

The Two-scale Interpretation: de Broglie and Schrödinger's External and Internal Wave Functions

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ABSTRACT. We propose an interpretative framework for quantum mechanics inspired by Louis de Broglie's double-solution theory. The principle is to decompose the evolution of a quantum system into two wave functions: an external wave function corresponding to the evolution of its center of mass along with other macroscopic degrees of freedom, and an internal wave function corresponding to the evolution of its internal variables in the center-of-mass system. These two wave functions will have different meanings and interpretations.

The external wave function "pilots" the center of mass of the quantum system: it corresponds to the de Broglie pilot wave. For the internal wave function, we argue in favor of the interpretation proposed by Erwin Schrödinger at the Solvay Congress in 1927: the particles are extended and the square of the module of the (internal) wave function of an electron corresponds to the density of its charge in space.

RÉSUMÉ. Nous proposons une grille de lecture de la mécanique quantique qui s'inspire de la théorie de la double solution de Louis de Broglie. Le principe est de considérer l'évolution d'un système quantique sous la forme de deux fonctions d'onde : une fonction d'onde externe correspondant à l'évolution de son centre de masse et de ces autres degrés de liberté macroscopique, et une fonction d'onde interne correspondant à l'évolution de ses variables internes dans le référentiel du centre de masse. Ces deux fonctions d'onde vont avoir des sens et des interprétations différentes.

La fonction d'onde externe pilote le centre de masse du système quantique : elle correspond à l'onde pilote de Louis de Broglie. Pour la fonction d'onde interne, nous défendons l'interprétation proposée par Erwin Schrödinger au congrès Solvay de 1927 : les particules sont étendues et le carré du module de la fonction d'onde (interne) d'un électron correspond à la densité de sa charge dans l'espace.

1 Introduction

The fathers of quantum mechanics strongly disagreed at the 1927 Solvay Congress regarding the interpretation of the wave function of a quantum system. And the debate continues today. Their respective points of view are recalled below.

Schrödinger's interpretation. As soon as his equation was defined, Erwin Schrödinger dreamed of the possibility of building a non-dispersive wave packet completely representing the particle. This was how he introduced the coherent states of the harmonic oscillator in 1926 [46]: *“Our wave packet always remains grouped, and does not spread over an increasingly large space over time, as do, for example, wave packets that we are used to consider in optics.”* But he fails to address the problem of the hydrogen atom by finishing his article in this way: *“It is certain that it is possible to construct by a process quite similar to the previous one, wave packets gravitating on Kepler ellipses at a large number of quanta and forming the wave image of the electron of a hydrogen atom; but in this case the difficulties of calculation will be much greater than in the particularly simple example that we have treated here and which from this point of view is almost a classroom exercise”.* He took up this interpretation [47] again in 1952 in a debate with the Copenhagen school.

de Broglie's interpretation. For Louis de Broglie, the double-solution theory is the true interpretation, which he outlined [17] in 1926 and sought to demonstrate his whole life [18, 20, 19], the pilot wave he presented at the Solvay Congress in 1927 being only a by-product: *“I was introducing, under the name of "double-solution theory" the idea that we had to distinguish between two solutions that remain distinct but intimately related to the wave equation, one that I called the u wave being a real and non-standard physical wave with a local accident defining the particle and represented by one singularity, the other, Schrödinger's Ψ wave, normalizable and devoid of singularity, which would only be a representation of probabilities [20]”.*

The 2017 special issue of the *Annales de la Fondation de Broglie*, contained a synthesis of recent work on Broglie's double-solution and its history over the past 90 years [28, 11, 45] since de Broglie first presented his ideas in 1927. Non-linear physics occupies an important part of this work as well as the concept of the soliton, described as a *“singular*

and persistent object, [a] materialization in wave form of the corpuscle concept” which calls for a reassessment of the double-solution program and offers promising prospects for the reconciliation of quantum theory with realism [11, 40]. The addition of non-linear terms to the Schrödinger equation is widely studied.

Of particular note is Thomas Durt’s article [25] which defines a very interesting double solution *à la de Broglie* for Schrödinger-Newton’s non-linear equation.

Our approach distinguishes two wave functions, but differs from de Broglie’s double-solution insofar as we do not introduce any non-linearity, the coherent states showing that the concept of soliton also appears in linear physics. Moreover, the two-level solution we present is more a distinction in scale (external/internal) than a search for an underlying non-linear wave equation [45].

Born’s interpretation. In his 1954 Nobel speech, Max Born [7] recalls his approach to defining the statistical interpretation of quantum mechanics: *“It was again Einstein’s idea that guided me. He had tried to make the duality of the waves and particles - light quanta or photons - comprehensible by considering the square of the light wave amplitudes as the probability density for the presence of photons. This idea could immediately extend to the function ψ : $|\psi|^2$ should be the probability density for the presence of electrons (or other particles). It was easy to assert this. But how can it be demonstrated? The atomic collision processes made it possible.”*

Heisenberg’s interpretation. In contrast to de Broglie and Schrödinger’s search for physical images, Werner Heisenberg presents a formal framework for quantum theory, theorized as a system of concepts [8] *“The new concept system at the same time yields the intuitive content of the new theory. From an intuitive theory in this sense we must therefore only ask that it be without contradiction and that it be able to predict unambiguously the results of every conceivable experiment in its field. Quantum mechanics will be in this sense an intuitive and complete theory of mechanical processes.”* Thus, Heisenberg only keeps the minimal, non-contradictory condition.

Bohr's interpretation. For Niels Bohr, the theory is based on the principle of complementarity [6] with which he hopes to “contribute to by reconciling the apparently contradictory conceptions defended by different physicists”. He argues that it is possible to express the essence of the theory using the “**quantum postulate**”. [...] “*This quantum postulate implies that any observation of atomic phenomena will involve an interaction with the agencies of observation; therefore, an independent reality in the ordinary physical sense can neither be ascribed to the phenomena nor to the agencies of observation. [...] we must regard them as complementary, but mutually exclusive features of our representation of the experimental findings.*”

Einstein's interpretation. Albert Einstein summed up this debate well in one of his final texts (1953), *Elementary Considerations on the Interpretation of the Foundations of Quantum Mechanics* in homage to Max Born: “*The fact that the Schrödinger equation combined with the Born interpretation does not lead to a description of the 'real state' of a single system, naturally gives rise to a search for a theory which is free of this limitation. So far there have been two attempts in this direction, which share the features that they maintain the Schrödinger equation, and give up the Born interpretation. The first effort goes back to de Broglie and has been pursued further by Bohm with great perspicacity [...] The second attempt, which aims at achieving a "real description" of an individual system, based on the Schrödinger equation, has been made by Schrödinger himself. Briefly, his ideas are as follows. The ψ -function itself represents reality, and does not stand in need of the Born interpretation....[...] From the previous considerations, it follows that the only acceptable interpretation of Schrödinger's equation up to now is the statistical interpretation given by Born. However, it does not give the 'real description' of the individual system, but only statistical statements related to sets of systems.*” [27]

In this paper, we propose an interpretation of quantum mechanics which attempts to synthesize the two realistic interpretations considered above by Einstein, that of the de Broglie pilot wave and that of Schrödinger. This interpretative framework is limited here to massive and non-relativistic particles.

This interpretation follows our work on the theory of double preparation [32] where we show that there are two types of interpretation depending on the preparation of the quantum system: the wave function

can either represent a quantum object inside a set of other identical and indistinguishable quantum objects (such as a particle beam), or it can represent a single and isolated (therefore necessarily distinguishable) quantum object. The basic idea is to study the evolution of a quantum system in terms of the evolution of its center of mass (external evolution) as well as its internal evolution. The use of the center-of-mass wave is not new, and has been mentioned very often by many authors. For example, it is very well explained in the book *Atomic Interferometry* by Baudon and Robert [1]: *“In the free evolution of an atom or molecule, the external motion (corresponding to the motion of the mass center \mathbf{R}) and the wave associated with it plays a separate role. Indeed, because of the separation of the Hamiltonian from the system (in the absence of any external interaction) in the form: $H = T + H_{int}$ where $T = -\frac{\hbar^2}{2m} \Delta_R$ is the kinetic energy operator and H_{int} the part of the Hamiltonian that involves only variables other than \mathbf{R} , there are system states whose wave function has the form $\Psi(\mathbf{R}, t)\Phi_{int}$, where $\Psi(\mathbf{R}, t)$ and Φ_{int} are proper states of T and H_{int} respectively. It is on the wave function Ψ of the external motion, which corresponds to a state of the continuum, that atomic interferometry will be carried out.”* A similar point of view is adopted by Claude Cohen-Tannoudji in the preface to this book [1]: *“A de Broglie wave is also associated with the movement of the center of mass of a more complex quantum system, such as an atom or molecule, composed of several protons, neutrons and electrons. The wavelength of the de Broglie wave associated with an object of mass M and velocity v is inversely proportional to the product Mv .”*

The approach is the same as in classical mechanics: it consists in studying the evolution of a system from its external variables such as the center of mass and its velocity, which correspond to the global motion of the quantum system, and the internal variables, which correspond to its motion in the reference frame of the center of mass. We study how the wave function of the system can be decomposed into two wave functions: the wave function of its center of mass (external evolution) and the internal wave function. These two wave functions are of a different nature and will have different interpretations. We argue in this paper that the distinction between the two wave functions (internal and external) is not only a mathematical trick to solve Schrödinger’s equation more simply but is real: the two wave functions are defined at different scales and have different physical meanings.

First, we show that the external wave function corresponds to the interpretation of the de Broglie-Bohm “pilot wave” (dBB). Indeed, we demonstrate mathematically that, when $\hbar \rightarrow 0$, the square of the module and the phase of the external wave function converge towards a density and a classical action verifying the Hamilton-Jacobi statistical equations. This interpretation of the external wave function by the de Broglie-Bohm pilot wave provides a physical explanation for the measurement results, both for diffraction and interference experiments as well as for spin measurements as in the Stern-Gerlach and EPR-B experiments. The reduction of the wave function of the quantum system is then controlled by the position of the center of mass at the time and position where the quantum system is captured.

For the internal wave function, several interpretations are possible. We will study two possibilities of interpretation: that of dBB and the interpretation proposed by Erwin Schrödinger in 1926 and then at the Solvay Congress in 1927. He argues that particles are extended and the square of the module of the (internal) wave function of an electron corresponds to the value of the distribution of its charge in space. The interpretation of the internal wave function is therefore deterministic, and our two-scale interpretation is also a deterministic theory.

The plan of the article is as follows. In Section 2, we study how the wave function of a N-body quantum system is decomposed between its external and internal wave functions. We also present two non-quantum analogies to better understand the interaction between external and internal wave functions. In Section 3, we demonstrate mathematically, by studying convergence towards classical mechanics, and experimentally, by simply explaining measurement in quantum mechanics, that the most plausible interpretation of the external wave function is that of de Broglie-Bohm “pilot wave”. In the appendices, we give three case studies where only the external wave function is considered: atomic interferometry, spin measurement, and the EPR-B experiment. In Section 4, we propose an interpretation of the internal function corresponding to the Schrödinger interpretation. Finally, in conclusion, we show that this two-scale interpretation clarifies debates on the interpretation of quantum mechanics and suggests the need to review the relationships between gravity and quantum mechanics.

2 External and internal wave functions

From the beginning of quantum mechanics, two types of variables have been distinguished for studying atomic or molecular dynamics: internal and external variables. The external variables concern the external dynamics of the atom, i.e. the movement of its center of mass and the orientation of the frame of reference linked to it. Internal variables describe, for example, the evolution of the structure of the atom or molecule.

An external variable of a massive object is a characteristic that emerges at the scale of its center of mass. For example, mass, charge, electric/magnetic dipole moments or spin are external variables. Moreover, all external variables also have an internal description. The mass of a molecule is the sum of the masses of its components. The same is true for electrical charge. The $1/2$ -spin of an atom with an odd number of nucleons comes from the $1/2$ -spin of its electron from its last shell. Internal variables do not occur on a higher level such as the spacial configuration of a molecule or the energy states configuration of electrons of an atom.

These internal and external degrees of freedom are not independent in general. The interactions between internal and external variables are indeed at the basis of the manipulation of atoms, in particular their cooling [15].

Depending on the experimental conditions, these interactions vary in size, making the decomposition of the total wave function into an external wave function and an internal wave function more or less approximated. Let us start by studying the case of a N -body system where this decomposition proves to be accurate.

2.1 Decomposition of a N -body system such as an atom or molecule

Let us consider a system of N particles without spin, with masses m_i and coordinates \mathbf{x}_i , subjected to a linear potential field $V_i(\mathbf{x}_i) = m_i \mathbf{g} \cdot \mathbf{x}_i$, and to mutual interactions described by the potentials $U_{ij}(\mathbf{x}_i - \mathbf{x}_j)$. This quantum system is therefore described by the wave function $\Psi^h(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, t)$, which satisfies the Schrödinger equation:

$$i\hbar \frac{\partial \Psi^h(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, t)}{\partial t} = H \Psi^h(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, t) \quad (1)$$

with the Hamiltonian:

$$H = \sum_i \left(\frac{\mathbf{p}_i^2}{2m_i} + V_i(\mathbf{x}_i) \right) + \sum_{ij} U_{ij}(\mathbf{x}_i - \mathbf{x}_j), \quad (2)$$

and the initial condition:

$$\Psi^h(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, 0) = \Psi_0^h(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N). \quad (3)$$

The movement of these N particles is separated from the movement of their center of mass as in classical mechanics: Let $\mathbf{x}_G = (\sum_i m_i \mathbf{x}_i) / (\sum_i m_i)$ be the position of the center of mass, $\mathbf{x}'_i = \mathbf{x}_i - \mathbf{x}_G$ the position of the particle i relative to the barycenter \mathbf{x}_G , $M = \sum_i m_i$ the total mass, $\mathbf{x}'_G = (\sum_i m_i \mathbf{x}'_i) / (\sum_i m_i) = \mathbf{0}$ the internal coordinates of the center of mass. Then the Hamiltonian H is written according to the total impulses ($\mathbf{p}_G = \sum_i \mathbf{p}_i$) and relative impulses ($\mathbf{p}'_i = \mathbf{p}_i - \frac{m_i}{M} \mathbf{p}_G$):

$$H = \frac{\mathbf{p}_G^2}{2M} + M\mathbf{g} \cdot \mathbf{x}_G + \sum_i \frac{\mathbf{p}'^2}{2m_i} + \sum_{ij} U_{ij}(\mathbf{x}'_i - \mathbf{x}'_j) = H_{ext} + H_{int}, \quad (4)$$

with $H_{ext} := \frac{\mathbf{p}_G^2}{2M} + M\mathbf{g} \cdot \mathbf{x}_G$ and $H_{int} := \sum_i \frac{\mathbf{p}'^2}{2m_i} + \sum_{ij} U_{ij}(\mathbf{x}'_i - \mathbf{x}'_j)$, and there is no interaction between internal and external parts of the Hamiltonian.

Proposition 1 - *If the initial wave function $\Psi_0^h(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N)$ is factorized in the form:*

$$\Psi_0^h(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) = \Psi_0^h(\mathbf{x}_G) \varphi_0^h(\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_N), \quad (5)$$

then $\Psi^h(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, t)$, a solution to (1), (2) and (3), is written as the product of an external wave function $\psi^h(\mathbf{x}_G, t)$ and an internal function $\varphi^h(\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_N, t)$:

$$\Psi^h(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, t) = \psi^h(\mathbf{x}_G, t) \varphi^h(\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_N, t) \quad (6)$$

where $\Psi^h(\mathbf{x}_G, t)$ is the solution to Schrödinger's external equations:

$$i\hbar \frac{\partial \Psi^h(\mathbf{x}_G, t)}{\partial t} = -\frac{\hbar^2}{2M} \Delta_{\mathbf{x}_G} \Psi^h(\mathbf{x}_G, t) + M\mathbf{g} \cdot \mathbf{x}_G \Psi^h(\mathbf{x}_G, t) \quad (7)$$

with the initial condition:

$$\Psi^h(\mathbf{x}_G, 0) = \Psi_0^h(\mathbf{x}_G) \quad (8)$$

and where $\varphi^h(\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_N, t)$ is the solution to Schrödinger's internal equations:

$$i\hbar \frac{\partial \varphi^h(\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_N, t)}{\partial t} = -\sum_i \frac{\hbar^2}{2m_i} \Delta_{\mathbf{x}'_i} \varphi^h(\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_N, t) + \sum_{i,j} U_{ij}(\mathbf{x}'_i - \mathbf{x}'_j) \varphi^h(\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_N, t) \quad (9)$$

with the initial condition

$$\varphi^h(\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_N, 0) = \varphi_0^h(\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_N). \quad (10)$$

The proposal is obtained simply by replacing in the Schrödinger equation (1), $\Psi^h(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, t)$ by $\psi^h(\mathbf{x}_G, t)\varphi^h(\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_N, t)$. This yields:

$$\begin{aligned} & \left(i\hbar \frac{\partial}{\partial t} - H \right) \Psi^h(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, t) \\ &= \psi^h(\mathbf{x}_G, t) \left[\left(i\hbar \frac{\partial}{\partial t} - H_{int} \right) \varphi^h(\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_N, t) \right] \\ & \quad + \varphi^h(\mathbf{x}'_1, \mathbf{x}'_2, \dots, \mathbf{x}'_N, t) \left[\left(i\hbar \frac{\partial}{\partial t} - H_{ext} \right) \psi^h(\mathbf{x}_G, t) \right] \\ &= 0. \end{aligned}$$

The decomposition of the total wave function as the product of an external wave function and an internal wave function is due to the exact decomposition of the Hamiltonian into its external and internal parts.

The fundamental property of the external wave function of a quantum system is that it can spread over time in space and be divided into several parts (see the interference experiment in Appendix A). It can be very large. On the contrary, the internal wave function of a system remains confined in space, it does not spread and cannot divide without changing the nature of the system; this is what happens during ionization, nuclear fission or chemical reaction. The size of the internal wave function is the size commonly given for an atom or molecule and is often much smaller than the size of the external wave function (i.e. the width of the wave packet) as we will see in the examples in Section 3.

When the quantum system is not composed of several particles but corresponds to an elementary particle such as a free electron, it can also be associated with an internal wave function and an external wave function. Its external wave function is the wave function usually associated with an electron coming out of an electron gun of an electron microscope or a tungsten tip of a scanning tunnel microscope. Although many physicists consider electrons to be material points, Schrödinger proposes to consider them as electronic clouds with a continuous charge distribution. The internal wave function of a free electron is not known but could correspond to Schrödinger's electronic cloud.

Remark 1 - The role of gravity - *The independence between the external and internal Hamiltonians of proposition 1 comes in part from the very special form of the linear gravitational potential $V_j(\mathbf{x}_j)$ that depends linearly on the product $m_j \mathbf{x}_j$, which yields $\sum_j V_j(\mathbf{x}_j) = M \mathbf{g} \cdot \mathbf{x}_G$. Indeed, it is assumed that the gravitational field is constant at the scale of the particle. In addition, with the linear potential $V_j(\mathbf{x}_j) = m_j \mathbf{g} \cdot \mathbf{x}_j$, gravity is transferred exactly to the external wave function.*

In the general case of a N -body quantum system, there is interaction between the external and internal Hamiltonians due to the role of the environment, and the external equations of Schrödinger (7) and (8) and internal variables of Schrödinger (9) and (10) are only approximated.

As many authors have already pointed out [35], the factorization of the wave function into different variables implies a physical independence of these variables. When the wavefunction is nonfactorizable, we said that the system is entangled. In this paper, only the simplest case, where the internal and external degrees of freedom of a quantum system are independent, is studied. The interest of the simple case study is to show clearly the very important differences of interpretation between internal and external wave functions (cf. sections 3 and 4). The factorization of the internal and external variables is not only a mathematical trick but separates two different scales of magnitude and gives two different physical meanings to the wave function. However, there are many cases where the internal and external degrees of freedom are entangled; these more complex cases remain to be studied. In this article, we focus on understanding the two scales separately from each other. In future work, we will study the interactions between these two scales. An example of interaction between the two scales is the repercussion on the center of mass of transitions between internal energy levels after the absorption or emission of a photon. This has already been taken into account by the quantum Monte Carlo simulation introduced by Dalibard, Castin and Molmer. [16].

Let us consider the case when it is assumed that each particle i admits a charge q_i and is also subjected to an electrical potential $q_i V_q(\mathbf{x}_i)$ that varies little on the scale of the quantum system: $V_q(\mathbf{x}_i) \sim V_q(\mathbf{x}_G)$. Under this assumption, the external field applying to the external wave function is written approximately:

$$V(\mathbf{x}_G) = M \mathbf{g} \cdot \mathbf{x}_G + \sum_i q_i V_q(\mathbf{x}_i) \simeq M \mathbf{g} \cdot \mathbf{x}_G + Q V_q(\mathbf{x}_G) \quad (11)$$

with $Q = \sum_i q_i$. We then consider a generalization of Schrödinger's external equation (7) by replacing $M\mathbf{g}\cdot\mathbf{x}_G$ by $V(\mathbf{x}_G)$. The solution obtained will no longer be accurate, but will be a good approximation if the quantum system is not too shaken and remains stable during its evolution. This will no longer be the case if it disintegrates. The case where there is an external magnetic field is taken into account in Appendices B and C for the Stern and Gerlach and EPR-B experiments.

Remark 2 - the N individual functions of a N -body system - In addition to the external and internal wave functions associated with a N -body system, each of the N individual particles in the system must also be associated with an individual internal wave function. In the general case, individual functions are not accessible because they are entangled with other individual functions. A simple case will be proposed in section 4.2.

We can certainly generalize the external wave function to mesoscopic and macroscopic quantum systems that are neither atoms nor molecules. An example of the entangled external wave function from two individual external wave functions is given in Appendix C for the EPR-B experiment.

2.2 Preparation of an internal wave function or an external wave function depending on the experiments

In many studies, knowledge of the quantum system does not correspond to a total wave function but only to an external wave function or an internal wave function. In particular, only the external wave function is considered for all measurement problems that are primarily related to the measurement of the position of the center of mass, namely cases of atomic interferometry, spin and energy measurements.

The internal wave function will explain the values related to the energy spectrum and quantum jumps, but they will be measured by position measurements via an external wave function. Thus in the debates of the 1927 Solvay Congress, de Broglie, with "*the pilot wave*", and Born, with "*the statistical wave*", implicitly considered the external wave function, while Schrödinger, with the coherent states [46], and Heisenberg, with its matrix representation of the transitions of the hydrogen atom, implicitly considered the internal wave function.

2.3 Classical external and internal analogies

We have just seen that there is not always a simple relationship between the total wave function of a quantum system and the external and

internal wave functions. The case of a simple product as in Proposition 1 is exceptional. In general, external variables influence internal variables and vice versa. The following two non-quantum analogies are instructive when it comes to understand the types of interaction between the external and internal wave functions: the analogy with Couder’s *walkers* and the analogy with the solar system.

2.3.1 Analogy with Couder’s “walkers”

The wave-particle duality seems to be a characteristic of the quantum world, having no equivalent in classical physics. Yves Couder and his team [13, 14] have shown “*that a drop bouncing off a vertically vibrating liquid surface can become self-propelled by its interaction with the surface wave it excites*”. The drop couples to the surface waves that it bounces generate and spontaneously moves on the surface. The resulting object, called a “walker”, combines the drop and its surface waves, possessing a dual nature that enables it to perform many of the textbook quantum experiments: Young’s slits [12], tunnel effect [26], quantized orbits [29].

Even if the walkers studied by Couder and his team present fundamental differences with the quantum case (system maintained by vibration, no Planck constant, existence of waves on a material medium), they show us that the wave-particle duality exists at the macroscopic level. The deformations of the drop during bounces constitute its internal evolution. The external evolution of the drop, i.e. its center of mass, is governed by the surface wave it creates during each bounce and the vibrating liquid exterior.

2.3.2 Analogy with the solar system

To study the evolution of a planet in the solar system, we break down the problem into two sub-problems: 1) the evolution of the center of mass of this planet (external evolution) for which the planet is reduced to a point, its center of mass. 2) the internal evolution of the planet in the reference frame of its center of mass, which takes into account the fact that the planet is not a point but has a gaseous and/or solid structure and possibly a rotational movement on itself. These two developments can be treated separately in the first approximation. But depending on the experimental conditions of each planet, we can observe significant effects of external evolution on the internal evolution; for example, terrestrial

tides due to solar attraction slightly deforming the solid structure of the Earth and slowing down the Earth's rotation on itself.

For a quantum system, we observe the same decomposition, an external evolution and internal evolution. We therefore have two functions: an external one that describes the evolution of the center of mass and an internal one defined in the reference frame of the center of mass.

However, depending on the experimental conditions, we will focus only on external variables or only on internal variables. This was the methodology we applied in the double-preparation theory [32]. In many experiments, it is indeed possible to separate these two wave functions. This is the case, for example, of free particles such as in the Young or Stern and Gerlach or EPR-B experiments where only the external evolution of particles (i.e. the evolution of the particle's center of mass) is studied, internal evolution being neglected. On the other hand, if we study the emission lines of a gas, only the internal evolution of gas particles is studied, external evolution being neglected.

3 Interpretation of the external wave function

To interpret the external wave function, we will study how it evolves when the Planck constant goes to zero: $\hbar \rightarrow 0$. Indeed, we show that, in this case, the wave function converges to the Hamilton-Jacobi action. However, in classical mechanics, the Hamilton-Jacobi action is not sufficient to determine a trajectory, we must complete the description of the system by adding the initial position of the center of mass. In quantum mechanics, the dBB theory proceeds from the same logic and completes the description of the system by also adding the initial position of the center of mass. The dBB theory is therefore logically applicable to the external wave function. The Hamilton-Jacobi action pilots the particle in classical mechanics as the wave does in quantum mechanics.

We consider the semi-classical variable change:

$\Psi^{\hbar}(\mathbf{x}_G, t) = \sqrt{\rho^{\hbar}(\mathbf{x}_G, t)} \exp(i \frac{S^{\hbar}(\mathbf{x}_G, t)}{\hbar})$, the density $\rho^{\hbar}(\mathbf{x}_G, t)$ and the action $S^{\hbar}(\mathbf{x}_G, t)$ being functions that in principle depend on \hbar .

Schrödinger's external equations (7) and (8) are decomposed by giving the Madelung equations [38] (1926), which correspond to two coupled equations:

$$\frac{\partial S^{\hbar}(\mathbf{x}_G, t)}{\partial t} + \frac{1}{2M} (\nabla S^{\hbar}(\mathbf{x}_G, t))^2 + V(\mathbf{x}_G) - \frac{\hbar^2}{2M} \frac{\Delta \sqrt{\rho^{\hbar}(\mathbf{x}_G, t)}}{\sqrt{\rho^{\hbar}(\mathbf{x}_G, t)}} = 0 \quad (12)$$

$$\frac{\partial \rho^{\hbar}(\mathbf{x}_G, t)}{\partial t} + \operatorname{div} \left(\rho^{\hbar}(\mathbf{x}_G, t) \frac{\nabla S^{\hbar}(\mathbf{x}_G, t)}{m} \right) = 0 \quad (13)$$

with the initial conditions:

$$\rho^{\hbar}(\mathbf{x}_G, 0) = \rho_0^{\hbar}(\mathbf{x}_G) \quad \text{and} \quad S^{\hbar}(\mathbf{x}_G, 0) = S_0^{\hbar}(\mathbf{x}_G). \quad (14)$$

Here, $V(\mathbf{x}_G)$ is a general potential as in the case of equation (11). In this section we study the convergence of the density $\rho^{\hbar}(\mathbf{x}_G, t)$ and of the action $S^{\hbar}(\mathbf{x}_G, t)$ of the Madelung equations when the Planck constant \hbar goes to 0.

We will limit ourselves to what we referred to as [32] quantum systems that are prepared as “*nondiscerned*”, i.e. such that the initial probability density $\rho_0^{\hbar}(\mathbf{x}_G)$ and the initial action $S_0^{\hbar}(\mathbf{x}_G)$ of the external wave function are the functions $\rho_0(\mathbf{x}_G)$ and $S_0(\mathbf{x}_G)$, independent of \hbar ($\rho_0^{\hbar}(\mathbf{x}_G) = \rho_0(\mathbf{x}_G)$ and $S_0^{\hbar}(\mathbf{x}_G) = S_0(\mathbf{x}_G)$). This is the case of a set of particles without interaction between them and prepared in the same way: beams of free particles or in a gravity field as in the Shimizu [48] experiment with cold atoms, or beams of fullerenes in a Young’s slits experiment (Appendix A).

THEOREM 1 [32]- *When $\hbar \rightarrow 0$, density $\rho^{\hbar}(\mathbf{x}_G, t)$ and action $S^{\hbar}(\mathbf{x}_G, t)$, the solutions to the Madelung equations (12-14) of quantum systems prepared as nondiscerned, converge to $\rho(\mathbf{x}_G, t)$ and $S(\mathbf{x}_G, t)$, the solutions to statistical Hamilton-Jacobi equations:*

$$\frac{\partial S(\mathbf{x}_G, t)}{\partial t} + \frac{1}{2m} (\nabla S(\mathbf{x}_G, t))^2 + V(\mathbf{x}_G) = 0 \quad (15)$$

$$S(\mathbf{x}_G, 0) = S_0(\mathbf{x}_G) \quad (16)$$

$$\frac{\partial \rho(\mathbf{x}_G, t)}{\partial t} + \operatorname{div} \left(\rho(\mathbf{x}_G, t) \frac{\nabla S(\mathbf{x}_G, t)}{m} \right) = 0 \quad (17)$$

$$\rho(\mathbf{x}_G, 0) = \rho_0(\mathbf{x}_G). \quad (18)$$

Thus, if the external wave function is prepared as non-discerned, the Madelung equations of this external wave function converge to the Hamilton-Jacobi statistical equations. These statistical Hamilton-Jacobi equations correspond to a set of classical particles, without interaction between them and subjected to an external potential field $V(\mathbf{x})$, and for which only the probability density $\rho_0(\mathbf{x})$ and the velocity field $\mathbf{v}_0(\mathbf{x})$ are

known at the initial moment due to knowledge of the initial action $S_0(\mathbf{x})$ ($\mathbf{v}_0(\mathbf{x}) = \frac{\nabla S_0(\mathbf{x})}{m}$). These non-discerned classical particles are prepared in the same way as non-discerned quantum particles. The velocity of the center of mass of the classical particle is given in each point (\mathbf{x}, t) by:

$$\mathbf{v}(\mathbf{x}, t) = \frac{\nabla S(\mathbf{x}, t)}{m} \tag{19}$$

Equation (19) shows that the $S(\mathbf{x}, t)$ of the Hamilton-Jacobi equations (15) defines the velocity field at any point (\mathbf{x}, t) from the speed field $\frac{\nabla S_0(\mathbf{x})}{m}$ to the initial moment. So, if we define the initial position \mathbf{x}_{init} of the center of mass of a classical particle, we deduce by (19) the trajectory of the particle's center of mass. The Hamilton-Jacobi action $S(\mathbf{x}, t)$ is therefore a field that “pilots” the movement of the classical particle.

The indetermination on the position of the center of mass of a quantum system therefore corresponds to the indetermination on the position of the center of mass of a conventional system of which only the initial distribution density has been defined. Like the Hamilton-Jacobi action for a classical particle prepared to be non-discerned, the external wave function of a quantum system is not sufficient to define the position of the center of mass of the quantum system; it is necessary to add its initial position and it is therefore natural to introduce the de Broglie-Bohm trajectories for the center of mass of a quantum system. Its speed at the moment t is given by [17, 3]:

$$\mathbf{v}^{\hbar}(\mathbf{x}_G, t) = \frac{\nabla S^{\hbar}(\mathbf{x}_G, t)}{m} \tag{20}$$

which satisfies the continuity equation (13).

Remark 3 - Convergence from \hbar to 0 - In the theorem 1, we assume that $\hbar \rightarrow 0$. However, physically \hbar is never equal to 0 and cannot tend towards 0 since it is a constant and more generally we never have a trajectory that can be called classic. All trajectories are quantum and the so-called classical trajectories are approximations of quantum trajectories for which the term in \hbar is negligible compared to the other terms. A very simple example is that of a quantum object defined by a Gaussian wave packet of center (x_0, y_0, z_0) and standard deviation $(\sigma_{0x}, \sigma_{0y}, \sigma_{0z})$ with a center of mass at the initial position $(x_G(0), y_G(0), z_G(0))$ and the initial

speed $\mathbf{v}_0 = (v_{0x}, v_{0y}, v_{0z})$. If $\mathbf{g} = (0, 0, g)$ is the only external force, then the particle's center of mass satisfies the following dBB trajectory [35, 31]:

$$X^h(t) = x_G(0) + v_{0x}t + (x_0 - x_G(0)) \left(1 - \frac{\sigma_{\hbar x}(t)}{\sigma_{0x}} \right) \quad (21)$$

$$Y^h(t) = y_G(0) + v_{0y}t + (y_0 - y_G(0)) \left(1 - \frac{\sigma_{\hbar y}(t)}{\sigma_{0y}} \right) \quad (22)$$

$$Z^h(t) = z_G(0) + v_{0z}t - \frac{gt^2}{2m} + (z_0 - z_G(0)) \left(1 - \frac{\sigma_{\hbar z}(t)}{\sigma_{0z}} \right) \quad (23)$$

with $\sigma_{\hbar i}(t) = \sigma_{0i} \sqrt{1 + \left(\frac{\hbar t}{2m\sigma_{0i}^2} \right)^2}$ and $i = x, y, z$. We verify that this dBB trajectory tends towards the trajectory we call classic (or Newton's trajectory) and the same starting point when we had \hbar tend to 0. In reality, it is not \hbar that tends towards 0, but the last term of the three equations that is negligible compared to the other terms if we consider a classical object as a stone, whereas if the object is an electron, an atom or a molecule, this term is not necessarily negligible. We observe that if m , the mass of the object increases, then the situation is the same as if \hbar decreases. It is these trajectories that we used to simulate Shimizu's experiment [48] on Young's slits with cold atoms; we thus showed [31] why such atoms with different wave functions were coherent enough to interfere.

Proposition 2 - Let us consider a set of quantum particles defined by a Gaussian wave packet in a constant gravity field as in remark 3. If the centers of mass of these particles follow the dBB trajectories, then $\Delta x(t)$ and $\Delta p_x(t)$, respectively the uncertainty in x on the position and on the momentum of the center of mass of these particles at time t , satisfy the Heisenberg inequalities:

$$\Delta x(t) \cdot \Delta p_x(t) = \frac{\hbar}{2} \frac{\sigma_{\hbar x}(t)}{\sigma_0} \geq \frac{\hbar}{2} \quad (24)$$

By designating $x(t) = x_0 + v_{0x}t$ the trajectory of the center of the external wave packet in x , we verify that $(\Delta x(t))^2 = \langle (x - x(t))^2 \rangle = \int (x - x(t))^2 (2\pi\sigma_{\hbar x}^2(t))^{-\frac{1}{2}} e^{-\frac{(x-x(t))^2}{2\sigma_{\hbar x}^2(t)}} dx = \sigma_{\hbar x}^2(t)$ and $(\Delta p_x(t))^2 = \langle (mv_x^{\hbar}(x, t) -$

$mv(t)^2) = \langle (\frac{\hbar^4 t^2}{16m^2 \sigma_0^4 \sigma_{\hbar}^2(t)} (x - x(t))^2) = \frac{\hbar^2}{4\sigma_0^2}$ [35]. The proposition is deduced from this and clearly leads to interpret Heisenberg's inequalities for the external wave function as **uncertainty relationships**. In this case, the meaning of Heisenberg's inequalities is that there is no source point for a particle. Only wave packets have a physical reality. The interpretation of Heisenberg's inequalities will not be the same for the internal wave function!

In appendices, we give three cases studies where only the external wave function is considered: atomic interferometry, spin measurement, and the EPR-B experiment. Indeed, in many studies, knowledge of the internal wave function is not necessary and can therefore be neglected. This is the case of particle beams with no interactions between them that we find in many experiments: diffraction, interference, spin measurement. This is particularly the case for all measurement problems, which are most often related to the measurement of the position of the center of mass.

4 Schrödinger's interpretation of the internal wave function: quantum chemistry

For the internal wave function, the interpretation is more open: should we take the dBB interpretation? Or the interpretation proposed since 1926 by Schrödinger. We will first consider the latter interpretation, which is little known, before examining the dBB interpretation at the end of the section.

First in 1926, then at the Solvay congress in 1927 and finally in 1952 [47] during a strong controversy with the Göttingen-Copenhagen school, Schrödinger proposed an interpretation of the wave function that was both realistic and deterministic, as reported by Born in his 1954 Nobel lecture [7]: *“Schrödinger thought that his wave theory made it possible to return to deterministic classical physics. He proposed (and he has recently [47] emphasized his proposal anew) to dispense with the particle representation entirely, and instead of speaking of electrons as particles, to consider them as continuous density distributions $|\psi|^2$ (or electric density $e|\psi|^2$).*” One can understand the criticisms of such an interpretation for the total wave function, but the arguments against it are no longer valid a priori for the internal wave function. The most important criticism related to the contradiction between what Schrödinger considers to be the most important: *“the particles are narrow wave packets”* and

the fact that the external function spreads over time as Born points out further on in his Nobel speech.

The great difficulty in defining a reliable interpretation of the internal wave function of a quantum system of N particles is that the internal wave function is defined in the space of $3N$ dimension configurations and no explicit solutions are known. However, we will use two specific examples of a single particle, the harmonic oscillator and the electron in the hydrogen atom as Schrödinger did, and then we will propose a generalization to the N -particle case.

4.1 Schrödinger's interpretation of the internal wave function for a single particle

The objective of Schrödinger's 1926 article, "*The continuous transition from micro-to macro-mechanics*" [46], is to "*demonstrate **in concreto** [for this chosen case of the harmonic oscillator] the transition to macroscopic mechanics by showing that a group of proper vibrations of high order-number n ('quantum number') and of relatively small order-number differences ('quantum number differences') may represent a 'particle', which is executing the 'motion', expected from the usual mechanics, i.e. oscillating with the frequency ν_0 .*"

He considers the classic problem of the Hamiltonian harmonic oscillator in dimension 1, $H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2$. He then looked for the solution to the Schrödinger equation in the case of a particular initial condition that can be written today as:

$$\Psi_0^h(x) = (2\pi\sigma_h)^{-\frac{1}{4}} e^{-\frac{(x-x_0)^2}{4\sigma_h^2}} \quad (25)$$

with $\sigma_h = \sqrt{\frac{\hbar}{2m\omega}}$ and $x_0 \gg \sigma_h$. It shows [46] that this initial wave packet corresponds to a small number of proper functions φ_n of the harmonic oscillator around the value $n \sim \frac{1}{2}(\frac{x_0}{\sigma_h})^2$. We then obtain the coherent state:

$$\Psi^h(x, t) = (2\pi\sigma_h)^{-\frac{1}{4}} e^{-\frac{(x-x(t))^2}{4\sigma_h^2} + i\frac{mv(t)x}{\hbar}} \quad (26)$$

where $x(t) = x_0 \cos \omega t$ and $v(t) = -x_0 \omega \sin \omega t$ correspond to the position and velocity of the center of mass of a classical particle. The coherent state is therefore based on two scales, one of the classical type with x_0 and one of the quantum type with σ_h , which corresponds to the size of the wave packet and oscillates with its center of mass without changing shape.

When $\hbar \rightarrow 0$, the density $\rho^{\hbar}(x, t) = (2\pi\sigma_{\hbar}^2)^{-\frac{1}{2}} e^{-\frac{(x-x(t))^2}{2\sigma_{\hbar}^2}}$ (depending on \hbar) converges to $\rho(x, t) = \delta(x - x(t))$, i.e. converges to the position of a single classical oscillator. This result on the internal wave function differs from theorem 1, which concerned the external wave function. According to Schrödinger’s interpretation, the internal wave function corresponds to an extended particle of a single quantum system. Therefore, the coherent state of the 1D harmonic oscillator satisfies the Heisenberg equation of an extended particle:

$$\Delta x(t) \cdot \Delta p_x(t) = \frac{\hbar}{2} \tag{27}$$

It corresponds to an equality related to the indetermination on the position and momentum of an extended particle. It is thus an indetermination equation. It is recalled that for the coherent state of a two-dimensional harmonic oscillator, the quasi-classical trajectory is an ellipse.

The extension of this interpretation to the electron in the hydrogen atom is done by Schrödinger in 1926: *“It is certain that we can build wave packets gravitating on Kepler ellipses at a large number of quanta and forming the wave image of the electron of a hydrogen atom”*. [46]. In this context, the Born rule that state *the square of the magnitude of a particle’s wavefunction at a given point $|\Psi(x, t)|^2$ is proportional to the probability density of finding the particle at that point x at the time t* is no longer valid and must by replace by what we call below the Schrödinger’s conjecture of 1926.

Conjecture 1 - Schrödinger’s conjecture: *The square of the magnitude of a particle’s internal wave function $|\Psi(x, t)|^2$ represents the continuous density of matter and electrical charge of the extended particle.*

We will see that this conjecture can be based on the analogy with the coherent states of the harmonic oscillator for the wave function of an electron in a Rydberg state. We still give ourselves two more scales, the Bohr radius a_0 and a very large radius $a \gg a_0$. We then look for a wave packet corresponding to a small number of proper functions $\Psi_{n,l,m}(r, \theta, \varphi)e^{-i\frac{E_n t}{\hbar}}$ of the hydrogen atom around the value $n \sim \sqrt{\frac{a}{a_0}}$.

It was not until 1995 that Schrödinger’s 1927 prediction was fulfilled for an electron in a Rydberg state. Bialynicki-Birula [36], Kalinski,

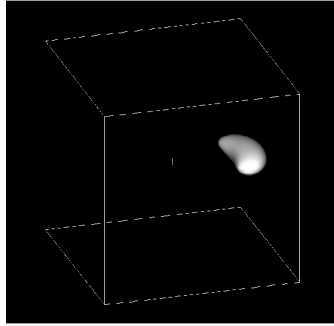


FIGURE 1 – Coherent wave packet of hydrogen atom, from the image of [9]. This internal wave function corresponds to a banana-shape extended electron revolving around the proton.

Eberly, Buchleitner and Delande [9] presented the first constructions of a non-dispersive wave packet in dimension 3 for the hydrogen atom in the presence of a circularly polarized microwave field. *“By passing through the rotating reference frame, the system becomes independent of time and a **stable** fixed point is located at a finite distance from the nucleus along the axis of the microwave electrical field. In the laboratory reference frame, it corresponds to a Bohr circular orbit passed through at a constant angular velocity equal to that of the microwave; this orbit is the center of the resonance island. In its vicinity, it is possible to build quantum wave packets that rotate around the nucleus without being distorted. These wave packets are not Gaussian, but totally non-dispersive.”* [21]. These non-dispersive wave packets correspond to periodic trajectories and are eigenvectors of the Floquet operator.

We note that these wave packets do not correspond to the usual solutions of quantum mechanics textbooks, which are stationary solutions of the electron and not wave packets located on a periodic trajectory.

Figure 1 shows such a packet of waves, in a banana shape, calculated in a frequency field of about 30 *Ghz* with a main quantum number $n = 60$. The package is at about 4,000 Bohr radius of the nucleus and revolves around it in the horizontal plane without deforming.

Non-dispersive waves were successfully formed in experimental conditions for the first time in 2004 by Maeda and Gallagher [39], with the

observation of a life cycle greater than 15,000 periods of the field, compared to the 10 periods observed in the absence of a field for the dispersion of the wave packet.

Remark 4 - Wave packets with or without a periodic field - *The numerically simulated wave packets in 1995 and those made experimentally in 2004 are created and maintained thanks to the presence of a periodic external field. One could therefore refute the conclusion of the existence of periodic wave packets without the presence of this external field. However, as Maeda and Gallagher point out, the field has no influence on the existence of periodic wave packets without deformation, but only on the number of periods without dispersion.*

As Schrödinger had noted, these dynamic wave packets are difficult to determine because there is no analytical representation of them. But we can analytically calculate the dynamics of the center of mass of these states using the Floquet and Ehrenfest theorems.

Thus, for a bound particle such as the electron in the hydrogen atom, Schrödinger's interpretation restricted to the internal wave function is considered valid: it is as if the electron were an electronic cloud with a charge density of $\rho_{int} = e|\varphi(\mathbf{x}, t)|^2$. This interpretation of the internal wave function is extended to free particles.

This is the conclusion Schrödinger drew at the Solvay congress in 1927:

“I found the following way of looking at things useful; it may be a little naïve but it is easy to grasp. The classical system of material points does not really exist, but there is something that continuously fills all the space [...] the real system is a composite image of the classical system in all its possible states, obtained by using $\phi\phi^$ as a 'weight function'. The systems to which the theory is applied are classically composed of a large number of charged material points. As we have just seen, the charge of each of these points is distributed continuously through space and each charge point e provides the contribution of the $e \int \phi\phi^* dx dy dz$ to the charge of the quarterly volume element $dx dy dz$. As $\phi\phi^*$ generally depends on time, these charges vary”.*

4.2 Schrödinger's generalized interpretation of the internal wave function

Let us consider a N -body quantum system. The evolution of the

quantum system is then given by Schrödinger's equation which calculates the internal wave function $\Psi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_N, t)$ from its initial condition $\Psi_0(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_N)$.

To generalize Schrödinger's interpretation to cases of a N -body quantum system, we consider the following conjecture 2:

Conjecture 2 - Schrödinger's generalized conjecture:

For a N -body quantum system, it exists for each body j ($\forall j = 1..N$) an individual wave function $\Psi^j(x, t)$ and $|\Psi^j(x, t)|^2$ represents the density of matter of the body j .

The meaning of the magnitude for the external wave function is given by the Born rule (and observed experimentally), but for the internal wave function, this conjecture gives a different meaning: the magnitude for the internal wave function represented the spatial extension of a N -body quantum system.

Then, the position of the center of mass of the particle j ($j = 1..N$) is:

$$X^j(t) = \int \mathbf{x} |\Psi^j(\mathbf{x}, t)|^2 d\mathbf{x} \quad (28)$$

Figure 2 is a topography of a carbon nanotube observed with a tunneling microscope; This image can be considered as a 2D representation of the nanotube's internal wave function. In the Figure, the red dots correspond to the individual internal wave functions of carbon atomic nuclei. In addition, the electrons of the carbon atoms that ensure the cohesion of the nanotube into a single object also have individual internal wave functions.

We now provide some additional arguments in favour of this generalized Schrödinger interpretation for the internal wave function:

- This is the simplest realistic interpretation for the internal wave function. The usual criticism (the measurement problem) is no longer valid because it is only intended for the interpretation of the external wave function.
- This interpretation is the basis of the **model of the elastically bound electron**.
- This interpretation is compatible with the Lorentz and Poincaré's extended and deformed electron model[44].

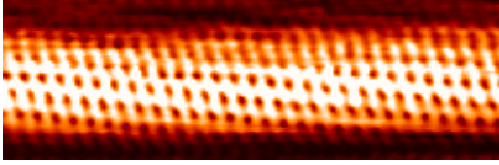


FIGURE 2 – Carbon nanotube observed with a tunneling microscope (STM) [source: Taner Yildirim (The National Institute of Standards and Technology - NIST) - Public Domain].

- The first theorem of Hohenberg and Kohn [34], which is the foundation of **the theory of functional density** and which stipulates that a given electron density corresponds to a single wave function, is basically compatible with Schrödinger’s interpretation of the internal wave function. The kinetic energy of electrons is then approximated as an explicit functional of density, while the contributions of core-electron attraction and electron-electron repulsion are treated in a classical way.
- The theory of the double solution with Schrödinger’s interpretation is therefore also fundamentally compatible with the methodology of **molecular dynamics**.
- Let us finish with the recent experiment (2019) by Mineev, Devoret et al. [41] on *“the jump from the fundamental state to an excited state of a three-level superconducting artificial atom”*. She seems to concur with Schrödinger in his 1952 discussion with the Copenhagen-Göttingen school on quantum jumps, because *“the experimental results show that the evolution of each jump performed is continuous, coherent and deterministic”*. As Devoret explains: *“Our experimental results show that quantum jumps are unpredictable and discrete (as Bohr thought) over long periods of time, they can be continuous (as suggested by Schrödinger) and predictable for a short period of time”*. This is the case because, just before a jump occurs, there is always a latency period (a few microseconds) during which it is possible to acquire a signal that alerts you to the next jump. These continuous and deterministic transitions are consistent with Schrödinger’s generalized interpretation. The same should apply to operators of continuous and deterministic creation and annihilation.

Remark 5 - Neglected external wave function - *The state of the transitions of the internal wave function are measured indirectly via the energy of the emitted particles (photons, electrons). This is the case for spectral measurements of atomic vapour lines (e.g. Balmer lines or Franck and Hertz experiments) or Dehmelt quantum jump experiments (fluorescence on a single three-level ion) and Mineev and Devoret. In these experiments, the external wave function (of the center of mass) of the quantum system is neglected. This is no longer the case in atom cooling experiments, where the recoil of the center of mass after each absorption/emission must be taken into account.*

The above arguments are not sufficient to exclude all other interpretations. For example, the dBB interpretation for the internal wave function is the most plausible realistic and deterministic alternative. It also remains in continuity with the interpretation of the external wave function.

Remark 6 - The dBB interpretation of the internal wave function - *Schrödinger's interpretation, which we have defended, is based on the existence of non-dispersive wave packets whose evolutions are deterministic. The dBB interpretation also applies very well to non-dispersive wave packets. Let us show it on the case of the coherent state of a harmonic oscillator. From the equation (26) we derive that the velocity of dBB is $v^h(x, t) = \frac{\nabla S^h(x, t)}{m} = v(t)$. For an initial position particle $x_0 + \eta$, where η is randomly drawn in a Gaussian of standard deviation σ_h , the dBB trajectory is $x_\eta(t) = x(t) + \eta$. When we tend \hbar towards 0, $x_\eta(t)$ also tends towards the classical trajectory $x(t)$ and the dBB interpretation is coherent to represent a single particle. The same is true for Schrödinger's generalized interpretation as Norsen et al. recently showed [43] : for particles without spin, it is also possible in dBB's pilot wave theory to replace the wave function $\Psi(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_N, t)$ in the configuration space by N wave functions $\Psi^j(\mathbf{x}_j, t)$ in the 3D physical space. These wave functions are the N conditional wave functions introduced by Dürr, Goldstein and Zanghi [24]:*

$$\Psi^j(x_j, t) = \Psi(x_1, x_2, \dots, x_N, t) \Big|_{x_i = \tilde{X}^i(t)} \quad \text{for } i \neq j \quad (29)$$

where $\tilde{X}^i(t)$ is the position of the particle i in Bohmian mechanics.

5 Conclusion and outstanding issues

We proposed, in the pre-relativist context, a two-scale interpretation inspired by Louis de Broglie's double solution. Like him, "*we had to distinguish between two solutions that are distinct, but intimately related to the wave equation*".

Our double solution corresponds to the synthesis of the two interpretations considered in 1953 by Einstein [27], that of Louis de Broglie's pilot wave "*pursued futher by Bohm with great perpicacity*" and that of Schrödinger's "*real description of an individual system*".

The principle of our two-scale solution is to consider the evolution of a quantum system, such as an atom or a molecule, as the product of two wave functions: an external wave function Ψ_{ext} for the evolution of its center of mass and an internal wave function Ψ_{int} for the evolution of its internal variables in the center-of-mass frame of reference. The exact mathematical decomposition is only possible in certain cases because of the many interactions between these two parts. These two wave functions correspond to different scales and have different meanings and interpretations.

The external wave function Ψ_{ext} represents the macroscopic view of the quantum system at the scale of the center of mass. Ψ_{ext} "*pilots*" the center of mass and the spin and corresponds to the de Broglie-Bohm theory. Indeed, the position of the center of mass, x_G , must be added in order to correctly describe the system. However, x_G is ignored by the observer who only knows its statistical distribution, which verifies the Born rule. Heisenberg's inequalities quantify the ignorance that the observer has of the precise position of the center of mass. If the positions of the center of mass are not included in the model, then all particles defined with the same Ψ_{ext} are indistinguishable particles prepared in the same way. The knowledge of x_G for each particle makes it possible to distinguish them. The dBB theory defines the underlying model that explains the statistical side of the Born rule. The external wave function spreads over time. The information on the external motion of the particle is given by the couple (Ψ_{ext}, x_G) .

The internal wave function Ψ_{int} represents the microscopic view of the quantum system: Schrodinger's interpretation seems to be the obvious one. The modulus square represents a continuous density of stuff as in the Poincaré electron model. The particle is not punctual but has a spatial extension. Ψ_{int} does not spread out in time; its size is fixed

and corresponds to the size usually given for the particle. It is much smaller than the size of the external wave (which can grow indefinitely). The Heisenberg inequalities are indetermination relations that can be explained because the particle is extended. By construction, the center of mass, x_G , of Ψ_{int} is equal to zero. To each particle $j = 1..N$ of an N -body system is associated an individual internal wave function ψ_j such that Ψ_{int} is the composition of all ψ_j .

The features of the two-scale interpretation is summarized in Table 1. This two-scale solution gives a simple explanation of wave-corpuscle duality; the external wave drives the system whereas the internal wave represents the corpuscular part of the system. This decomposition also makes it possible to see the relationships between quantum mechanics and general relativity in terms of a new paradigm, gravitation only appearing in the external wave function.

However, there are still many outstanding issues that we have not addressed in this article. The most important seem to be:

- How can this interpretation be generalized to special relativity?
- How does one take into account photons and massless particles?
- How does one take into account the many cases where the internal and external degrees of freedom are entangled?
- How does one take into account cases where quantum systems are remote?
- How does one introduce creation and annihilation operators in the two-scale solution?

Annexe A Case study 1: interference of the external wave of the molecule C_{60}

Young or Mach-Zehnder interferometry experiments are examples where only the external wave function of a particle interferes with itself. The internal structure is not necessary to understand the experiment and is omitted in the calculations.

The figures 3 and 4 represent a simulation of Young's slits experiment with fullerene molecules C_{60} under the conditions of the experiment conducted by Nairz, Arndt and Zeilinger [42]. The two slits are spaced 100 nm (center to center), each with a width of 55 nm. The average speed of the molecules is 200 m/s, which corresponds to a wavelength of 2.8 pm. The standard of the external wave function is represented in the blue figures: the lighter the blue, the higher the density.

TABLE 1 – Summary of the two-scale interpretation

External wave function Ψ_{ext}	Internal wave function Ψ_{int}
General features	
<p>Global motion of the quantum system (center of mass)</p> <p>Ψ_{ext} can spread over time in space and be divided into several parts. Its size is that of the wave packet, it can grow to infinity.</p> <p>Ψ_{ext} represents a set of indistinguishable quantum objects prepared in the same way.</p> <p>(Ψ_{ext}, x_G) represents the external part of a single quantum object.</p>	<p>Internal motion/structure in the center-of-mass reference frame (proper frame).</p> <p>Ψ_{int} is a non-dispersive wave packet, always remains confined in space. Its size is fixed and corresponds to the size usually given for the quantum object.</p> <p>Ψ_{int} represents the internal part a single quantum object.</p>
Reference Frame	
<p>Ψ_{ext} is defined in the laboratory reference frame.</p> <p>Ψ_{ext} pilots, x_G, the center of mass. We need to add x_G to the model in order to describe the system correctly.</p>	<p>Ψ_{int} is defined in the proper reference frame.</p> <p>The center of mass is stationary. We have no location information.</p>
Interpretation	
<p>dBB theory: (Ψ_{ext}, x_G)</p> <p>dBB trajectories converge towards Newton's trajectories</p>	<p>Schrödinger interpretation</p> <p>Extended particle (non-punctual particle)</p>
Approximation of quantum mechanics	
<p>Semi-classical approximation.</p>	<p>Quasi-classical approximation.</p>
Statistical interpretation of Born	
<p>Valid: $\Psi_{ext} ^2$ represents the statistical distribution of positions.</p>	<p>Non valid: $\Psi_{int} ^2$ represents a continuous density of matter like in Poincaré's electron model.</p>
Heisenberg's inequalities	
<p>Uncertainty (removed by the dBB theory)</p>	<p>Indeterminism (extended particle)</p>
Measurement	
<p>Possibility to measure the position of the center of mass</p> <p>Interference, Stern and Gerlach, EPR-B experiments, all experiments with particle beams</p>	<p>Experiments</p> <p>No direct measurement is possible (requires measuring another external wave function)</p> <p>Experiments with spectral lines (emission or absorption lines)</p>

In figure 3, the external wave function of a molecule of C_{60} is represented fifteen different times, from two millimeters before the slits (far left) to 5 millimeters after the slits (far right). The red line corresponds to the dBB trajectory of the center of mass of a molecule C_{60} whose initial position (before the slits) was randomly drawn in the initial wave packet¹.

The internal wave function of a molecule C_{60} that defines the internal structure is schematically represented on the figure by a drawing of the arbitrarily magnified molecule ($\times 13$) because its size is only 1 nm. The internal wave function remains unchanged throughout the experiment, its size remains 1 nm before and after the slits, and it does not interact with the external wave function that it transforms deeply. This experiment can be considered as crucial because it is difficult to imagine that the internal wave function of a molecule C_{60} does not pass through only one of the slits. It is its external wave function that passes through both slits at the same time. When measuring the impact of the molecule C_{60} to 5 mm after the slits, it is the internal wave function that interacts with the detection screen and produces the impact.

Figure 4 describes the same experiment, the density of the external wave function is continuously represented from 2 mm before the slits to 5 mm after. 24 dBB trajectories of the center of mass of a molecule of C_{60} are represented by a red line corresponding to 24 different starting points of the center of mass.

This experiment clearly shows that the external wave function pilots the internal wave function; they have very different spatial scales. The external wave can propagate indefinitely and divide, while the internal wave remains grouped and has a fixed size. We recall the point of view of Dirac who, in 1930 writes [23]: *“In quantum mechanics, particles are connected to waves that direct them and give rise, under appropriate conditions, to phenomena of interference and diffraction”*, and he adds that it is only a question of *“one and the same reality”*.

Thus, the external wave function corresponds, within the semi-classical limit, to the evolution of the center of mass of quantum systems. In relation to the scale of de Broglie wavelength, we return to the classical world.

1. Videos from this experiment are available here: vimeo.com/350139153 and vimeo.com/350132498

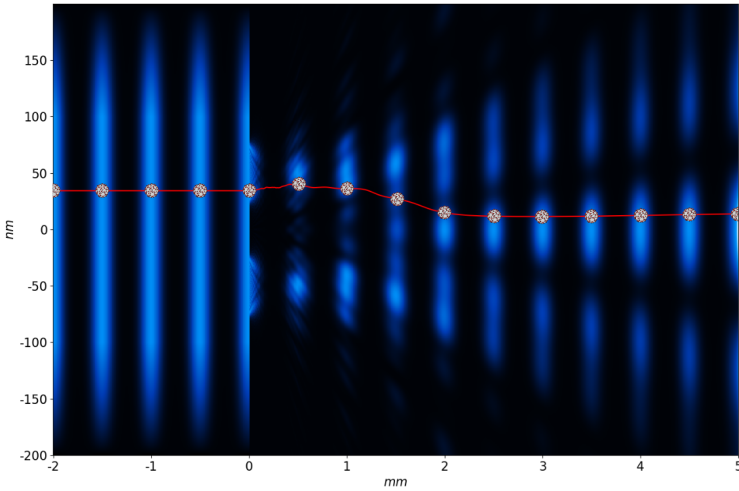


FIGURE 3 – Simulation of the evolution of the external (blue) and internal (white, magnified 13 times) wave functions of a molecule of C_{60} under the experimental conditions of [42] at fifteen different times every $2.5 \mu s$ (i.e. every 0.5 mm). The slits are placed at 0 mm and are spaced 100 nm and have a width of 55 nm .

Annexe B Case study 2: spin measurement by the external wave for the Stern and Gerlach experiment

Let us study the practical case of the spin which is a property also carried by the external wave function of the particle. Indeed, the measurement of the spin of a particle is a measure of the position of its center of mass. Let us consider the Stern and Gerlach experiment in measuring the spin of a silver atom. These atoms have, at the exit of the source \mathbf{E} , a velocity \mathbf{v} parallel to (Oy) . They cross an electromagnet \mathbf{A}_1 before condensing on a plate \mathbf{P}_1 (Fig. 5).

The magnetic moments of these silver atoms have been prepared in a pure state $(\theta_0, \varphi_0)^2$ at the initial instant $t = 0$ each atom can be

2. To prepare atoms all in the same state, the beam of atoms is first passed through a Stern and Gerlach apparatus, and only one of the two outputs is kept, judiciously oriented to obtain the desired pure state.

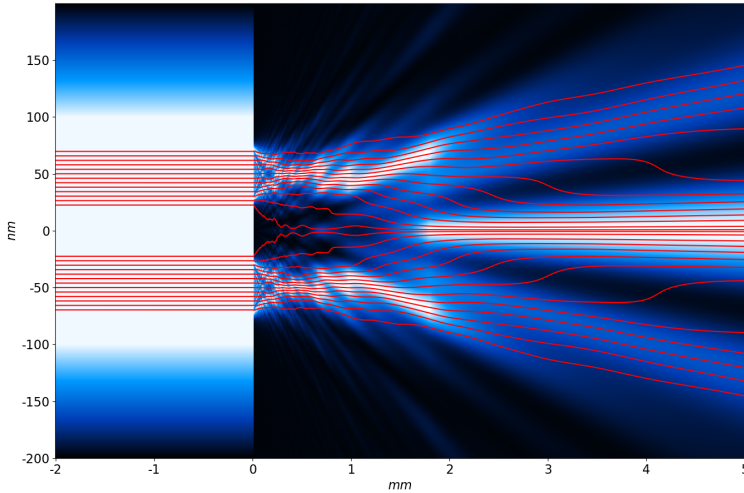


FIGURE 4 – Simulation of 24 dB trajectories of C_{60} under the experimental conditions of [42] corresponding to 24 different starting points of the center of mass of a C_{60} .

described by the Gaussian spinor in x and z :

$$\Psi^0(x, z, \theta_0, \varphi_0) = (2\pi\sigma_0^2)^{-\frac{1}{2}} e^{-\frac{(z^2+x^2)}{4\sigma_0^2}} \begin{pmatrix} \cos \frac{\theta_0}{2} e^{i\frac{\varphi_0}{2}} \\ i \sin \frac{\theta_0}{2} e^{-i\frac{\varphi_0}{2}} \end{pmatrix} \quad (30)$$

with $\mathbf{r} = (x, z)$. The variable y is processed in the classic way with $y = vt$. For the silver atom [10], we have $m = 1.8 \times 10^{-25}$ kg, $v = 500$ m/s, $\sigma_0 = 10^{-4}$ m. In the (30), θ_0 and φ_0 are the angles polarities characterizing the initial orientation of the vector representing the magnetic moment, θ_0 corresponds to the angle with (Oz) . Here, we have a pure state and all silver atoms have the same orientation of the magnetic moment.

Most quantum mechanics textbooks do not take into account the spatial extension $f(\mathbf{r}) = (2\pi\sigma_0^2)^{-\frac{1}{2}} e^{-\frac{(z^2+x^2)}{4\sigma_0^2}}$ of the spinor (30) and simply take the wave function in Hilbert's space of dimension 2 generated by $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$|\psi^0\rangle = \cos \frac{\theta_0}{2} e^{i\frac{\varphi_0}{2}} |0\rangle + i \sin \frac{\theta_0}{2} e^{-i\frac{\varphi_0}{2}} |1\rangle \quad (31)$$

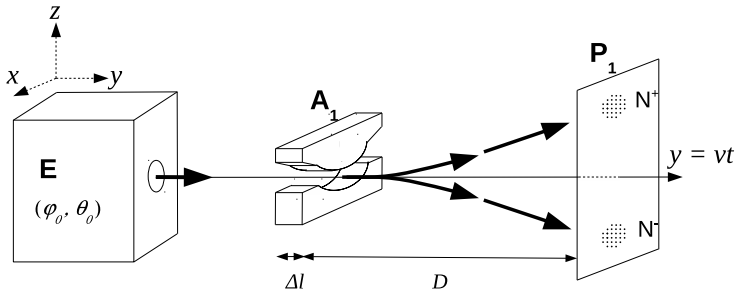


FIGURE 5 – Diagram of the Stern and Gerlach experiment: a stream of silver atoms, prepared in a pure state (φ_0, θ_0) and coming from the enclosure **E** passes through an inhomogeneous magnetic field (magnet **A**₁), then separates into two distinct beams resulting on the plate **P**₁ in two distinct spots of intensity N^+ and N^- .

By only retaining the wave function (31) with orientation (θ_0, φ_0) for the quantum system, part of the spinor (30) is lost, and only the statistical character is kept. The spatial extension of the spinor (30) takes into account (x_0, z_0) , the initial position of the center of mass of the particle (external variable) and makes the system quantum's evolution (wave function + position) deterministic.

The initial spinor (30) is actually only the external wave function of the silver atom. The internal wave function of the silver atom is not useful to describe the experiment. However, the spin of the silver atom seems to be only an internal degree of freedom since it comes from the addition of all the spin moments of its internal electrons. Atoms with an odd number of electrons, such as the silver atom, which has 47 electrons, have a half spin. Each pair of electrons has a total spin of zero. It is thus the $\frac{1}{2}$ -spin of the 47th electron that gives the atom its $\frac{1}{2}$ -spin. The spin is therefore an internal property but it also has an external modeling via the definition of the spinor. It is this external property that interests us in the experiment of Stern and Gerlach; Obviously the modification of the external model (spinor) has consequences on the internal model, but we do not take this aspect into account. The size of the internal wave function is about twice that of the atomic radius of the silver atom, or about 30 nm. while the initial size of the external wave function, i.e. the

width of the initial wave packet, is $3 \times \sigma_0 = 3 \times 10^5 \text{ nm}$, which is 4 orders of magnitude larger than the internal one.

The evolution of spinor $\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$ in a magnetic field $\mathbf{B} = (B_x, B_y, B_z)$ is then given by the Pauli equation [10]:

$$i\hbar \begin{pmatrix} \frac{\partial \psi_+}{\partial t} \\ \frac{\partial \psi_-}{\partial t} \end{pmatrix} = -\frac{\hbar^2}{2m} \Delta \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} + \mu_B \mathbf{B} \boldsymbol{\sigma} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \quad (32)$$

where $\mu_B = \frac{e\hbar}{2m_e}$ is Bohr's magneton, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and corresponds to Pauli's three matrices and where $\mathbf{B} \boldsymbol{\sigma}$ corresponds to the $B_x \sigma_x + B_y \sigma_y + B_z \sigma_z$.

Silver atoms pass through an electromagnetic field \mathbf{B} oriented mainly along the (Oz) axis, $B_x = B'_0 x$; $B_y = 0$; $B_z = B_0 - B'_0 z$, with $B_0 = 5$ Tesla, $B'_0 = \left| \frac{\partial B}{\partial z} \right| = -\left| \frac{\partial B}{\partial x} \right| = 10^3$ Tesla/m over a length $\Delta l = 1 \text{ cm}$. Upon exiting the magnetic field, the particles are free up to the plate P_1 placed at a distance $D = 20 \text{ cm}$. The particle passes the time $\Delta t = \frac{\Delta l}{v} = 2 \times 10^{-5} \text{ s}$ in the magnetic field. Upon exiting this field, we show [2, 22, 30, 33] that at the moment $t + \Delta t$ ($t \geq 0$), the external spinor is equal to:

$$\begin{aligned} \Psi(x_G, z_G, t + \Delta t) &= \begin{pmatrix} R_+ e^{i \frac{S_+}{\hbar}} \\ R_- e^{i \frac{S_-}{\hbar}} \end{pmatrix} \\ &\simeq \begin{pmatrix} \cos \frac{\theta_0}{2} (2\pi\sigma_0^2)^{-\frac{1}{2}} e^{-\frac{(z_G - z_\Delta - ut)^2 + x_G^2}{4\sigma_0^2}} e^{i \frac{mu z_G + \hbar \varphi_+}{\hbar}} \\ i \sin \frac{\theta_0}{2} (2\pi\sigma_0^2)^{-\frac{1}{2}} e^{-\frac{(z_G + z_\Delta + ut)^2 + x_G^2}{4\sigma_0^2}} e^{i \frac{-mu z_G + \hbar \varphi_-}{\hbar}} \end{pmatrix} \end{aligned} \quad (33)$$

with:

$$z_\Delta = \frac{\mu_B B'_0 (\Delta t)^2}{2m} = 10^{-5} m, \quad u = \frac{\mu_B B'_0 (\Delta t)}{m} = 1 m/s. \quad (34)$$

In the de Broglie-Bohm interpretation, the external spinor will define the trajectory $X_G(t) = (x_G(t), z_G(t))$ of the silver atom's center of mass from its initial position $X_G(0) = (x_G(0), z_G(0))$ by the formula [5, 49]:

$$\frac{dX_G(t)}{dt} = \frac{\hbar}{2m\rho} \text{Im}(\Psi^\dagger \nabla \Psi)|_{\mathbf{x}=X_G(t)} \quad (35)$$

where $\Psi^\dagger = (\Psi^{+*}, \Psi^{-*})$ and $\rho = \Psi^\dagger \Psi$. Bohm et al. [5] define a spin vector field \mathbf{s} as:

$$\mathbf{s}(\mathbf{x}, t) = \frac{\hbar}{2\rho} \Psi^\dagger(\mathbf{x}, t) \sigma \Psi(\mathbf{x}, t) = \frac{\hbar}{2} (\sin\theta \sin\varphi, \sin\theta \cos\varphi, \cos\theta). \quad (36)$$

The spin vector of an individual particle is evaluated along its trajectory as follows: $\mathbf{s} = \mathbf{s}(X_G(t), t)$. This spin vector is totally defined by the spinor and the position of the particle's center of mass.

Figure 6 represents $\rho(z_G, t) = \int \Psi^\dagger(x_G, z_G, t) \Psi(x_G, z_G, t) dx_G$, the probability density of presence of the silver atom for the values $\theta_0 = \pi/3$ and $\phi_0 = 0$. The axis (Oy), of the jet propagation, is on the abscissa ($y = v_y t$) and the axis (Oz) on the ordinate (the variable x is not represented because the wave remains Gaussian along this axis). The magnet A_1 is represented on the figure by the two white rectangles, it is $\Delta l = 1cm$ long and there is $D = 20cm$ of free travel before the atom measurement on the P_1 detection screen. A trajectory is also represented in figure 6 with its spin $\mathbf{s}(X_G(t), t)$ along this trajectory. If the position of the particle's center of mass is located at the top of the wave packet, as shown in the figure, then the particle will be *measured* in spin UP; if the initial position is lower, it will be measured DOWN.

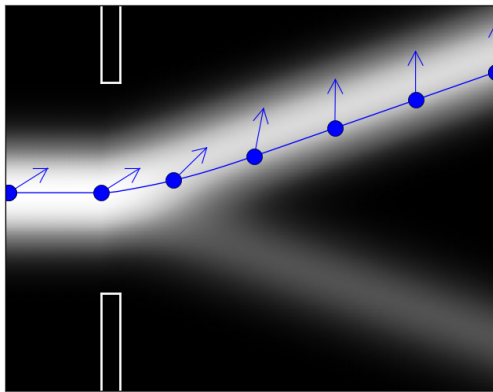


FIGURE 6 – The arrows indicate the θ orientation of the spin vector \mathbf{s} (initially $\theta_0 = \pi/3$) along the path. The position of the particle exists before the measurement; the particle then follows a deterministic trajectory and the impact on the screen only reveals its position.

The space (x, z) and spin (θ, φ) variables are independent in the initial external spinor Ψ_0 of Eq. (30). Indeed, Ψ_0 is factorized as the product of a (x, z) -dependent function and a (θ, φ) -dependent function. After passing through the inhomogeneous magnetic field, the variables of space and spin are entangled, cf. Eq. (33). Indeed, it is no longer possible to factorize Ψ as the product of a (x, z) -dependent function and a (θ, φ) -dependent function. The Stern and Gerlach experiment is not the measure of spin projection along the (Oz) axis, but the straightening of the spin orientation either in the direction of the magnetic field gradient or in the opposite direction. The result depends on the initial position of the particle's center of mass in the external wave function. It is a simple explanation of the non-contextuality of measuring spin along different axes. The duration of the measurement is the time required for the particle to straighten its spin in the final direction. The value “*measured*” (the spin) is not a pre-existing value like the mass and charge of the particle but a contextual value in accordance with the Kochen and Specker theorem [37].

The spin variables θ and φ are external degrees of freedom of the atom related to the external wave function, like the spatial variables x and z . The modification of the external spin variables obviously comes from an unknown underlying model involving the interaction of the photons of the inhomogeneous magnetic field with the spin of the 47th electron of the silver atom. The functioning of this sub-model concerns the level of the internal wave function.

Annexe C Case study 3: simulation of the particles involved in the EPR-B experiment

The EPR-B experiment corresponds to two Stern and Gerlach measurements of two particles entangled by their spin but very distant from each other. All measurements are made, as we have seen in Appendices A and B, through the position of the center of mass of the quantum system which is driven by the external wave function.

For the EPR-B experiment, we have 3 external wave functions: an external wave function for each of the two particles A and B (Ψ^A for particle A and Ψ^B for particle B), and an external wave function corresponding to the entanglement of the two particles, Ψ^{EPRB} . The Pauli equation enables to define the evolution in time of this entangled wave function. In this way, several dBB approaches [35, 4] have already calculated the evaluation of the entangled wave function, Ψ^{EPRB} , from the

initial singlet wave function:

$$\Psi_0^{EPRB}(\mathbf{x}_G^A, \mathbf{x}_G^B) = \frac{1}{\sqrt{2}} f(\mathbf{x}_G^A) f(\mathbf{x}_G^B) (|+A\rangle|-B\rangle - |-A\rangle|+B\rangle) \quad (37)$$

where \mathbf{x}_G^A and \mathbf{x}_G^B are the centers of gravity of the particles A and B and where $f(\mathbf{x}) = (2\pi\sigma_0^2)^{-\frac{1}{2}} e^{-\frac{\mathbf{x}^2}{4\sigma_0^2}}$. As in Stern and Gerlach's experiment (Appendix B), the initial function of the singlet wave must have a spatial extension in order to solve Pauli's equation.

Our difference with the usual Bohmian approach is that we assume that the two other external functions corresponding to particles A and B also exist, particularly at the initial instant. To take into account the fact that they are entangled by the (external) spin, we apply Pauli's principle and obtain for the total external wave function, the singlet state. Indeed, we assume that at the moment of the creation of the two entangled particles A and B, each of the particles has an initial wave function $\Psi_0^A(\mathbf{x}_G^A, \theta_0^A, \varphi_0^A)$ and $\Psi_0^B(\mathbf{x}_G^B, \theta_0^B, \varphi_0^B)$ of type (30) with opposite spins: $\theta_0^B = \pi - \theta_0^A$, $\varphi_0^B = \varphi_0^A - \pi$. If then we apply the Pauli principle, which stipulates that the entangled two-body must be antisymmetric, this yields:

$$\begin{aligned} \Psi_0^{EPRB}(\mathbf{x}_G^A, \theta_A, \varphi_A, \mathbf{x}_G^B, \theta_B, \varphi_B) &= \Psi_0^A(\mathbf{x}_G^A, \theta_A, \varphi_A) \Psi_0^B(\mathbf{x}_G^B, \theta_B, \varphi_B) \\ &\quad - \Psi_0^A(\mathbf{x}_G^B, \theta_B, \varphi_B) \Psi_0^B(\mathbf{x}_G^A, \theta_A, \varphi_A) \\ &= -e^{i\varphi_A} f(\mathbf{x}_G^A) f(\mathbf{x}_G^B) (|+A\rangle|-B\rangle - |-A\rangle|+B\rangle) \end{aligned} \quad (38)$$

which is the singlet state with spatial extension (37).

Then we measure the spin of the two particles one after the other. We show mathematically [33] that the first measured particle, particle A, behaves in the first Stern-Gerlach apparatus in the same way as if it were not entangled.

During the *measure* of A, the density of the particle B also evolves as if it were not entangled. These two properties can be experimentally tested as soon as the EPR-B experiment with atoms is feasible. During the *measure* of A, the spin of the particle B straightens up to always be in opposition to the spin of the particle A [33]. So there is an instantaneous action at a distance between the spins of A and B that always keeps them opposite. The second measure is a Stern-Gerlach measure with specific orientations. We then find perfectly the results of quantum mechanics and the violation of Bell's inequalities. Our approach differs from the

usual Bohmian approaches [35, 4] because in their modeling, the spin of each of the two particles A and B is initially zero and it is only after passing through the magnetic field that each of the spins increases from zero to its final value (with an opposite value). In our approach, each particle always has a 1/2-spin value and the two spin vectors are always opposite. Everything happens as if the total external wave function is replaced by two external wave functions (one for each particle) and an action-at-a-distance on the spins that always keeps them in opposite directions:

$$\text{Pauli eq. with } \Psi_{EPRB}^0 \iff \left\{ \begin{array}{l} \text{Pauli eq. with } \Psi_A^0 \\ \text{Pauli eq. with } \Psi_B^0 \\ \text{Action-at-a-distance on spins:} \\ \theta^B = \pi - \theta^A, \varphi^B = \varphi^A - \pi \end{array} \right.$$

As with the Stern and Gerlach experiment, the external spinor of the entangled state, which uses only the resolution of Pauli's equation on the external variables of the two particles yields, for the EPR-B, the same statistical results as the usual quantum mechanics. Quantum particles have a local position like a conventional particle, but also have a non-local behaviour due to the entangled external wave function.

In our article “*Replacing the Singlet Spinor of the EPR-B Experiment in the Configuration Space with two Single-particle Spinors in Physical Space*” [33] we show precisely how these three external spinors interfere, the singlet spinor with spatial extension which verifies the Pauli equation and the spinors of the two particles entangled with their spatial extensions.

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