \otimes \overrightarrow{\alpha}^B)\mathbf{\Psi}_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2),$$

$$\begin{aligned} &\rho_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2) = \\ &\mathbf{\Psi}^\dagger_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2)(\alpha_0^A \otimes 1\!\!1^B + 1\!\!1^A \otimes \alpha_0^B)\mathbf{\Psi}_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2) = \\ &2(|\mathbf{\Psi}_{11}^{AB}|^2 + |\mathbf{\Psi}_{22}^{AB}|^2 + |\mathbf{\Psi}_{12}^{AB}|^2 + |\mathbf{\Psi}_{21}^{AB}|^2 \\ &\quad - |\mathbf{\Psi}_{33}^{AB}|^2 - |\mathbf{\Psi}_{44}^{AB}|^2 - |\mathbf{\Psi}_{34}^{AB}|^2 - |\mathbf{\Psi}_{43}^{AB}|^2), \end{aligned} \quad (22)$$

the difference of positive terms in eqn.(23) showing clearly that the density is not definite positive. It is worth noting at this level that in the non-relativistic limit, one can separate the 4-components spinor into two large components (1,2) and two small components (3,4), respectively associated to the positive and negative energy sectors, where the small amplitudes go to zero in the limit where v/c tends to zero. This treament is explictly carried out in [6] and can be generalised to the Bohm-Hiley equation [3]. The idea is to project the evolution along the sector of positive energies, in a first time by considering a separable state, in which case we obtain the Pauli equation with two components spinors only (the large components). By linearity the Pauli equation will be valid for entangled states too. A similar treatment is valid in the present case and we find again the Pauli multiparticle equation in the non-relativistic limit. This is so because when the full state factorizes into products of individual states, the individual Dirac equation is valid for each of them and the usual treatment is valid [6]. Neglecting the small components we get the usual expression of the density $\rho_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2) \approx 2(|\mathbf{\Psi}_{11}^{AB}|^2 + |\mathbf{\Psi}_{22}^{AB}|^2 + |\mathbf{\Psi}_{12}^{AB}|^2 + |\mathbf{\Psi}_{21}^{AB}|^2$ which is thus definite positive in the non-relativistic limit.

5.2.1 Non-identical fermions.

Let us consider two non-identical fermions. Even when the full state factorizes ($\mathbf{\Psi}_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2)$=$\mathbf{\Psi}_A(t, \overrightarrow{x}_1)\otimes\mathbf{\Psi}_B(t, \overrightarrow{x}_2)$), which corresponds to the situation where the two fermions are independent (not entangled), the density does not factorize:

$\rho_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2) \neq \rho_A(t, \overrightarrow{x}_1) \cdot \rho_B(t, \overrightarrow{x}_2)$, but instead

$\rho_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2) = \rho^+_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2) - \rho^-_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2)$

Where

$\rho^\pm_{AB}(t, \overrightarrow{x}_1, \overrightarrow{x}_2) = \rho^\pm_A(t, \overrightarrow{x}_1) \cdot \rho^\pm_B(t, \overrightarrow{x}_2)$

with

$\rho^+_{AB} = 2(|\mathbf{\Psi}_{11}^{AB}|^2 + |\mathbf{\Psi}_{22}^{AB}|^2 + |\mathbf{\Psi}_{12}^{AB}|^2 + |\mathbf{\Psi}_{21}^{AB}|^2)$

$\rho^-_{AB} = 2(|\mathbf{\Psi}_{33}^{AB}|^2 + |\mathbf{\Psi}_{44}^{AB}|^2 + |\mathbf{\Psi}_{34}^{AB}|^2 + |\mathbf{\Psi}_{43}^{AB}|^2)$,

$\rho^+_{A(B)} = \sqrt{2}(|\mathbf{\Psi_1}^{A(B)}(t, \overrightarrow{x})|^2 + |\mathbf{\Psi_2}^{A(B)}(t, \overrightarrow{x})|^2)$

and $\rho^-_{A(B)} = \sqrt{2}(|\mathbf{\Psi_3}^{A(B)}(t, \overrightarrow{x})|^2 + |\mathbf{\Psi_4}^{A(B)}(t, \overrightarrow{x})|^2)$,

which suggests an interpretation according to which two species of fermions coexist, of positive and negative "weights".

5.2.2 Identical fermions.

Now, we have to properly antisymmetrize the full wave function. Even when it makes sense to describe the wave function as a wave function in region A and a wave function in region B, which occurs when

$\mathbf{\Psi}_{AB}(t,\overrightarrow{x}_1,\overrightarrow{x}_2) = \frac{1}{\sqrt{2}}\cdot(\mathbf{\Psi}_A(t,\overrightarrow{x}_1)\otimes\mathbf{\Psi}_B(t,\overrightarrow{x}_2) - \mathbf{\Psi}_A(t,\overrightarrow{x}_2)\otimes\mathbf{\Psi}_B(t,\overrightarrow{x}_1))$ with supports of $\mathbf{\Psi}_A$ and $\mathbf{\Psi}_B$ located in remote regions of space, ρ_{AB} does not "factorize modulo undiscernability"[5]:

$\rho_{AB}(t,\overrightarrow{x}_1,\overrightarrow{x}_2) \neq \rho_A(t,\overrightarrow{x}_1)\cdot\rho_B(t,\overrightarrow{x}_2) + \rho_A(t,\overrightarrow{x}_2)\cdot\rho_B(t,\overrightarrow{x}_1)$, but instead

$$\rho_{AB}(t,\overrightarrow{x}_1,\overrightarrow{x}_2) = \rho^+_{AB}(t,\overrightarrow{x}_1,\overrightarrow{x}_2) - \rho^-_{AB}(t,\overrightarrow{x}_1,\overrightarrow{x}_2)$$

Where

$$\rho^+_{AB}(t,\overrightarrow{x}_1,\overrightarrow{x}_2) = \rho^+_A(t,\overrightarrow{x}_1)\cdot\rho^+_B(t,\overrightarrow{x}_2) + \rho^+_A(t,\overrightarrow{x}_2)\cdot\rho^+_B(t,\overrightarrow{x}_1)$$

and

$$\rho^-_{AB}(t,\overrightarrow{x}_1,\overrightarrow{x}_2) = \rho^-_A(t,\overrightarrow{x}_1)\cdot\rho^-_B(t,\overrightarrow{x}_2) + \rho^-_A(t,\overrightarrow{x}_2)\cdot\rho^-_B(t,\overrightarrow{x}_1)$$

with

$$\rho^+_{AB} = 2(|\mathbf{\Psi}^{AB}_{11}|^2 + |\mathbf{\Psi}^{AB}_{22}|^2 + |\mathbf{\Psi}^{AB}_{12}|^2 + |\mathbf{\Psi}^{AB}_{21}|^2)$$
$$\rho^-_{AB} = 2(|\mathbf{\Psi}^{AB}_{33}|^2 + |\mathbf{\Psi}^{AB}_{44}|^2 + |\mathbf{\Psi}^{AB}_{34}|^2 + |\mathbf{\Psi}^{AB}_{43}|^2),$$

and, as before, $\rho^+_{A(B)} = \sqrt{2}(|\mathbf{\Psi_1}^{A(B)}(t,\overrightarrow{x})|^2 + |\mathbf{\Psi_2}^{A(B)}(t,\overrightarrow{x})|^2)$ and $\rho^-_{A(B)} = \sqrt{2}(|\mathbf{\Psi_3}^{A(B)}(t,\overrightarrow{x})|^2 + |\mathbf{\Psi_4}^{A(B)}(t,\overrightarrow{x})|^2)$,

which suggests again an interpretation according to which two species of fermions coexist, of positive and negative "weights".

Related to the fact that the density is not definite positive the de Broglie-Bohm velocities are not strictly confined inside the light cone as is the case in the single particle case. As is the case with the (single particle) Klein-Gordon equation, supraluminal de Broglie-Bohm velocities are likely to occur in the present case [11]. This is not so amazing: the difficulties related to a Lorentz covariant formulation of de Broglie-Bohm's so-called causal interpretation have never been solved in a satisfactory

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[5]This is a property called remoteness by Asher Pérès [10]. Consider for instance an electron of state $\mathbf{\Psi}_A$ in a terrestrial lab. and an electron of state $\mathbf{\Psi}_B$ at the surface of the moon. The properly antisymmetrized state describing the pair of electrons $\mathbf{\Psi}_{AB}(t,\overrightarrow{x}_1,\overrightarrow{x}_2) = \frac{1}{\sqrt{2}}\cdot(\mathbf{\Psi}_A(t,\overrightarrow{x}_1)\otimes\mathbf{\Psi}_B(t,\overrightarrow{x}_2) - \mathbf{\Psi}_A(t,\overrightarrow{x}_2)\otimes\mathbf{\Psi}_B(t,\overrightarrow{x}_1))$ and $\rho_{AB}(t,\overrightarrow{x}_1,\overrightarrow{x}_2) = \rho_A(t,\overrightarrow{x}_1)\cdot\rho_B(t,\overrightarrow{x}_2) + \rho_A(t,\overrightarrow{x}_2)\cdot\rho_B(t,\overrightarrow{x}_1)$ which is de facto factorizable into the product of a terrestrial wave function with a lunar one excepted that we symmetrize the labels 1 and 2 due to undiscernability.

manner up to now, excepted in the single particle (Dirac fermion) case [12]. Moreover, the status of antimatter, considered as holes in the Dirac sea of negative energy states is still unclear at this level.

6 Aparté: Implications in relation with instantaneous Newtonian gravitational interaction.

Although it is in principle required to invoke general relativity if we wish to discuss about gravity, it is temptating to consider the non-relativistic (slow velocities) limit of the mutlitime multiparticle equation (18) in connection with the Newtonian two-body problem, by imposing in (18) $V(\overrightarrow{r}_A, \overrightarrow{r}_B, t_a = t_B = t) = -Gm_a \cdot m_B / r_{AB}$. In the classical limit, when particles A and B move slowly relatively to c and relatively to each other, gravitational interaction can be considered in good approximation to be instantaneous relatively to the frame attached to their centre of mass [9]; moreover the time associated to this frame plays the role of Newtonian, absolute, time so that the single time restriction of (18) naturally imposes itself.

As we noted before (section 5.2) in the regime of low velocities considered here, the spinorial components 1 and 2 associated to positive weights (3 and 4 in the case of negative weights) in the conserved density (23) correspond in good approximation to the positive energy and negative energy fermions, because it is well-known that if we consider plane wave solutions of the single particle Dirac equation, the ratio of the weigth of the 1 and 2 components to their 3 and 4 counterparts is of the order of $p/2mc$ for positive energy fermions and $2mc/p$ for negative energy fermions.

Imposing thus in (18) $V(\overrightarrow{r}_A, \overrightarrow{r}_B, t_a = t_B = t) = -Gm_a \cdot m_B / r_{AB}$, we observe that, due to the multiplication of V by α_0 matrices, positive (resp. negative) mass/energy states will attract each other but will repel negative (resp. positive) mass/energy states.

It is worth noting that in the ultrarelativistic regime (velocities close to c, and rest mass energy negligible in comparison to the kinetic energy) which characterizes neutrinos and/or electrons moving nearly at the speed of light, it has been shown that positive energy corresponds to positive masses and negative energy to negative masses by a totally different argument: the ratio between the momentum and the de Broglie-Bohm velocity of a plane wave function of a single massless Dirac-Weyl particle has the same sign as its energy [13]. This is reminiscent of the picture according to which negative energy states move backwards in time.

In conclusion, we suggest to interpret the conserved density (23) as a density of mass, positive (negative) components being associated to positive (negative) masses. Positive (resp. negative) mass states are predicted to attract each other while they repel negative (resp. positive) mass states. This remark deserves to be put in correspondence with several recent proposals [14] aimed at checking whether anti-matter would be attracted towards the conter of earth or would be repulsed from it, violating thereby the equivalence principle. What we shall not tackle here is the answer to the following question: ...If matter (positive energy states essentially) gravitationally repulses negative energy states, will anti-matter (holes in the Dirac sea of negative energy states) get attracted or repulsed by matter?...it is with this open question that we conclude the present section.

7 Conclusions.

We proposed a new multitime multiparticle equation for which Lorentz covariances is restored. Similar to the Klein-Gordon equation its single time version is associated to a non definite positive conserved density which is the sum of a positive contribution associated, in the non-relativistic limit, to the positive mass/energy states and of a negative contribution associated to the negative mass/energy states. We speculatively propose to interpret these properties in terms of positive and negative gravitational mass which leads to the prediction of attractive and repulsive gravitation.

In virtue of relativistic covariance, the multiparticle equation presents appealing potential applications like e.g. describing a repulsive dissociation process in the center of mass coordinates and translating the result in the lab. frame by a Lorentz transformation. More generally, it could be useful for analysing trajectories associated to collision processes in an accelerator.

Last but not least the new multiparticle equation opens promising perspectives regarding a reformulation of de Broglie's ideas regarding the so-called fusion theory [15] according to which a photon can be described as a juxtaposition of two fermions (work in progress; see also [16]).

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Appendix: de Broglie's necessary condition for Lorentz invariance of the new multitime equation.

A1: Constraints on Lorentz invariance: one particle.

It is possible to tackle the problem of Lorentz invariance by asking the following question [1]:

..."If the Dirac equations in the new frame assigned to the boosted observer are given by the Dirac matrix (5), can we find a linear spinorial transformation T such that (9) is satisfied"...?

T is actually univoquely constrained by the requirement (9). For instance, the following system of equations must be satisfied if the answer to de Broglie's question appears to be positive:

$$(T^{-1})^\dagger M'_{Dirac}(T^{-1})T\begin{pmatrix}\Psi_1(t,z)\\ \Psi_3(t,z)\end{pmatrix} = (T^{-1})^\dagger M'_{Dirac}\begin{pmatrix}\Psi_1(t,z)\\ \Psi_3(t,z)\end{pmatrix}$$
$$= \begin{pmatrix}P'_0 - mc & -P'_3\\ -P'_3 & P'_0 + mc\end{pmatrix} T\begin{pmatrix}\Psi_1(t,z)\\ \Psi_3(t,z)\end{pmatrix}$$

In other words certain linear combinations of the Dirac equations expressed in terms of the ancient spinorial components in the new frame through $M'_{Dirac}\begin{pmatrix}\Psi_1(t,z)\\ \Psi_3(t,z)\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$ must be equivalent to the covariantly transformed Dirac equation
$\begin{pmatrix}P'_0 - mc & -P'_3\\ -P'_3 & P'_0 + mc\end{pmatrix}\begin{pmatrix}\Psi'_1(t,z)\\ \Psi'_3(t,z)\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$, where $\Psi'_1(t,z)$ and $\Psi'_3(t,z)$ are linear combinations of $\Psi_1(t,z)$ and $\Psi_3(t,z)$.

In particular,

$$\begin{aligned}(T^{-1})_{11}(M'_{11}\Psi_1 + M'_{12}\Psi_3) + (T^{-1})_{21}(M'_{21}\Psi_1 + M'_{22}\Psi_3)\\ = (P'_0 - mc)(T_{11}\Psi_1 + T_{12}\Psi_3) - P'_3(T_{21}\Psi_1 + T_{22}\Psi_3)\end{aligned}$$

Let us denote $(T^{-1})_{11} = e_1$, $(T^{-1})_{21} = e_2$, $T_{11} = a_1$, $T_{12} = a_2$, $T_{21} = a_3$, $T_{22} = a_4$. We obtain thus the constraint

$$\begin{aligned}e_1\{(P'_0ch\gamma + P'_3sh\gamma - mc)\Psi_1 + (-P'_3ch\gamma - P'_0sh\gamma)\Psi_3\}\\ +e_2\{(-P'_0sh\gamma - P'_3ch\gamma)\Psi_1 + (P'_3sh\gamma + P'_0ch\gamma + mc)\Psi_3\}\\ = (P'_0 - mc)(a_1\Psi_1 + a_2\Psi_3) - P'_3(a_3\Psi_1 + a_4\Psi_3) \qquad (23)\end{aligned}$$

Identifying the coefficient of the mass term at the left and at the right of (23) imposes $e_1(-mc) = -a_1 mc$ and $e_2 mc = -a_2 mc$ thus $e_1 = a_1$ and $e_2 = -a_2$.

Identifying now the coefficient of P'_0 imposes $e_1 ch\gamma - e_2 sh\gamma = a_1$ and $-e_1 sh\gamma + e_2 ch\gamma = a_2$.

Combining both constraints we get

$e_1 ch\gamma - e_2 sh\gamma = e_1$

and

$-e_1 sh\gamma + e_2 ch\gamma = -e_2$.

The determinant of this system of equations is equal to 0, and its non-trivial solutions are of the type $e_1 = \lambda ch(\gamma/2)$, $e_2 = \lambda sh(\gamma/2)$ where λ is an arbitrary non-null complex number. Taking account of the other constraints fixes the value $\lambda = 1$. Similar results hold or a_2 and a_4

These results are of course compatible with the spinorial transformation laws (6,10,11). The interest of this approach is that when the constraints imposed by Lorentz covariance are not compatible (which is expressed by the non-null value of certain determinants associated to the constraints) Lorentz covariance is de facto impossible. As we wrote before, making use of de Broglie's method, it can be shown [1] that the two particles multitime Dirac equation (1) is not Lorentz invariant [1],at least under transformations that do not mix the 1-3 components with the 2-4 components as one should expect in a direct generalisation of the single particle case treated above, which motivated the present paper.

A2: Constraints on Lorentz invariance: two particles.

By construction the invariance of (18) under Lorentz transformation is guaranteed; therefore it is not absolutely necessary to repeat the method of section 7 [1, 4] in order to check whether or not Lorentz invariance is possible. However, it constitutes an argument in favor of the consistence of our approach so that we shall sketch the procedure below. Following [1], let us thus reconsider the Lorentz invariance of the new multitime equation (18); for reasons of simplicity, we shall confine ourselves to the restriction of the 16 equations (18) to four equations involving the 4-spinor defined through (13).

The first four equations read now

$$
\begin{aligned}
(P_0^A + P_0^B - (m_A + m_B))\Psi_{11} - P_3^A\Psi_{31} - P_3^B\Psi_{13} = 0 \\
(-P_0^A + P_0^B + (m_A + m_B))\Psi_{13} + P_3^A\Psi_{33} - P_3^B\Psi_{11} = 0 \\
(P_0^A - P_0^B + (m_A + m_B))\Psi_{31} - P_3^A\Psi_{11} + P_3^B\Psi_{33} = 0 \\
(-P_0^A - P_0^B - (m_A + m_B))\Psi_{33} + P_3^A\Psi_{13} + P_3^B\Psi_{31} = 0
\end{aligned} \tag{24}
$$

Differently from the multitime equations (1) studied in [1], these ones have only terms of masses with the sum of the two particle masses,and in the second and third of these equations the difference $P_0^A - P_0^B$.

Let us impose (in full similarity with (23)) that a linear combination with parameters e_1, e_2, e_3, e_4 of these equations, in the boosted frame, is equivalent to the first non-boosted equation, expressed in terms of a linear combination of the four spinor components with $\Psi'_{11} = a_1\Psi_{11} + a_2\Psi_{13} + a_3\Psi_{31} + a_4\Psi_{33}$, $\Psi'_{13} = b_1\Psi_{11} + b_2\Psi_{13} + b_3\Psi_{31} + b_4\Psi_{33}$, $\Psi'_{31} = c_1\Psi_{11} + c_2\Psi_{13} + c_3\Psi_{31} + c_4\Psi_{33}$ and $\Psi'_{33} = d_1\Psi_{11} + d_2\Psi_{13} + d_3\Psi_{31} + d_4\Psi_{33}$.

As in the single particle case there is a direct connection between the parameters e1,e2,e3,e4, a1,a2,a3,a4 and $T^A \otimes T^B$ (equation (12)). The a's parameters constitute the first line of the matrix $T^A \otimes T^B$, while the e's constitute the first line of the inverse of this matrix.

Identifying the mass term and the coefficient of $(P'_0)^A$ in the first Dirac equation as we did in the single particle case we find, through a similar derivation, the constraints

$e_1 = a_1$, $-e_2 = a_2$, $-e_3 = a_3$ and $e_4 = a_4$, together with

$e_1 ch\gamma - e_3 sh\gamma = a_1$

$-e_2 ch\gamma + e_4 sh\gamma = a_2$

$e_3 ch\gamma - e_1 sh\gamma = a_3$

and $-e_4 ch\gamma + e_2 sh\gamma = a_4$.

Combining all constraints we derive two independent systems:

$e_1 ch\gamma - e_3 sh\gamma = e_1$ and $-e_1 sh\gamma + e_3 ch\gamma = -e_3$;

$-e_2 ch\gamma + e_4 sh\gamma = -e_2$ and $e_2 sh\gamma - e_4 ch\gamma = e_4$

The determinants of these system of equations are both equal to 0, and they lead to non-trivial solutions of the type $e_1 = \lambda ch(\gamma/2)$, $e_3 = \lambda sh(\gamma/2)$, $e_2 = \lambda' ch(\gamma/2)$, $e_4 = \lambda' sh(\gamma/2)$.

These constraints are compatible with (15) provided we impose the values $\lambda = ch(\gamma/2), \lambda' = sh(\gamma/2)$.

As expected, the a's parameters constitute the first line of the matrix $T^A \otimes T^B$, while the e's constitute the first line of the inverse of this matrix, which we obtained by direct construction (15) in a previous section.

As already demonstrated (17), the generalisation of the single spinor matricial transformation T_{4x4} (11) to the two particle case reads $T^{AB}_{16x16} = T^A_{4x4} \otimes T^B_{4x4} = (ch\frac{\gamma}{2}\mathbb{1}^A - sh\frac{\gamma}{2}\alpha^A_z) \otimes (ch\frac{\gamma}{2}\mathbb{1}^B - sh\frac{\gamma}{2}\alpha^B_z)$. Due to the block structure of the α_z matrix which acts separately on the $1-3$ components and the $2-4$ components of the Dirac quadrispinor, T^{AB}_{16x16} can be split into 4 blocks, one of them connecting the $1-3$ components of Ψ^A and the $1-3$ components of Ψ^B, one of them connecting the $1-3$ components of Ψ^A and the $2-4$ components of Ψ^B, one of them connecting the $2-4$ components of Ψ^A and the $1-3$ components of Ψ^B, and the last block connecting the $2-4$ components of Ψ^A and the $2-4$ components of Ψ^B.

One can show in a similar fashion, for these four cases, that the constraints imposed by Lorentz covariance are compatible with our construction. As we see, de Broglie's analysis allows us to check in an independent fashion that the spinorial transformation law that we obtained in the section 4 by a constructive approach is indeed Lorentz invariant.

Unicity of the linear spinorial transformation having been demonstrated in the single particle case [5, 6], it is de facto guaranteed in the two particles case provided we also impose that the spinorial transformation factorizes into a product of a A transformation with a B transformation.